

[Physics 216]

Lecture II - Ideal Fluids
(Read Landau)

- Equations
- Basic Concepts, especially { Kelvin's Thm
Potential Flow
- Induced Mass

I.) Euler Equations / Ideal Fluids

Ideal - "The Flow of Dry Water"
blok (Feynman)

$\vec{v}$ ρ Volume $\bar{V}$
density ρ
 $\vec{v} \cdot \vec{s}$

- argue macroscopically but really derive from Boltzmann Equation
- viscosity brings additional time scale.

① - mass conservation

$$\frac{dM}{dt} = \frac{\partial}{\partial t} \int d^3x \rho(\vec{x}, t) = - \int d^3x \cdot \nabla \rho$$

$$= - \int d^3x \nabla \cdot (\rho \vec{v})$$

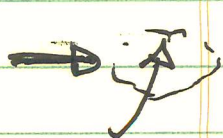
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$$\partial_t \rho + \nabla \cdot (\rho \underline{v}) = 0$$

↑ - mass flux
↓

$$\partial_t \rho + \nabla \cdot \underline{f} = 0$$

② - Momentum Conservation



blob/element

$$\underline{f} = -\nabla p + \underline{f}_{\text{body}}$$

↓
net force
density on element

↓
pressure
gradient

↳ body force

i.e. $\rho \underline{g}$

$\underline{I} \times \underline{B} / c$

⋮

For $\underline{f}_{\text{body}} \Rightarrow$

$$\rho \underline{a} = -\nabla p + \underline{f}$$

↓
acceleration.

$$\underline{a} = \frac{d\underline{v}}{dt}$$

↳ "substantive derivative"

○ →



now

$$d\underline{v} = \frac{\partial \underline{v}}{\partial t} dt + d\underline{r} \cdot \nabla \underline{v}$$

↓
increment

↓
local
acceleration

↓
displacement

↳ particle moves
in inhomogeneous
velocity field

so

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \frac{d\underline{r}}{dt} \cdot \underline{\nabla} \underline{v} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v}$$

so

$$\rho \frac{d\underline{v}}{dt} = \rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right) = -\underline{\nabla} p + \underline{f}$$

Euler Egn.

? Momentum Flux

Will show:

$$\partial_t (\rho v_i) = - \frac{\partial \pi_{ik}}{\partial x_k}$$

so

$$\begin{aligned} \partial_t (\rho v) &= v \partial_t \rho + \rho \frac{\partial v}{\partial t} \\ &= -v (\rho (\underline{\nabla} \cdot \underline{v}) + \underline{v} \cdot \underline{\nabla} \rho) \\ &\quad + \rho (-\underline{v} \cdot \underline{\nabla} v - \frac{\underline{\nabla} p}{\rho}) \\ &= - \left(\rho [\underline{v} (\underline{\nabla} \cdot \underline{v}) + \underline{v} \cdot \underline{\nabla} v] \right. \\ &\quad \left. + v (\underline{v} \cdot \underline{\nabla} \rho) \right) - \underline{\nabla} p \end{aligned}$$

$$\Rightarrow \partial_t(\rho \underline{v}) = - \underline{\nabla} \cdot (\underbrace{\rho \underline{v} \underline{v}}_{\substack{\text{Reynolds} \\ \text{stress tensor}}} + \underbrace{\underline{\underline{I}} P}_{\text{identity}})$$

(analogue to Maxwell stress tensor).

So

$$\boxed{\pi_{ik} = \rho v_i v_k + \delta_{ik} P} \quad \text{momentum flux}$$

$$\partial_t \int d^3x \rho \underline{v} = \frac{d}{dt} \underline{P} = - \int d\underline{S} \cdot (\rho \underline{v} \underline{v} + \underline{\underline{I}} P)$$

change in momentum of blob

$\pi_{ik} dS_k \equiv$ momentum flux in i th direction.

→ Beyond Euler, viscous stress appears due to momentum flux from collisions, interacting with microscopic flow gradients.

For incompressible flow ($\underline{\nabla} \cdot \underline{v} = 0$), continuity and Euler/Nav-Stokes describe flow.

→ Mass, Momentum and Energy!

In ideal fluid, no heat exchanged
between fluid elements $\Rightarrow$ motion
adiabatic - i.e. entropy conserved
along trajectories

$$\frac{ds}{dt} = 0$$

S = entropy/mass

$$\boxed{\frac{\partial S}{\partial t} + \underline{v} \cdot \nabla S = 0}$$

→ adiabatic equation
for fluid

For energy flux

$$\underbrace{\mathcal{E}}_{\substack{\downarrow \\ \text{total} \\ \text{energy} \\ \text{density} \\ \text{of fluid}}} = \underbrace{\frac{\rho \underline{v}^2}{2}}_{\substack{\downarrow \\ \text{kinetic} \\ \text{energy} \\ \text{density}}} + \underbrace{\rho \mathcal{E}}_{\substack{\downarrow \\ \text{internal} \\ \text{energy density} \\ (\text{i.e. thermal})}}$$

then use dynamics + thermo to
derive total energy balance equation

$$\partial_t \left(\rho \frac{v^2}{2} + \rho \epsilon \right) + \nabla \cdot \left(\rho \underline{v} \left(\frac{v^2}{2} + w \right) \right) = 0$$

$$w = \epsilon + \frac{p}{\rho}$$

enthalpy.

$$\partial_t \int d^3x \left(\rho \frac{v^2}{2} + \rho \epsilon \right)$$

$$= - \int d\underline{s} \cdot \left[\rho \underline{v} \left(\frac{v^2}{2} + w \right) \right]$$

What does this mean?

$$\underline{Q} = \rho \underline{v} \left(\frac{v^2}{2} + w \right)$$

energy flux density

What does it mean?

