Quasi-Linear Theory

210b

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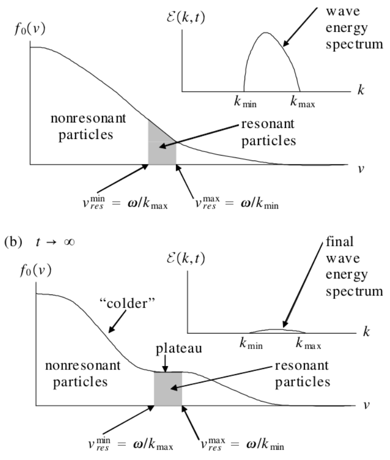
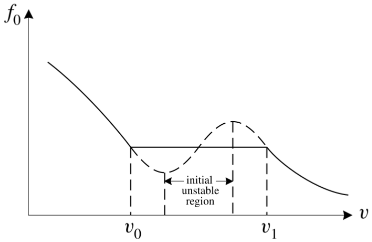
Motivations





Two types of initial value problem when we use QLT:

1. Wave spectrum in resonance with stable part of distribution function, $\partial f_0/\partial v < 0$
2. Bump-on-tail distribution with an interval of unstable waves where $\partial f_0/\partial v > 0$

Assumptions and Other Considerations

- Quasilinear theory (QLT) applies for a broad and dense spectrum of modes.
- Phases of modes are random and do not couple nonlinearly (weak turbulence assumption, more subtle assumptions and limitations - later)
- However, modes are dense $\delta\left(\omega/k\right) < \sqrt{e\phi_k/m}$ and resonant particles respond strongly to a few waves, not one. Particle motion becomes chaotic. They migrate in velocity space from one group of resonant waves to the next

The main subject of the QL theory is the backreaction of the excited modes onto the particle distribution function.
However, it is treated self-consistently with the mode evolution. The fundamental physical process here is the emission and absorption of waves by resonant particles at (Landau, or Cerenkov resonance)

$$\omega = kv$$

(limit treatment to 1D, $kv \to \mathbf{k} \cdot \mathbf{v}$, etc. in simple 3D cases)

Qualitative Arguments

Mode overlapping means the phase velocities of the neighbor modes satisfy

$$\delta\left(\frac{\omega}{k}\right) < \sqrt{\frac{e\bar{\phi}_k}{m}} \tag{1}$$

$\bar{\phi}_k$ is the characteristic wave potential, associated with the scale k^{-1}
the main variable of the QLT is the energy spectral density E_k^2, relates to $\overline{\phi}_k$:

$$\bar{\phi}_k = k^{-1}\sqrt{E_k^2 \Delta k}$$

Δk is the mode spacing in the wave number space in a system of length L, $\Delta k = 2\pi/L$.

- particle motion may become chaotic if only two modes are present
- Thus particles undergo Brownian motion in velocity space
- random walk from one resonance to the next

expect them to become evenly distributed over the interval

$$\Delta\left(\frac{\omega}{k}\right) = \left(\frac{\omega}{k}\right)_{max} - \left(\frac{\omega}{k}\right)_{min}$$

Expectations and more subtle assumptions

- plateau forms is the waves from the interval maintain their amplitudes to meet the condition (1) in course of the evolution
- However, waves in the packet should not be too strong to create their own potential well to make the particle dynamics quasi-coherent, akin to trapping phenomenon in a single wave. This limitation means

$$\Delta\left(\frac{\omega}{k}\right) \gg \sqrt{\frac{e\phi_0}{m}} = \sqrt{\frac{e}{m}\left(\int E_k^2 dk/k^2\right)^{1/2}}$$

Otherwise, most of the particles from this velocity interval will be involved in a collective quasi-coherent motion in the effective potential ϕ_0.

