

→ BBGKY to Boltzmann,
+ H Thm.

- N particle Hamiltonian, $N \gg 1$.

system described by:

$$F(t, \underline{x}_1, \underline{p}_1; \underline{x}_2, \underline{p}_2; \dots \underline{x}_N, \underline{p}_N)$$

→ N particle distribution

This satisfies full Liouville Equation

$$\frac{\partial F^N}{\partial t} + \sum_{i=1}^N \left\{ \frac{\partial}{\partial \underline{x}_i} \cdot (\dot{\underline{x}}_i F^N) + \frac{\partial}{\partial \underline{p}_i} (\dot{\underline{p}}_i F^N) \right\} = 0$$

$$\nabla \cdot \underline{v}_F = 0$$

$$\frac{\partial F^N}{\partial t} + \sum_{i=1}^N \left(\dot{\underline{x}}_i \cdot \frac{\partial F^N}{\partial \underline{x}_i} + \dot{\underline{p}}_i \cdot \frac{\partial F^N}{\partial \underline{p}_i} \right) = 0$$

Liouville Thm: F^N conserved along N particle orbits

$F^N \rightarrow$ exact, useless $N \sim 10^{23}$

seek: $P^G, P^A \rightarrow$ pdf for a particle

$F(\underline{x}, \underline{v}, t) \Rightarrow$ phase space density

approach: integrate out additional particles $\Leftrightarrow$ reduce description

catch: basic interaction is 2 body!

$$\dot{\underline{x}}_i = \underline{v}_i$$

$$\dot{\underline{p}}_i = -2 \sum_{j \neq i} \nabla_{\underline{x}_i} V_{ij}$$

$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^N \left(\underline{v}_i \cdot \frac{\partial f^N}{\partial \underline{x}_i} - \frac{\partial f^N}{\partial \underline{p}_i} \cdot \sum_{j < i} \frac{\partial \underline{V}_{ij}}{\partial \underline{x}_i} \right) = 0$$

Define: 1-particle distribution

$$F(t, \underline{x}_1, \underline{p}_1) = \int d\underline{\Gamma}_2 d\underline{\Gamma}_3 \dots d\underline{\Gamma}_N f^N$$

$\downarrow$
 $\underline{x}_1, \underline{p}_1$

integrate out
other dependencies

$$F(t, \underline{x}_1, \underline{p}_1, \underline{x}_2, \underline{p}_2) = \int d\underline{\Gamma}_3 \dots d\underline{\Gamma}_N f^N$$

→ 2 particle distribution

so, for $N=1$ total deriv

$\downarrow$

$$\int d\underline{\Gamma}_2 d\underline{\Gamma}_3 \dots d\underline{\Gamma}_N \left(\frac{\partial f^N}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial \underline{x}_i} \cdot (\underline{v}_i f^N) \right)$$

$$+ \sum_{i=1}^N \frac{\partial}{\partial \underline{p}_i} \cdot \left(\sum_{j < i} \left(\frac{\partial \underline{V}_{ij}}{\partial \underline{x}_i} \right) f^N \right) = 0$$

$\downarrow$
 $N \neq 1$ killed by
surface term

↳ 2 particle interaction.

V_{ij} is 2-particle interaction $\Rightarrow$
necessarily enters with f_2 .

Need treat all possible pairs $\Rightarrow$

$$\frac{\partial f^{(1)}}{\partial t} + \underline{v}_1 \cdot \frac{\partial f^{(1)}}{\partial \underline{x}_1}$$

$$= (N-1) \int d\Gamma_2 \frac{\partial \underline{V}_{12}}{\partial \underline{x}_1} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_1}$$

$\downarrow$
binary
pairs
 N particles

$\downarrow$
2 particle
interaction

$\downarrow$
2 body
distribution
not

N.B.: $\frac{\partial f^{(1)}}{\partial t} = \int (1) f^{(2)}$

$\rightarrow$ hierarchy problem
- how couple?

Need $f^{(2)}$ eqn.!

$\rightarrow$ how? - integrate out from 3, on.