$$w = \epsilon + p/\rho$$

flux of KE and internal energy thru surface

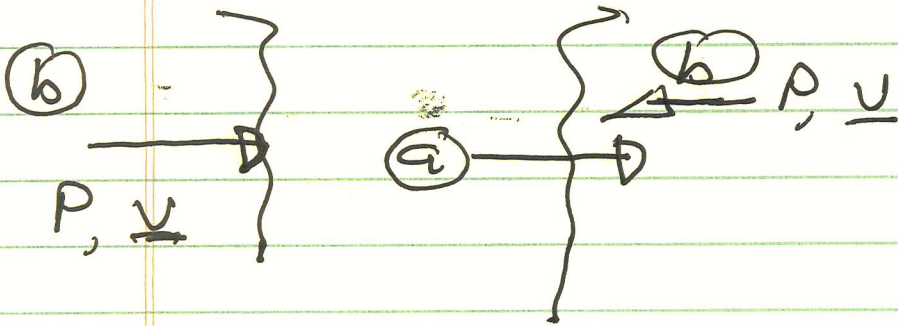
$$\int d\underline{s} \cdot \underline{Q} = \int d\underline{s} \cdot \rho \underline{v} \left(\frac{v^2}{2} + \epsilon \right)$$

$$+ \int d\underline{s} \cdot \rho \underline{v} \frac{p}{\rho}$$

$$\textcircled{b} = \int d\vec{s} \cdot \underline{v} P$$

$$= \int (\underline{v} \cdot d\vec{s}) P \rightarrow PdV \text{ work by pressure on fluid in blob}$$

$\textcircled{a} \equiv$ transport of energy thry the surface of the blob



Rate change of energy density

$$= \textcircled{a} + \textcircled{b}$$

To show:

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial t} \left(\overset{\textcircled{1}}{\frac{\rho v^2}{2}} + \overset{\textcircled{2}}{\rho \phi} \right)$$

$$\textcircled{1} = \frac{v^3}{2} \frac{\partial \rho}{\partial t} + \rho \underline{v} \cdot \frac{\partial \underline{v}}{\partial t}$$

$$= - \frac{\underline{v}^2}{2} \underbrace{\underline{\nabla} \cdot (\rho \underline{v})}_{\text{continuity}} - \underbrace{\underline{v} \cdot \underline{\nabla} p}_{\text{mom. balance}} - \rho \underline{v} \cdot (\underline{v} \cdot \underline{\nabla} \underline{v})$$

but

$$\underline{v} \cdot \underline{\nabla} \underline{v} = - \underline{v} \times \underline{\omega} + \underline{\nabla} \left(\frac{\underline{v}^2}{2} \right)$$

$$\downarrow$$

$$\underline{\omega} = \underline{\nabla} \times \underline{v} \rightarrow \text{vorticity}$$

$$\begin{aligned} \rho \underline{v} \cdot (\underline{v} \cdot \underline{\nabla} \underline{v}) &= \rho \underline{v} \cdot \left(- \underline{v} \times \underline{\omega} + \underline{\nabla} \frac{\underline{v}^2}{2} \right) \\ &= \rho \underline{v} \cdot \underline{\nabla} \frac{\underline{v}^2}{2} \end{aligned}$$

To deal with pressure:

$$d\underbrace{w}_{\text{Enthalpy}} = dE + d(pV)$$

$$= Tds - p d\underline{v} + v dp + p d\underline{v}$$

$$= Tds + \underline{d}p$$

$$\Rightarrow \boxed{\underline{\nabla} p = \rho \underline{\nabla} w - \rho T \underline{\nabla} s}$$

thus:

$$\textcircled{1} = \partial_t \left(\rho \frac{v^2}{2} \right) = -\frac{v^2}{2} \nabla \cdot (\rho \underline{v}) - \rho \underline{v} \cdot \nabla \left(\frac{v^2}{2} + w \right) + \rho T \underline{v} \cdot \nabla \underline{S}$$

$$\textcircled{2} \quad \partial_t (\rho \epsilon) = \dots$$

Useful to note:

$$d\epsilon = dQ - p dv$$

$$= T ds - p dv$$

$$v = 1/\rho, \quad dv = -d\rho/\rho^2$$

$$d\epsilon = T ds + \frac{p}{\rho^2} d\rho$$

$$\underline{\underline{\partial}} \quad d(\rho \epsilon) = \rho d\epsilon + \epsilon d\rho$$

$$d(\rho \epsilon) = \left(\frac{p}{\rho} + \epsilon \right) d\rho + \rho T ds$$

$$W = \epsilon + pV = \epsilon + p/\rho$$

$$d(\epsilon\rho) = Wd\rho + \rho Tds$$

and

$$\textcircled{1} = \partial_t (\rho\epsilon) = W \frac{\partial \rho}{\partial t} + \rho T \frac{\partial s}{\partial t}$$

$$= -W \underline{\nabla} \cdot (\rho \underline{V}) - \rho T \underline{V} \cdot \underline{\nabla} s$$

and so, combining $\textcircled{1}$, $\textcircled{2}$

$$\partial_t \left(\frac{\rho v^2}{2} + \rho\epsilon \right) = - \left(\frac{v^2}{2} + W \right) \underline{\nabla} \cdot (\rho \underline{V})$$

$$- \rho \underline{V} \cdot \underline{\nabla} \left(\frac{v^2}{2} + W \right)$$

$$= - \underline{\nabla} \cdot \left(\rho \underline{V} \left(\frac{v^2}{2} + W \right) \right)$$

$\Rightarrow$

$$\boxed{\partial_t \left(\frac{\rho v^2}{2} + \rho\epsilon \right) + \underline{\nabla} \cdot \left(\rho \underline{V} \left(\frac{v^2}{2} + W \right) \right) = 0}$$

→ Basic Laws and Concepts

What about vorticity $\underline{\omega} = \underline{\nabla} \times \underline{V}$?

Convenient to note:

$$\begin{aligned} dE &= \delta Q - p dV \\ &= T dS - p dV \end{aligned}$$

$W = E + pV \rightarrow$ enthalpy
then

$$dW = T dS + V dp = T dS + dP/\rho$$

and for isentropic flow ($dS = 0$)

$$dP/\rho = dW$$

thus can write (in isentropic case)
RHS of Euler as perfect derivative

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla} \underline{V} = -\underline{\nabla} W$$

Then consider circulation

$$\Gamma = \oint \underline{v} \cdot d\underline{l}$$

then

$$\begin{aligned} \frac{d}{dt} \oint \underline{v} \cdot d\underline{l} &= \oint \frac{d\underline{v}}{dt} \cdot d\underline{l} + \oint \underline{v} \cdot \frac{d}{dt} d\underline{l} \\ &= \oint (-\nabla w) \cdot d\underline{l} + \oint \underline{v} \cdot d\underline{v} \\ &= 0 \end{aligned}$$

so

$$\Gamma = \oint \underline{v} \cdot d\underline{l} = \text{const.}$$

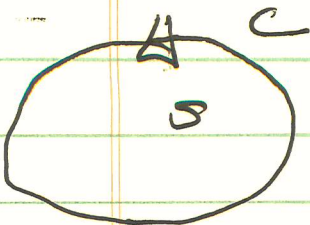
for ideal, isentropic fluid.