Formal Description-1

- decompose the particle distribution into
  - averaged, slowly varying part $f_0(v,t)$
  - and a small perturbation $f'(v,x,t)$ (averaging is over the random phases of modes that are responsible for the perturbation f'), $\langle f' \rangle = 0$

$$f = f_0 + f' = \langle f \rangle + f'$$

  - Alternatively, averaging can be performed over rapid oscillations of the modes either in time or in space, or both, whereas f_0 changes only slowly in time and, possibly, in space (ignore the latter for simplicity). Substituting the decomposition $f = f_0 + f' = \langle f \rangle + f'$ into the kinetic equation

$$\frac{\partial f_0}{\partial t} = \frac{e}{m} \left\langle E \frac{\partial f'}{\partial v} \right\rangle \tag{2}$$

$$\frac{\partial f'}{\partial t} + v \frac{\partial f'}{\partial x} = \frac{e}{m} E \frac{\partial f_0}{\partial v} \tag{3}$$

Formal Description-2

- 'quasilinear' character of approximation: the equations for f' and for E (given below) are linear
- nonlinearity is crucial in equation for f_0, but not w.r.t. f_0
  - describes backreaction of waves on particles, but phase- (or otherwise) averaged
- decomposition

$$f = \frac{1}{\sqrt{2}} \sum_k f_k \exp\left(-i\omega_k t + ikx\right) + c.c.$$

$$E = \frac{1}{\sqrt{2}} \sum_k E_k \exp\left(-i\omega_k t + ikx\right) + c.c.$$

Here $\omega_k = \Re\omega_k + i\gamma_k \approx \Re\omega_k$ is the solution of the appropriate dispersion equation, $\varepsilon(\omega, k) = 0$, with $\gamma_k \ll \omega_k$. From (3) we thus have

$$f_k = i\frac{e}{m}\frac{\partial f_0}{\partial v} E_k \left(\omega_k - kv\right)^{-1}$$

QL - derivation

- random phase approximation

$$\langle E_k E_{k'}^* \rangle = |E_k|^2 \, \delta\left(k - k'\right).$$

Thus, eq.(2) rewrites

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial f_0}{\partial v} \tag{4}$$

where

$$D = \frac{e^2}{m^2} \sum_k \tau_k \, |E_k|^2 \tag{5}$$

where a normalized particle scattering rate

$$\tau_k \equiv \frac{\gamma_k}{(\omega_k - kv)^2 + \gamma_k^2}, \quad \gamma_k \geq 0.$$

QL -derivation-2

- Eq.(4) describes diffusion in velocity space
- need add an equation for D, which is

$$\frac{\partial}{\partial t}|E_k|^2 = 2\gamma_k |E_k|^2 \tag{6}$$

where

$$\gamma_k = \left(\frac{\partial \Re \varepsilon}{\partial \omega_k}\right)^{-1} \frac{\omega_p^2}{n_0 k} \int \frac{\partial f_0}{\partial v} \pi \delta(\omega_k - kv)\, dv \tag{7}$$

- NB: growth rate of the waves is determined by the current distribution $f_0(v,t)$ which itself evolves in time under the wave backreaction on particles.
- τ_k has two simple limits:

QL- non-resonant heating

1. $$\tau_k \to \pi\delta(\omega - kv), \quad \gamma_k \to 0+ \tag{8}$$

   Physically, this means that particles are kicked by the resonant waves when they are near the points $v = \omega/k$.

2. $\omega_k \gg kv$, e.g. bulk electrons and the spectrum of Langmuir waves with $V_{ph} \gg V_{Te}$. We have $\tau_k \approx \gamma_k/\omega_p^2$. (Similarly, one can consider $kv \gg \omega_k$)

   $$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\frac{1}{2m}\left(\frac{\partial}{\partial t}\sum_k \frac{|E_k|^2}{4\pi n_0}\right)\frac{\partial f_0}{\partial v}$$

   This equation can be easily solved by introducing a new variable instead of t

   $$\tilde{T} = \sum_k \frac{|E_k|^2}{4\pi n_0} + T_e \tag{9}$$

   where T_e is the electron temperature, so that the last equation has the solution

QL-non-resonant heating-2

$$f_0(v,t) = \sqrt{\frac{m}{2\pi\tilde{T}}}\exp\left(-\frac{mv^2}{2\tilde{T}}\right)$$

- Without waves $\tilde{T} = T_e$, electrons have a Maxwellian distribution with the temperature T_e
- When waves are excited, the only effect is a reversible increase of the temperature. It is caused by electron nonresonant oscillations in the wave field,
  Indeed, from the equation of motion of electrons in nonresonant plasma waves with $\omega \approx \omega_p$, $\dot{v} = -(e/m)E$,

$$\overline{mv^2} = \sum_k m|v_k|^2 = \sum_k |E_k|^2/4\pi n_0$$

Therefore, the heating effect disappears together with the waves as there is **no entropy production**

