Physics 210B

<u>L3</u> Beyond Boltzmann ... to <u>Euler</u>

So far: – derived Boltzmann equation

$$\partial_t f + \underline{v} \cdot \underline{\nabla} f = C(f)$$

$$\left[\partial_t f + \underline{v} \cdot \underline{\nabla} f + \underline{a} \cdot \underline{\nabla}_v f = C(f) \right]$$

$$\frac{q}{m}\underline{E} + \frac{q}{mc}\underline{v} \times \underline{B}$$ – Vlasov for $C(f) \to 0$

– proved H-Thm. { Phase space / Fluid continuity

$$\frac{dS}{dt} \geq 0$$

→ identified Eqbrm. Distribution f_{Max}

→ $f \to f_{Max}$ on ν_c time scale

→ <u>local</u> Maxwellian.

Now what?

What to do with Boltzmann Equation?

→ real problems:

- dynamics
- inhomogeneity

} combining collective and collisional processes

i.e. heat — conduction
— convection.

→ two roads forward:

- low collisionality, work with modified Vlasov equation

⇒ see Plasmas, Galactic Dynamics

$$d < n^{-1/3} < l_{mfp} < L$$

– Fluid Equations

i.e. $f \approx f_{Max}[T(\underline{x}), n(\underline{x}), \underline{V}(\underline{x})]$

here

⇒ evolution equations for:
$n(\underline{x})$, $\underline{V}(\underline{x})$, $T(\underline{x})$ etc.

⇒ macroscopic

⇒ Derived from Boltzmann . .

Related: Inhomogeneity – i.e. $T(\underline{x})$

⇒ Transport

$$\underline{Q} = -\chi \underline{\nabla} T$$ {constitutive relation

↓

thermal diffusivity

→ transport coefficient

$$\rightarrow f = f_{0Max.} + \delta f$$

i.e. distribution 'close to', but not equal to Maxwellian.

$l_{mfp}/L \ll 1$, but not $\rightarrow 0$

viscosity, heat conductivity..

- how compute transport coefficients
  ⇒ needed for real fluid equations
- point:

$$\partial_t f + \underline{v} \cdot \underline{\nabla} f = C(f)$$

For inhomogeneous f_{eq},

$$\cancel{\partial_t f_{eq}} + \underline{v} \cdot \underline{\nabla} f_{eq} = \cancel{C(f_{eq})}$$

- does not satisfy Boltzmann equation!

- $f = f_{eq} + \underbrace{\delta f}_{\text{collisional flux}}$

The Rest:

On Fluid Equations:

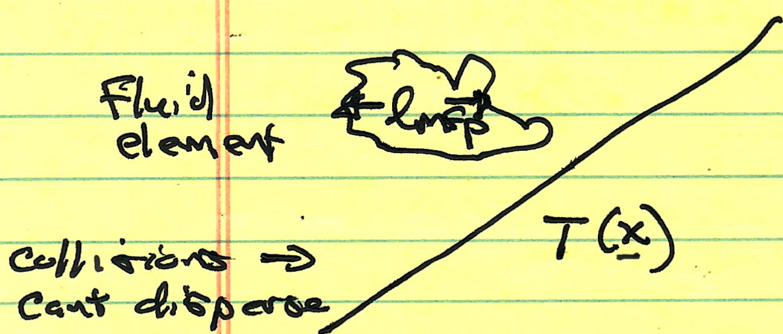
- Euler from Boltzmann ("ideal")
- Euler from macroscopics
- On Fluids (general thoughts).

Fluid Eqns

- replace B.E. by set of equations which evolve thermodynamic parameters.

- local Eulerian description (lab frame) i.e $n(\underline{x},t)$, $\underline{V}(\underline{x},t)$ etc

- describes 'blobs' of gas held together by collisions. *



$L > l_{mfp}$

$\omega < \nu_{col.}$

- parametrizes dynamics in terms structure of distribution

$$f = \frac{n(\underline{x})}{(2\pi)^{3/2} v_{Th}(x)^3} \exp\left[-(\underline{v} - \underline{V}(x,t))^2 / v_{Th}(x,t)^2\right]$$

- works for slight deviation from equilibrium

i.e.

$f = f_{eq} + \delta f$ will see.

$\downarrow$ $\hookrightarrow$ $\delta f \simeq - \frac{\underline{v} \cdot \underline{\nabla} f_{eq}}{\nu}$

local Maxwellian

$\downarrow$ $\simeq \ell_{mfp} \nabla f_{eq}$

Ideal Equations $\sim \boxed{\frac{\ell_{mfp}}{L} f_{eq}}$

(Perfect fluid)

$\downarrow$ $\downarrow$

Euler — viscous, dissipative equations

$\downarrow$

— sacrifices higher moments of f. — Navier - Stokes

Ideal Equations:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f = C(f)$$

demand:

$$\int d^3v \; C(f) = 0$$

$$\int d^3v \; mv \; C(f) = 0$$

$$\int d^3v \; \epsilon \; C(f) = 0$$

[easily shown. Collision operator conserves mass, energy, momentum.