Kelvin's Thm.

Circulation
conserved

- n.b. : - broken ^{by} viscosity
 - $\nabla \cdot \underline{v} \neq 0$ irrelevant.

- Analogy in mechanics is
 Poincaré - Cartan invariant



$$I = \oint \underline{p} \cdot d\underline{q}$$

$$dI/dt = 0 \quad \text{for Hamiltonian system.}$$

and elementary vector calculus,

$$\Gamma = \oint_C \underline{v} \cdot d\underline{\ell} = \int_A \underline{\omega} \cdot d\underline{S}$$

$\uparrow$ normal to plane enclosed area.
 $\downarrow$
 $\nabla \times \underline{v} = \underline{\omega}$

What is vorticity:

- describes rotation of fluid element
- $\underline{\omega}$ is 2x effective local angular velocity of the fluid

$$d\underline{v} = (\underline{\omega} \times \underline{r}) / 2$$

* Vorticity is the non-trivial element in fluid dynamics / Vorticity is central to all interesting topics.

How evolve vorticity?

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla w$$

$$\begin{aligned} \underline{v} \cdot \nabla \underline{v} &= -\underline{v} \times (\nabla \times \underline{v}) + \nabla \frac{v^2}{2} \\ &= -\underline{v} \times \underline{\omega} + \nabla \frac{v^2}{2} \end{aligned}$$

so

$$\frac{\partial \underline{v}}{\partial t} - \underbrace{\underline{v} \times \underline{\omega}}_{\text{Magnus Force}} = -\nabla \left(w + \frac{v^2}{2} \right)$$

then $\nabla \times$

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{v} \times \underline{\omega})} \rightarrow \text{induction equation}$$

$$= -\underline{v} \cdot \nabla \underline{\omega} + \underline{\omega} \cdot \nabla \underline{v} - \underline{\omega} \nabla \cdot \underline{v}$$

$$\frac{d\underline{\omega}}{dt} = \underline{\omega} \cdot \nabla \underline{v} - \underline{\omega} (\nabla \cdot \underline{v})$$

and with continuity:

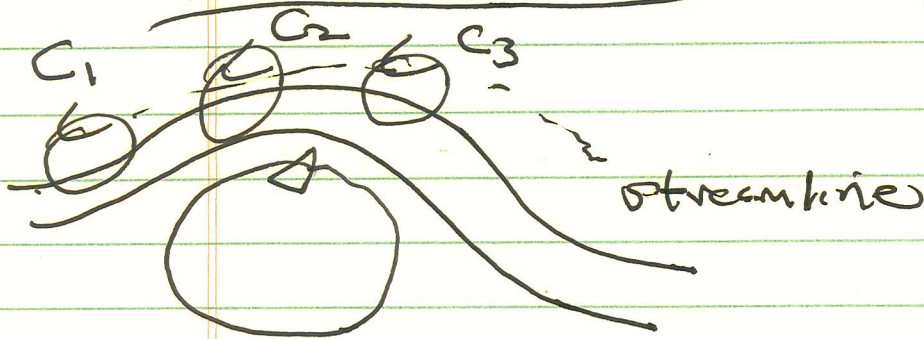
$$\frac{d}{dt} \left(\frac{\underline{\omega}}{\rho} \right) = \frac{\underline{\omega}}{\rho} \cdot \nabla \underline{v} \rightarrow \frac{\underline{\omega}}{\rho} \text{ "Frozen-in"}$$

Can derive Kelvin's Thm from induction eqn.

TBS

→ Potential Flow

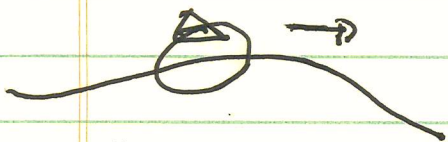
(copious analogies with electrostatics)

- Consider streamlines

Fluid flows along there, so

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}$$

If $\underline{\omega} = 0$ at any point on streamline, Kelvin's thm $\Rightarrow \underline{\omega} = 0$ everywhere on line,



tiny loop, then
pull along line, and
invoke Kelvin's theorem.

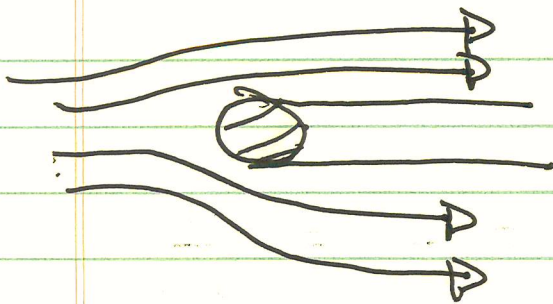
$$\oint_{C_1} \underline{v} \cdot d\underline{l} = \int_{A_1} \underline{\omega} \cdot d\underline{s} = 0 \quad \underline{\underline{so}}$$

$$\oint_{C_n} \underline{v} \cdot d\underline{l} = \int_{A_n} \underline{\omega} \cdot d\underline{s} = 0, \quad \text{all } C_n \text{ along line}$$

- Flow with $\omega = 0$ everywhere is potential or irrotational flow.

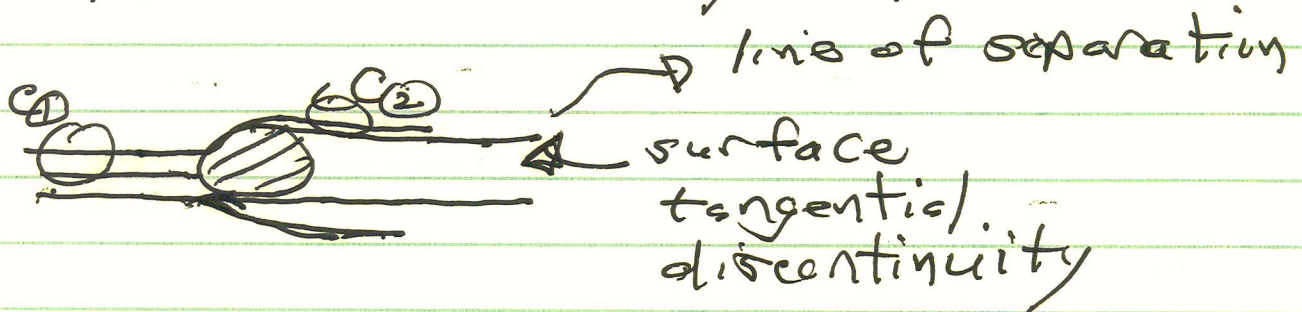
Important: Fails for Separation

e.g. consider flow around sphere



- streamlines separate from the body
- surface of tangential discontinuity appears in velocity component

e.g.