QL-resonant interaction

Consider resonant wave-particle interaction in $\gamma_k \to 0$ limit (8)
δ function in the QLT equations (4,6) allows one to eliminate k as an independent variable

$$\frac{\partial f_0}{\partial t} = \pi \frac{\omega_p^2}{m n_0} \frac{\partial}{\partial v} \frac{W(v)}{\left| v - V_g(v) \right|} \frac{\partial f_0}{\partial v}$$

$$\frac{\partial}{\partial t} W = 2\pi \frac{\omega_p^2}{n_0} \left(\omega_k |k| \frac{\partial \Re \varepsilon}{\partial \omega_k} \right)^{-1}_{k=\omega_k/v} v \frac{\partial f_0}{\partial v} W(v)$$

Here $V_g(v) = \partial \omega_k / \partial k$ and $W(v)$

$$W_k = \omega_k \left(\frac{\partial \Re \varepsilon}{\partial \omega} \right)_{\omega=\omega_k} \frac{|E_k|^2}{8\pi} \qquad (10)$$

is the wave energy density (10), both calculated at $k = \omega_k / v$.

QL-Enefgy and Momentum Conservation

- total energy
  $\int mv^2 f_0 dv/2 + \int W_k dk = \int mv^2 f_0 dv/2 + \omega_p \int W(v)\, dv/v^2$
  and momentum
  $\int mvf_0 dv + \int \mathscr{P}_k dk$
  are conserved. The wave momentum density is
  $\mathscr{P}_k = kW_k/\omega_k = W(v)/v$ and $\omega \partial \Re \varepsilon / \partial \varepsilon \approx 2$.
  Note, that the quantity $N_k = W_k/\omega_k$ may be interpreted as a number of wave quanta. Indeed, the energy of a quantum with the wave number k (sometimes called *plasmon*) can be written as $\Delta W_k = \hbar\omega_k$, so that the number of such quanta is $N_k = W_k/\Delta W_k$. The momentum of the quantum is then $\Delta \mathscr{P}_k = \hbar k$. As the phenomena, we are interested in are purely classical, we may use the units in which $\hbar = 1$.

“Quasi-linear” integral

Consider Langmuir waves propagating at phase velocities $\omega_k/k \approx \omega_p/k \gg V_g \sim k^2\lambda_D^2\omega_p/k$. The QL equations are simplified to

$$\frac{\partial f_0}{\partial t} = \pi \frac{\omega_p^2}{mn_0} \frac{\partial}{\partial v} \frac{W(v)}{v} \frac{\partial f_0}{\partial v} \tag{11}$$

$$\frac{\partial}{\partial t} W = \pi \frac{\omega_p}{n_0} v^2 \frac{\partial f_0}{\partial v} W(v) \tag{12}$$

Apart from the velocity-integrated phase space density, particle energy, and momentum, also the following ’local’ quantity is conserved

$$\frac{\partial}{\partial t}\left(f_0 - \frac{\omega_p}{m} \frac{\partial}{\partial v} W/v^3 \right) = 0 \tag{13}$$

The latter result is the so-called QL integral, a powerful tool in solving wave-particle interaction problems

Q-L integral -2

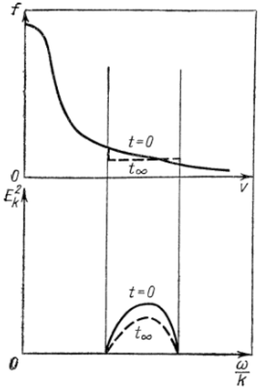
- from the initial conditions, $f_0(t=0) = f_0^0(v)$, $W(t=0) = W_0(v)$, also the final stage of the evolution, when the system reaches a steady state, can in many practical cases be determined without further calculations.

1. The first such case is when $f_0(t=\infty) = f_0^\infty(v) = const$ where $W(v) \neq 0$ (this nulls both r.h.s. in [11,12]).
2. The second case is when $W(v) = 0$, where $f_0^\infty(v) \neq const$. Then, integrating (13) in time, we can determine either W or f_0 in the final state, or we can relate them at any moment of time.