So natural to define:

$$n = \int d^3v \, f \quad \rightarrow \text{density}$$

$$\underline{V} = \underline{V}(\underline{x}, t) = \frac{1}{n}\int d^3v \, \underline{v} f \quad \rightarrow \text{fluid / momentum.}$$

$$\bar{\varepsilon} = \frac{1}{n}\int d^3v \, \varepsilon f \quad \rightarrow \text{energy density}$$

Now,

$$\boxed{\partial_t f + \partial_{x_i}(v_i f) = C(f)}$$

Conservative Form

Taking moments:

$$\int d^3 \underline{v} \quad *$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n \underline{V}) = 0}$$

Continuity, # Cons.

number density flux

cons.

$$\int d^3v \; m \, v_\alpha \; *$$

$\Rightarrow$

$$\partial_t (m \bar{V}_\alpha) + \frac{\partial}{\partial x_\beta} \Pi_{\alpha,\beta} = 0$$

$\Downarrow$

mom. cons

$$\Pi_{\alpha\beta} = \int d^3v \; m \, v_\alpha v_\beta \, f$$

$\Downarrow$

momentum flux

and $\int d^3v \; \epsilon \; *$

$\Rightarrow$

$$\partial_t (n \bar{\epsilon}) + \underline{\nabla} \cdot \underline{q} = 0$$

$$\underline{q} = \int d^3v \; \underline{v} \, \epsilon f$$

Note: $\left.\begin{array}{c} n \\ \underline{V} \\ \bar{\epsilon} \end{array}\right\}$ equations have the form:

① $\partial_t(\text{Stuff}) + \underline{\nabla}\cdot(\text{Flux of Stuff}) = 0$

i.e. of form:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0$$

$\leftarrow$ Macroscopic conservation

$\Downarrow$

② all rest upon conservation properties of the collision operator.

③ Key is <u>constitutive relation</u>, i.e.

relating $\underline{J}$ to something <u>useful</u>.

i.e. Fick's Law : $\underline{J} = -D\,\underline{\nabla}\rho$

$$\Rightarrow \frac{\partial \rho}{\partial t} - D\,\nabla^2\rho = 0$$

usually not so simple ....

$\leftrightarrow$ closure problem.

(4) Essence of fluid equation construction is <u>calculation</u> of <u>fluxes</u>

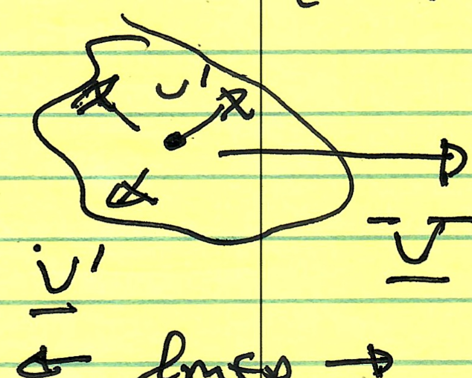
Further simplify:

$$\underline{v} = \underline{V}(\underline{x},t) + \underline{v}'$$

$\underline{v}$ ↓ particle velocity (micro)

$\underline{V}(\underline{x},t)$ ↓ mean bulk flow (macro) - linked to body forces

$\underline{v}'$ ⟶ thermal fluctuation abt mean $(\sim\sqrt{T/m})$



Realistically, $|\underline{V}| < |\underline{v}'|$

but $\underline{v}'$'s cancel ⇒ random

<u>so</u>

$$\Pi_{\alpha,\beta} = \int d^3v \; m \left(\underline{V}_\alpha(\underline{x},t) + v'_\alpha\right)\left(\underline{V}_\beta(\underline{x},t) + v'_\beta\right) f$$

Now here,

$$f = f_{eq} + \cancel{\delta f} \quad \Rightarrow \underline{\underline{\text{ideal}}} \text{ fluid}$$

$\downarrow$ local Maxwellian

$\hookrightarrow$ would have ∇F dependence.

and taking out $\underline{V}$, f_{eq} is $\underline{\text{even}}$ in v'.