- cannot infer $\oint_{C_2} \underline{v} \cdot d\underline{l}$ from $\oint_{C_1} \underline{v} \cdot d\underline{l}$, due to separation - induced tangential discontinuity

- viscosity important in boundary layer. (No slip B.C.)

Now, for isentropic fluids:

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla W$$

potential flow
↓

$$\text{if } \underline{\omega} = 0, \quad \underline{u} = \nabla \phi$$

↑
stream function

$$\begin{aligned} \underline{u} \cdot \nabla \underline{u} &= -\underline{u} \times \underline{\omega} + \nabla (v^2/2) \\ &= \nabla (v^2/2) \end{aligned}$$

$$\frac{\partial \underline{u}}{\partial t} + \nabla (v^2/2) = -\nabla W$$

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} + W \right) = 0$$

So

have dynamical equation for potential flow:

$$\frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} + w = f(t)$$

defined for each stream line

- $\frac{\partial \phi}{\partial t} = 0$, recover ($ds=0$)

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{const.} \quad (\text{Bernoulli Law})$$

- Potential not uniquely defined, as $\underline{v} = \nabla \phi$.

Consider incompressible potential flow:

- $\underline{v} = \nabla \phi, \quad \nabla \cdot \underline{v} = 0$

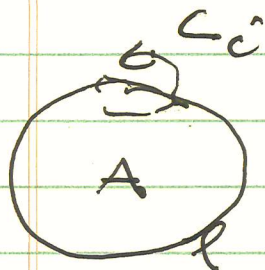
$$\nabla^2 \phi = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} + \frac{p}{\rho} = f(t)$$

For static Flow, with gravity:

$$\boxed{\frac{v^2}{2} + \frac{p}{\rho} + g z = \text{const}}$$

N.B.: In potential Flow,
streamlines must be open



$$\oint_{C_i} \underline{v} \cdot d\underline{l} = \int_{A_i} \underline{\omega} \cdot d\underline{S}_i = 0$$

$\underline{\omega} = 0$ along line

but then,

$$\int_A \underline{\omega} \cdot d\underline{S} = 0$$

but

$$= \oint_{\gamma_L} \underline{v} \cdot d\underline{l}$$

but $\oint \underline{v} \cdot d\underline{l} \neq 0 \rightarrow$ Fluid flow

$\Rightarrow$ contradiction.

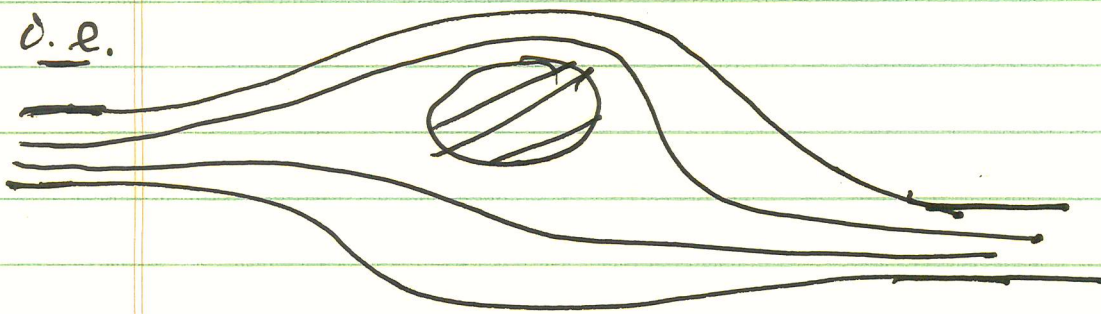
$\Rightarrow$

Streamlines
must be open.

Also, streamlines (for potential flow) should not intersect boundaries.

Generally, potential flow problems apply to infinite media, some distance from ~~surfaces~~ surfaces, boundaries.

d.e.



sphere in $\underline{V} = V_0 \hat{z}$ flow, for locations away from sphere, is typical, Flow problem, potential

Aside: What does "incompressibility" mean? When is $\nabla \cdot \underline{v} = 0$ a good approximation?

$$\leadsto |\underline{v}| \ll c_s$$

$$c_s^2 = dp/d\rho$$

$$\left(\frac{l}{\tau}\right)^2 \ll c_s^2$$

$\leadsto$ length, time scale ratio

v.e compare terms in continuity equation:

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \underline{v}$$

$$\int \quad \int$$

$$\frac{\Delta \rho}{\lambda} \quad \rho \frac{\underline{v}}{l}$$

For $\underline{v}$:

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\nabla p}{\rho}$$

$$\frac{\underline{v}}{\lambda} \sim \frac{c_s^2}{l \rho} \Delta \rho$$

so

$$\frac{\Delta \rho}{\lambda} \text{ vs } \frac{\lambda c_s^2}{l \rho} \Delta \rho$$

$$\text{Now, } |\nabla \cdot \underline{v}| \gg \left| \frac{1}{\rho} \frac{d\rho}{dt} \right|$$

means $\nabla \cdot \underline{v} \approx 0$, to good approximation.

so, incompressible if:

$$\frac{\gamma c_s^2}{\ell^2} \Delta \rho \gg \frac{\Delta \rho}{\rho}$$

$$\Rightarrow \boxed{c_s^2 \gg \frac{\ell^2}{\gamma^2}}$$

→ criteria in terms length time scale of flow.

$$\Leftrightarrow c_s^2 \gg \frac{\omega^2}{k^2}$$

Note: Long time favors incompressible

so $\underline{v} \cdot \underline{v} \approx 0$ if

- flow speeds subsonic

- times slow compared to time to traverse 1 spatial scale at acoustic speed.

$$\underline{\omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{z} = \hat{z} (-\nabla^2 \psi)$$

$$\frac{d\underline{\omega}}{dt} = 0 \Rightarrow \begin{cases} + \frac{\partial}{\partial t} \nabla^2 \psi + \underline{\nabla} \psi \times \underline{z} \cdot \nabla \nabla^2 \psi = 0 \\ \text{2D incompressible fluid eqn.} \end{cases}$$

iv.) Problems in Potential Flow

a.) Incompressible Potential Flow Around Sphere

Consider ^{rigid} sphere in motion at $\underline{u}$ in infinite fluid



Flow Pattern ?