Assume at $t=0$, a Langmuir wave packet, having phase velocities $0 < v_1 < \omega_p/k < v_2$, is launched into a plasma with $\partial f_0/\partial v < 0$ for $v_1 < v < v_2$ (therefore, the waves can only be damped). From (13) we obtain

$$W(v, t=\infty) = W^\infty(v) = W_0(v) + \frac{m}{\omega_p} v^3 \int_{v_1}^{v} \left[f_0^\infty - f_0^0 \right] dv \tag{14}$$

QL-integral, wave damping

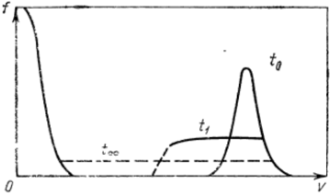


f_0^∞ is obtained from the final 'plateau' and particle conservation requirement

$$f_0^\infty = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} f_0^0 \, dv, \tag{15}$$

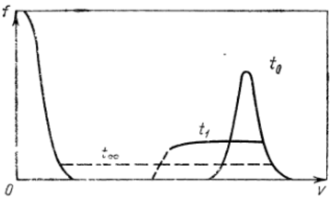
provided that $W^\infty > 0$.
The opposite case, $W^\infty < 0$, simply means that the initial wave energy is insufficient to form a plateau.
Time asymptotic solution is $W^\infty = 0$, while $f_0^\infty(v) \neq const$.
The latter quantity can be determined from (14)

QL Integral: Beam Relaxation



- consider a 'warm' beam instead of a wave packet
- assume waves to be initially weak for all v's they resonate with
- those waves where $\partial f_0/\partial v > 0$ are unstable.
- amplification of unstable waves will result in diffusion of particles to lower velocities, accompanied by the further wave excitation
- Finally, a **plateau** forms where $\partial f_0/\partial v$ was positive during the beam relaxation.
- Its height is (15), with v_1 being an intersection point of the f_0^∞ level with the thermal core distribution ($v_1 \simeq V_T$). v_2- is the intersection point with the right side of the beam distribution, where $\partial f_0/\partial v < 0$
- The wave energy density $W^\infty(v)$ comes from (14) where W_0 can be neglected in most cases.

Time-dependent Beam Relaxation: Qualitative Considerations



Fact

wave energy in region is very small, $\sim W_{th}$

Corollary

sharp front $f(v)$ *propagates to lower velocities*

- waves are rapidly excited within the front where $\partial f_0/\partial v$ is large
- plateau forms behind the front,

$$f_0 = f_0^+ = f_0(u+0,t)$$

  where $u(t)$ is the front coordinate in velocity space
- at $v \approx u$, the time dependence of f_0 takes the form $f_0(v,t) = f_0(v-u)$ and similarly for $W(v-u)$

Time-dependent Beam Relaxation: Dynamics-1

- dividing eq.(12) by W and integrating across the front we approximately have

$$-\dot{u}\ln\frac{W^+}{W_{th}} = \pi\frac{\omega_p}{n_0}u^2 f_0^+ = \pi\frac{\omega_p}{n_0}\frac{u^2 n_b}{v_2 - u}$$

where $W^+ = W(v = u + 0, t)$, v_2 is the right edge of the plateau (approximately equal to v_2 discussed earlier for a narrow beam) and n_b is the beam density. We also used the particle conservation to obtain the plateau height f_0^+.

- Integrate the last equation

$$\frac{v_2}{u} + \ln\frac{u}{v_2} = \pi\frac{n_b}{n_0}\omega_p\int_0^t\frac{dt}{\Lambda(t)} + 1 \approx \pi\frac{n_b}{n_0}\omega_p\frac{t}{\Lambda} + 1$$

Time-dependent Beam Relaxation: Dynamics-2

Proof.

quantity $\Lambda(t) = \ln\left(W^{+}(t)/W_{th}\right) \gg 1$ is large, so we can neglect the slow dependence $W^{+}(t)$ and took it out of the integral $\square$

- characteristic beam relaxation time, when u becomes $u \sim V_T \ll v_2 \simeq V_b$ is $\tau_{rel} = \omega_p^{-1}\Lambda n_0/n_b$. The factor Λ is close to the Coulomb logarithm L_C (see lecture on binary collisions in plasmas)
- The total energy transferred from the beam to the waves can be obtained from (14) and it is equal to beam energy loss:

$$\Delta E_b = W_{tot} = n_b \frac{mV_b^2}{2} - \int_{v_{min}}^{v_{max}} f_\infty \frac{mv^2}{2} dv \simeq n_b \frac{mV_b^2}{3} \qquad (16)$$