$\underline{\text{so}}$

$$\Pi_{\alpha,\beta} = mn\left(V_\alpha(\underline{x},t)\, V_\beta(x,t) + \langle v'_\alpha v'_\beta \rangle \right)$$

$$f = \frac{n(x)}{(2\pi)^{3/2}\left[v_{th}(x)\right]^3} \exp\left[-\frac{v'^2}{v_{th}(x)^2} \right]$$

$$v' = \underline{v} - \underline{V}(\underline{x},t)$$

l.o. in $\dfrac{\ell_{mfp}}{L}$

$\underline{\text{so}}$

$$\langle v'_\alpha v'_\beta \rangle = 1/3\, v'^2 \delta_{\alpha,\beta} \qquad (\text{isotropic } f_{eq})$$

$$\langle v'^2 \rangle = 3\,T/m.$$

3×3

so, can define: $\underline{\underline{P}} = [\quad]$

$\underline{\underline{P}} = mn \langle v_\alpha' v_\beta' \rangle$ → stress tensor (pressure)

$= \frac{1}{3} mn \langle v'^2 \rangle \delta_{\alpha,\beta}$ → diagonal for ideal fluid

⇒ 1st moment:

$$\partial_t (n\underline{V}) + \underline{\nabla} \cdot (n\underline{V}\underline{V} + \underline{\underline{I}} P) = 0$$ * (Euler)

↓
Reynolds stress tensor

recall: $\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n\underline{V}) = 0$ **

and ~~subtract~~ eliminate ** from *:

Momentum $- \underline{V}$ (Continuity)

$$\boxed{n\left(\frac{\partial}{\partial t}\underline{V} + \underline{V} \cdot \underline{\nabla}\underline{V}\right) = -\underline{\nabla} P}$$

"Euler Eqn."

→ microscopically, Euler Eqn.

corresponds to $\underline{l_{mfp}/L} \to 0$.

→ $\underline{\underline{\Pi}} = n\underline{V}\,\underline{V} + P\underline{\underline{I}}$ is constitutive relation for

$$\partial_t(n\underline{V}) + \underline{\nabla}\cdot\underline{\underline{\Pi}} = 0$$

→ note C.R. contains no derivatives

⇒ ideal.

Contrast: Viscous fluid:

$$\underline{\underline{\Pi}} = n\underline{V}\,\underline{V} + P\underline{\underline{I}} - \overset{\text{viscosity}}{\nu}\,\underbrace{\underline{\nabla}\,\underline{V}}_{\text{viscous stress}}$$

Viscous fluid

⇒ Navier Stokes

→ What is p ?

$\underline{\nabla} \cdot \underline{\text{Euler}} = 0$

for $\underline{\nabla} \cdot \underline{v} = 0$ → incompressible

$\underline{\nabla} \cdot \underline{v} \neq 0$ ⇒ Equation of state

$\underline{\text{i.e}}$ p eqn.

Similarly;

$\epsilon = \frac{1}{2} m v^2 + \epsilon'$

$\frac{1}{2} m v^2$ ↓ KE; ϵ' ↓ internal dof

$= \frac{1}{2} m \left(\underline{V}(\underline{x}, t) + \underline{v}' \right)^2 + \epsilon'$

$q = \int \epsilon \, \underline{v} f \, d^3 v$

$\cong \int \epsilon \, v \, f_{eq} \, d^3 v$

so

$$\underline{q} = \int d^3v \, \left(\underline{V}(\underline{x},t) + \underline{v}'\right)\left(\epsilon' + 1/2\left(\underline{V}(\underline{x},t) + \underline{v}'\right)^2\right) f$$

$f = f_{eq.}$ PV work

$$\underline{q} = \underline{V}(\underline{x},t)\left(\frac{1}{2} n m V^2 + \underbrace{P + n\bar{\epsilon}'}_{\text{enthalpy}}\right)$$

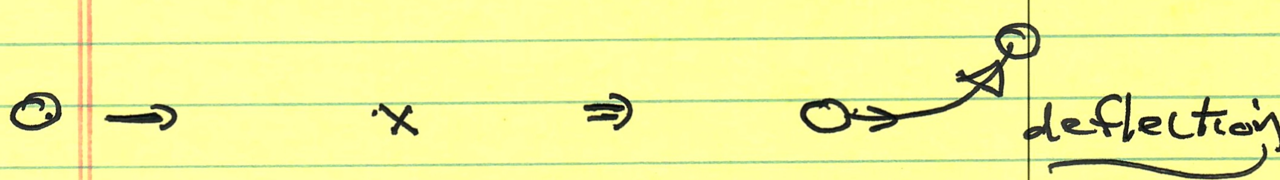
so

$$\boxed{\partial_t (n\bar{\epsilon}) + \underline{\nabla}\cdot\left[\underline{V}(\underline{x},t)\left(\frac{1}{2} n m V^2 + P + n\epsilon'\right)\right] = 0}$$

can simplify, as for momentum.

N.B:

- angular momentum not conserved by $C(f)$



→ Most truncations stop at 3rd moment.