Now :

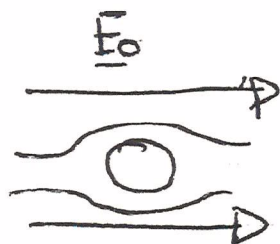
- intuitively, expect :



i.e. equivalent to $\begin{cases} \text{sphere at rest} \\ \underline{v}|_{\text{fluid}} = -\underline{u} \\ \infty \end{cases}$

Electrostatic analogy: Conducting sphere in uniform electric field

i.e.



$$\phi = -\underline{E}_0 \cdot \underline{r} + \phi_{\text{sphere}}$$

ϕ_{sphere} is dipole field.

Dipole moment determined by b.c.

i.e. $\phi = \text{const} = 0$ on sphere surface

Now, for potential flow (incompressible):

$$\nabla^2 \phi = 0$$

$$\underline{v} = \nabla \phi$$

$$v_n = \underline{v} \cdot \hat{n} = \underline{u} \cdot \hat{n} \Big|_{\text{surface}}$$

(i.e. normal velocity = sphere velocity on surface)

By analogy with electrostatics, can solve via:

- multipole expansion
- b.c.'s determine effective "charge" distribution

Recall e.s. $\Rightarrow \nabla^2 \phi = -4\pi\rho$

$$\phi = \int d^3x' \frac{\rho(x')}{|\underline{x} - \underline{x}'|}$$

For $\underline{x}$ outside region ρ :



$$\phi(\underline{x}) = \int d^3x' \frac{\rho(x')}{|\underline{x} - \underline{x}'|}$$

$$= \int d^3x' \frac{\rho(x')}{|\underline{x} - \underline{x}'|} = \int d^3x' \rho(x') \cdot \nabla \left(\frac{1}{|\underline{x}|} \right) + \dots$$

$$= \frac{Q}{|\underline{x}|} - \underline{d} \cdot \nabla \left(\frac{1}{|\underline{x}|} \right) + \dots$$

$\downarrow$
monopole

$\downarrow$
dipole

$\downarrow$
quadrupole

Thus, can write down general solution for potential flow streamlines around body as multipole expansion.

$Q = 0$ (no sources, sinks)

$\therefore$ in general dipole dominates

→ in 2D, same story with $\ln|x-x'| \rightarrow 1/|x-x'|$

Here: $u = u \hat{z}$ (spherical symmetry)
(flow velocity) (body velocity)



$$V_n|_R = V_r|_R = u \hat{z} \cdot \hat{n} = u \cos \theta$$

boundary condition

Now,

$$\phi(\underline{x}) = \underline{A} \cdot \underline{\nabla} (1/|\underline{x}|)$$

$u \rightarrow \hat{z}$

$\underline{A} = A \hat{z}$ (dipole moment in $\hat{z}$ direction)

$$\phi = -A \frac{\cos \theta}{r^2}$$

$$V_r = 2A \cos \theta / r^3$$

$V_r = u \cos \theta$
on surface

$$\Rightarrow \frac{2A \cos \theta}{R^3} = u \cos \theta$$

$$\Rightarrow A = \frac{R^3}{2} u$$

$$\phi = -u R^3 \cos \theta / 2 r^2$$

$$\underline{V} = \underline{\nabla} \phi$$

determined
general flow
field

Note:

regularity at ∞

- can recover from $\phi = \sum_{l=0}^{\infty} \left(a_l r^l + \frac{b_l}{r^{l+1}} \right) P_l(\cos \theta)$

expansion and b.c.'s.

- if sphere in uniform field:

$$\phi = U_0 r \cos \theta + \phi_{\text{sphere}}$$

ϕ
determine from $V_n = 0$

to determine pressure distribution on sphere,

$$\text{Recall: } \left. \begin{aligned} \rho \frac{\partial \phi}{\partial t} + \frac{\rho v^2}{2} + p &= p_0 \\ \downarrow \\ &\text{ambient pressure at } \infty \end{aligned} \right\} \begin{aligned} &\text{incompressible} \\ &\text{Bernoulli's Egn.} \end{aligned}$$

Thus, can immediately write:

$$p(x) = p_0 - \frac{\rho}{2} \nabla \phi \cdot \nabla \phi - \rho \frac{\partial \phi}{\partial t}$$

$\phi(x) \equiv$ determined a/c' above via $\nabla^2 \phi = 0$
and b.c.'s.

As sphere in motion (but uniform) :

$$\frac{\partial \phi}{\partial t} = -\underline{u} \cdot \nabla \phi + \frac{\partial \phi}{\partial y} \dot{y}$$

$\dot{y} = 0$

so

$$p(x) = p_0 - \frac{\rho}{2} \nabla \phi \cdot \nabla \phi - \underline{u} \cdot \nabla \phi$$

Generally, leads to concept of stagnation point

i.e. for Bernoulli Egn. for incompressible fluid :

$$\frac{p}{\rho} + \frac{V^2}{2} = \text{const.} = p_0$$

Now, consider fixed body in fluid with $\begin{cases} V_{\infty} = u_0 \\ p_{\infty} = p_0 \end{cases}$

As $V = 0$ on surface body :

$$p_{\max} = p|_{\text{bdy}} = p_0 + \frac{1}{2} \rho u^2$$

- stagnation point ($V=0$) on body is point of maximal pressure

- maximal pressure determined by $\begin{cases} p_0 \\ \text{speed} \end{cases}$



→ Fish skeleton strongest on front face, weakest elsewhere

→ front face is point of maximal pressure (head)

→ eye lens adjusts to allow for speed-induced pressure changes.

b.) Drag Force and Induced Mass

bubble

→ Heuristics: Consider rigid body in water.



Slow body motion $\Rightarrow$ potential flow around sphere
 $\Rightarrow$ energy in fluid motion, too!

Thus, for F_{ext} to move body in fluid, need work against

- inertia of body (obvious)
- inertia of fluid, excited into potential flow

Thus, for body in water, need interpret Newton's 2nd Law as:

$$\underline{F}_{\text{ext}} = M_{\text{eff}} \frac{d\mathbf{y}}{dt}$$

$$M_{\text{eff}} = \underbrace{M}_{\substack{\downarrow \\ \text{mass of body}}} + m_{\text{induced}}$$

induced mass of fluid in potential flow around body
(mass of fluid flow which 'dresses' the body)

in water

see plot
from
reference

To calculate induced mass:

① - calculate energy in potential flow around rigid body in uniform motion in fluid

② - use $dE = d\mathbf{p} \cdot \mathbf{y}$ to determine momentum in fluid

$$\text{as } \underline{p} = \underline{p}(\mathbf{y}) \Rightarrow p_i = m_{ik} U_k$$

$\therefore m_{ik}$ is induced mass tensor!

→ Calculation: Consider rigid body moving in fluid

i.e.



Now, for flow field outside body, multipole expansion solution to $\nabla^2 \phi = 0$ yields

$$\phi = \underbrace{\frac{A}{r}}_{\substack{\text{monopole} \\ (\text{vanishes} \rightarrow \\ \text{no sources})}} + \underbrace{A \cdot \underline{D} \left(\frac{1}{r} \right)}_{\substack{\text{dipole} \\ (\text{dominant multipole} \\ \text{at large radius})}} + \dots$$

$\rightarrow$ dipole moment: $A = c R^3 \underline{u}$

$\therefore \phi = A \cdot \underline{D} \left(\frac{1}{r} \right) \quad (c = 1/2, \text{ sphere})$

$$= -A \cdot \underline{r} / r^3 = -A \cdot \underline{\hat{n}} / r^2$$

$$\underline{V} = \underline{\nabla} \phi = A \cdot \underline{\nabla} \left(\frac{1}{r} \right)$$

$$= (A \cdot \underline{\nabla}) \left(-\frac{1}{r^3} \right)$$

$$\underline{V} = (3(A \cdot \underline{\hat{n}}) \underline{\hat{n}} - A) / r^3$$

$$\Phi = -A \frac{\cos \theta}{r^2}$$

$$u_r = \frac{2A \cos \theta}{r^3}$$

$$u_{\cos} = \frac{2 \sin \theta}{r^3}$$

$$A = \frac{4}{3} R^3$$

Now, for energy, seek calculate fluid energy
in Volume V enclosed within radius R
around body. Take $R^3 \gg V_0 \equiv$ volume of body.

Thus:

$$E = \frac{1}{2} \rho \int dV |\underline{\nabla} \phi|^2$$

$$= \frac{1}{2} \rho \int d^3x (\underline{u}^2 + |\underline{\hat{n}}|^2 - \underline{u}^2)$$

$$\begin{aligned}
 \text{out } |\underline{V}|^2 - u^2 &= (\underline{V} + \underline{u}) \cdot (\underline{V} - \underline{u}) \\
 &= \nabla(\phi + \underline{u} \cdot \underline{r}) \cdot (\underline{V} - \underline{u}) \\
 &= \nabla \cdot [(\phi + \underline{u} \cdot \underline{r})(\underline{V} - \underline{u})]
 \end{aligned}$$

$$\begin{aligned}
 \text{as } \underline{V} &= \nabla \phi & \nabla \cdot \underline{V} &= 0 \\
 \underline{u} &= \text{const.} & \nabla \cdot \underline{u} &= 0
 \end{aligned}$$

$$\therefore E = \frac{1}{2} \rho \int d^3x \left[u^2 + \nabla \cdot [(\phi + \underline{u} \cdot \underline{r})(\underline{V} - \underline{u})] \right]$$

$$= \frac{1}{2} \rho u^2 (V - V_0) + \frac{1}{2} \rho \int d\underline{S} \cdot [(\phi + \underline{u} \cdot \underline{r})(\underline{V} - \underline{u})]$$

$\int$ $\hookrightarrow$ Volume object/body
 volume space

$$V = \frac{4\pi}{3} R^3$$



$$\begin{cases} (\underline{V} - \underline{u}) \cdot d\underline{S} = 0 \\ \text{on } R_0 \text{ surface} \end{cases}$$

Now, $d\underline{S} = \underline{\hat{n}} R^2 d\Omega$, on outer surface

$$E = \frac{1}{2} \rho u^2 (V - V_0)$$

$$+ \frac{1}{2} \rho \int R^2 d\Omega [(\underline{\hat{n}} \cdot \underline{V} - \underline{\hat{n}} \cdot \underline{u})(\phi + \underline{u} \cdot \underline{r})]$$

$$E = \frac{1}{2} \rho u^2 (V - V_0) + \frac{1}{2} \rho \int R^2 d\Omega \left[\left(2 \frac{(\underline{A} \cdot \underline{\hat{n}})}{R^3} - \underline{u} \cdot \underline{\hat{n}} \right) \left(\frac{-\underline{A} \cdot \underline{\hat{n}}}{R^2} + R \underline{u} \cdot \underline{\hat{n}} \right) \right]$$

$$= \frac{1}{2} \rho u^2 (V - V_0) + \frac{1}{2} \rho \int R^2 d\Omega \left[-2 \frac{(\underline{A} \cdot \underline{\hat{n}})^2}{R^5} \right] \quad \text{vanishes for large } R$$

$$+ \frac{(\underline{u} \cdot \underline{\hat{n}})(\underline{A} \cdot \underline{\hat{n}})}{R^2} + \frac{2(\underline{A} \cdot \underline{\hat{n}})(\underline{u} \cdot \underline{\hat{n}})}{R^2} - R(\underline{u} \cdot \underline{\hat{n}})^2 \Big]$$

$$= \frac{1}{2} \rho u^2 (V - V_0) + \frac{1}{2} \rho \int R^2 d\Omega \left[\frac{3(\underline{A} \cdot \underline{\hat{n}})(\underline{u} \cdot \underline{\hat{n}})}{R^2} - R^3 (\underline{u} \cdot \underline{\hat{n}})^2 \right]$$

Thus finally,

$$E = \frac{1}{2} \rho u^2 (V - V_0) + \frac{1}{2} \rho \int d\Omega \left[3(\underline{A} \cdot \underline{\hat{n}})(\underline{u} \cdot \underline{\hat{n}}) - R^3 (\underline{u} \cdot \underline{\hat{n}})^2 \right]$$

$$d\Omega = d\theta \sin\theta d\phi$$

$$\text{if } \int d\Omega () = \langle () \rangle$$

$$\Rightarrow \langle (\underline{A} \cdot \underline{\hat{n}})(\underline{B} \cdot \underline{\hat{n}}) \rangle = \frac{1}{2} \delta_{ij} A_i B_j = \frac{1}{3} \underline{A} \cdot \underline{B}$$

$$E = \frac{1}{2} \rho u^2 (V - V_0) + \frac{1}{2} \rho \left[4\pi \underline{A} \cdot \underline{u} - \frac{4\pi}{3} R^3 u^2 \right]$$

$$= \frac{1}{2} \rho \left[4\pi \underline{A} \cdot \underline{u} - u^2 V_0 \right]$$

Thus finally,

$$E = \frac{\rho}{2} \left[4\pi \underline{A} \cdot \underline{u} - u^2 V_0 \right] \quad \left\{ \begin{array}{l} \text{energy in} \\ \text{potential} \\ \text{flow and} \\ \text{body} \end{array} \right.$$

Now, $\underline{A} = \underline{A}(u) \Rightarrow \left\{ \begin{array}{l} E = \frac{1}{2} m_{ik} u_i u_k \\ \text{defines induced mass} \\ \text{tensor} \end{array} \right.$

$$dE = \underline{u} \cdot d\underline{P}$$

$$\Rightarrow \underline{P} = \frac{\rho}{4\pi\sigma} \left[4\pi \underline{A} - V_0 \underline{u} \right] \quad \left\{ \begin{array}{l} \text{momentum in} \\ \text{potential flow} \end{array} \right.$$

Now, consider external force acting system,
where system = body + fluid (in Pot. flow)

$$\text{i.e. } \underline{F}_{\text{ext}} = \frac{dP_{\text{fluid}}}{dt} + M_{\text{body}} \frac{dU}{dt}$$

$$\Rightarrow F_i = (M \delta_{ik} + m_{ik}) \frac{dU_k}{dt}$$

∴ effective mass of "system" is sum
of - body mass

- induced mass of fluid in
potential flow around body

→ Note induced mass is determined purely
by body shape (i.e. via volume and dipole
moment)

$$\text{i.e. for sphere } A = \frac{R_0^3}{2} U$$

$$P = \rho \left[4\pi \frac{R_0^3}{2} U - \frac{4\pi}{3} R_0^3 U \right]$$

$$= \rho \frac{2}{3} \pi R_0^3 U$$

$$m_{\text{induced}} = \rho \frac{2}{3} \pi R_0^3$$

In general $M_{\text{induced}} \sim \# \rho R^3$

$$\sim \# \rho V$$

↓ $\rightarrow$ displaced mass
 Numerical fluid
 factor, shape dependent

$\rightarrow$ Example of "renormalization" in classical physics
 "dressing field" in continuum i.e. $\left\{ \begin{array}{l} \text{renorm.} \\ \text{polaron} \\ \text{debye shield} \\ \text{etc} \end{array} \right.$

i.e. in quantum electrodynamics $\rightarrow$ electron polarizes Vacuum

$$\rightarrow \frac{d}{dt} \frac{d}{dt}$$

$$\rightarrow m_e = m_e^{\text{bare}} + m_e^{\text{V.P.}} \\ (E=mc^2)$$

in classical potential flow $\rightarrow$ moving a sphere in H_2O requires that some energy go into surrounding media (the water!)

(skip)

$\rightarrow$ Enhanced inertia due induced mass may alternatively, be viewed as drag force on body
 mom. transmittal to fluid (careful of phase!)

i.e.
$$F_{\text{ext}} = \frac{dP_{\text{fluid}}}{dt} + M \frac{dy}{dt}$$

$$M \frac{d\vec{u}}{dt} = \vec{f}_{ext} - \frac{d\vec{P}_{fluid}}{dt}$$

$$= \vec{f}_{ext} + \vec{f}_{drag, lift}$$

$$\vec{f}_{drag} \sim \vec{u}$$

$\vec{f}_{drag} = -\frac{d\vec{P}_{fluid}}{dt}$, along direction motion.

$\vec{f}_{lift} = -\frac{d\vec{P}_{fluid}}{dt}$, $\perp$ direction of motion.

Note: $\rightarrow$ if body in uniform motion in ideal (fantasy) fluid $\vec{f}_{drag} = \vec{f}_{lift} = 0$ { D'Alembert's paradox }

$\rightarrow$ need external force to maintain uniform motion

as no = no dissipation (ideal fluid)
 = no loss of energy to ∞ ($V \sim 1/R^3$)

$\rightarrow$ but if body near surface



body will radiate surface waves to ∞ (wake) $\Rightarrow$ wave drag induced energy loss!

→ Related Problem:

- consider body in fluid, which is set in motion by external agent



Relate $\underline{u}$ body to $\underline{v}$ fluid !!

- Now $\underline{v} \equiv$ velocity of unperturbed flow

$$\frac{\|\nabla \underline{v}\| R_0}{\|\underline{v}\|} \ll 1 \Rightarrow \underline{v} \sim \text{const over scale of body} \\ (\text{potential flow valid})$$

so if body fully carried along by fluid ($\underline{v} = \underline{u}$), then force on it would equal force on volume of displaced fluid

$$\text{i.e. } \frac{d}{dt} (M \underline{u}) = \rho V_0 \frac{d\underline{v}}{dt}$$

but body moves relative to fluid, so that fluid acquires momentum → due to relative motion

$$\text{i.e. } d\underline{p}_{\text{fluid}}/dt = -\underline{m} \cdot \frac{d}{dt} [\underline{u} - \underline{v}]$$

∴ so really,

$$\frac{d}{dt} (M \underline{u}) = \rho V_0 \frac{d\underline{v}}{dt} = \frac{d}{dt} (m (\underline{u} - \underline{v}))$$

$$\frac{d}{dt} (M u_i) = \rho V_0 \frac{d v_i}{dt} - m_{ik} \frac{d}{dt} (u_k - v_k)$$

⇒

$$M u_i = \rho V_0 v_i - m_{ik} (u_k - v_k)$$

$$(M \delta_{ik} + m_{ik}) u_k = (\rho V_0 \delta_{ik} + m_{ik}) v_k$$

$$u_k = \left(\frac{\rho V_0 \delta_{ik} + m_{ik}}{M \delta_{ik} + m_{ik}} \right) v_k$$

Note: $\rho V_0 < M$ (body heavier than displaced fluid) → body lags

$\rho V_0 > M$ → body leads

$$\rho V_0 = M \quad u_k = v_k.$$

Thus

$$M \frac{du}{dt} = \rho_f V \frac{dv}{dt} - \underline{m} \cdot \frac{d}{dt} [y - v]$$

$$(M \delta_{ij} + m_{ij}) \frac{du_j}{dt} = M_f \delta_{ij} + m_{ij} \frac{dv_j}{dt}$$

$$\therefore u_j = \left[(M_f \delta_{ij} + m_{ij}) / (M \delta_{ij} + m_{ij}) \right] v_j$$

$$M_f = \rho_f V_0$$

$$M = \rho V_0$$

$$\Rightarrow u = v \quad \text{if} \quad \rho_f = \rho$$

$$u < v \quad \text{if} \quad \rho_f < \rho \quad \rightarrow \text{heavy object / eqs}$$

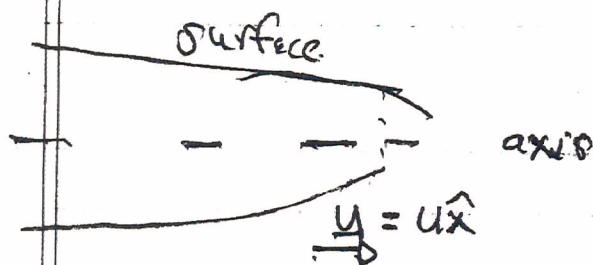
$\rho_f \equiv$ fluid density

$\rho \equiv$ body density

$$u > v \quad \text{if} \quad \rho_f > \rho \quad \rightarrow \text{light object / eqs.}$$

c.) Potential Flow - General Slender Body

- Till now, have considered simple body potential flows, i.e. sphere, cylinder,
Here consider general body from surface of revolution



- i.e.
- generally axially symmetric slender body
 - slender $\Rightarrow w/L \ll 1$

Now, observe analogy with electrostatics again,

i.e. e.s. $\Rightarrow \phi(\underline{x}) = \int d^3x' \rho(\underline{x}') / |\underline{x} - \underline{x}'|$

potential flow ($A \sim u V$)

$$\phi(\underline{x}) = \frac{1}{4\pi} \int d^3x' (\dot{\rho}(\underline{x}') / \rho_0) / |\underline{x} - \underline{x}'|$$

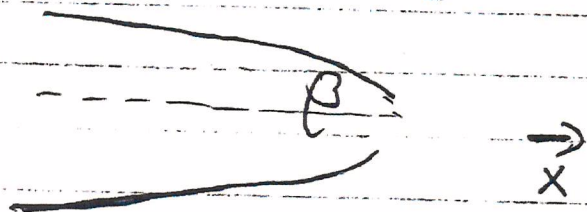
$\frac{\dot{\rho}(\underline{x}')}{\rho_0} \equiv$ normalized density of fluid flowing across cross-section of body

$\rightarrow$ yields $A \sim V_0 U$ etc.

$$\phi(\underline{x}) = \frac{1}{4\pi |\underline{x}|^2} \int d^3x' \frac{\dot{\rho}(\underline{x}')}{\rho_0} \underline{x}' + \text{h.o.t.}$$

$\downarrow$
 dipole term dominates

Flow, - body slender $\rightarrow \frac{w}{L} < 1 \Rightarrow \beta \ll 1$



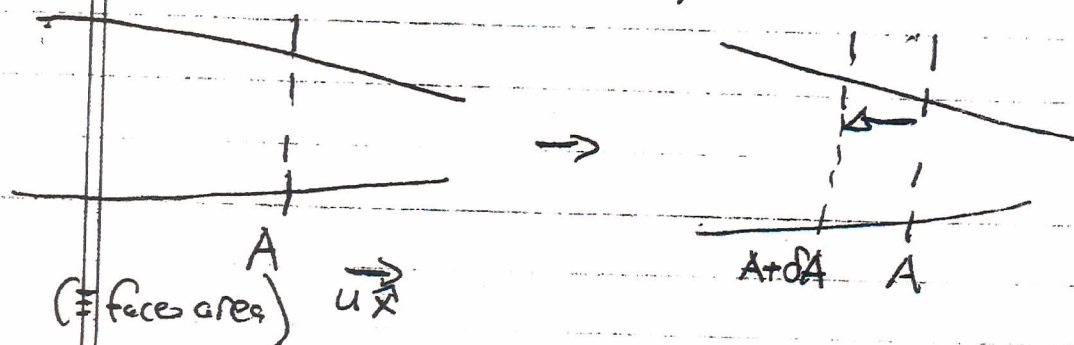
- $\nabla \cdot \underline{V} = 0$ and axial symmetry $\Rightarrow$

$$\frac{\partial V_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r V_r) = 0$$

$$\therefore \frac{V_r}{V_x} \sim \frac{\Delta r}{\Delta x} \sim \beta \sim \frac{w}{L} \ll 1$$

$\Rightarrow$ need only consider $\hat{x}$ fluid motion

$\therefore$ to compute dipole moment, need $\rho(x)/\rho_0$ for fluid flow across body



$$\text{Net } \frac{\rho}{\rho_0} = u \left[A + \delta A - A \right] = u \frac{\partial A}{\partial x} dx$$

$$\Rightarrow \rho(x)/\rho_0 = u \frac{\partial A}{\partial x}$$

$$\therefore \phi(x) = \frac{1}{4\pi/x^2} \int dx' x' u \frac{\partial A(x')}{\partial x'}$$

$$= \frac{-u}{4\pi/x^2} \int dx' A(x')$$

$$= \frac{-u V}{4\pi/x^2}$$

$$V \equiv \text{volume of body} = \int dx' A(x')$$

$\Rightarrow$ yields intuitive result:

$$\phi(x) = \frac{-u V_{\text{body}}}{4\pi r^2}$$

effective dipole moment for slender body.