→

$$\frac{\partial f^{(2)}}{\partial t} + \underline{v}_1 \cdot \frac{\partial f^{(2)}}{\partial \underline{x}_1} + \underline{v}_2 \cdot \frac{\partial f^{(2)}}{\partial \underline{x}_2} - \frac{\partial \bar{V}_{12}}{\partial \underline{x}_1} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_1} - \frac{\partial \bar{V}_{12}}{\partial \underline{x}_2} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_2}$$

$$= \underbrace{(N-2)}_{\substack{\text{\# triplets} \\ \text{\# particles}} \int d\Gamma_3 \left[\frac{\partial f^{(3)}}{\partial \underline{p}_1} \cdot \frac{\partial \bar{V}_{13}}{\partial \underline{r}_1} + \frac{\partial f^{(3)}}{\partial \underline{p}_2} \cdot \frac{\partial \bar{V}_{23}}{\partial \underline{r}_2} \right] f^{(4,2)}$$

Can we simplify this?

① ①

$$\frac{\partial f^{(2)}}{\partial t} + \underline{v}_1 \cdot \underline{\nabla}_1 f^{(2)} = \frac{\partial \bar{V}_{12}}{\partial \underline{x}_1} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_1} + (1 \leftrightarrow 2)$$

$$= (N-2) \int d\Gamma_3 \left[\frac{\partial f^{(3)}}{\partial \underline{p}_1} \cdot \frac{\partial \bar{V}_{13}}{\partial \underline{r}_1} + \frac{\partial f^{(3)}}{\partial \underline{p}_2} \cdot \frac{\partial \bar{V}_{23}}{\partial \underline{r}_2} \right]$$

Look at ②/①

→ exploit low volume filling of interaction - $nd^3 \ll 1$.

$$\textcircled{2} \sim N \int d\underline{x}_3 \int d\underline{A} \frac{\partial f^{(3)}}{\partial \underline{A}} \cdot \frac{\partial \underline{V}_{2,3}}{\partial \underline{r}_2}$$

ign $N-2 \rightarrow$ lose N

$$f^{(6)} \sim \underbrace{\frac{1}{(\Delta p)^3} \frac{1}{\text{Vol.}}}_{\text{normalization}} f^{(2)}$$

$$\textcircled{2} N \int d\underline{x}_3 \int d\underline{A} \frac{1}{(\Delta p)^3 \text{Vol.}} \frac{\partial f^{(6)}}{\partial \underline{A}_2} \frac{\partial \underline{V}_{2,3}}{\partial \underline{r}_2}$$

$\rightarrow \int d\underline{p}_0$ cancels $1/\Delta p^3$ normalization

$$\sim \int d\underline{x}_3 \frac{\partial f^{(2)}}{\partial \underline{A}_2} \frac{\partial \underline{V}_{2,3}}{\partial \underline{r}_2} \frac{N}{\text{Vol}}$$

∇ Fill $\int d^3$ volume

$$\textcircled{2} \sim d^3 \frac{N}{\text{Vol.}} \left(\frac{\partial \underline{V}_{2,3}}{\partial \underline{r}_2} \right) \left(\frac{\partial f^{(2)}}{\partial \underline{p}_2} \right)$$

$$\sim n d^3 \left(\frac{\partial \underline{V}_{2,3}}{\partial \underline{r}_2} \right) \left(\frac{\partial f}{\partial \underline{p}_2} \right)$$

180

$$\boxed{\textcircled{2}/\textcircled{1} \sim n d^3 \ll 1}$$

$$\text{RHS/LHS} \sim d^3/\bar{r}^3 \ll 1.$$

$$\boxed{\frac{d}{dt} f^{(2)}(t, \vec{r}_1, \vec{r}_2) = 0}$$

- constitutes truncation of BBGKY hierarchy, for dilute gas

- key is $d^3/\bar{r}^3 \ll 1$

$\Rightarrow$ Boltzmann scale ordering.

$\rightarrow \frac{d}{dt} f^{(2)} = 0$ is straightforward for dilute.

$\rightarrow$ if post statistical independence of colliding particles, aka
Molecular chaos.

$$f(t, 1, 2) = f(t, 1) f(t, 2)$$

then,

$$F(t, \Gamma_1, \Gamma_2) \stackrel{+}{=} f(t, \Gamma_1) f(t, \Gamma_2)$$

verifies as c.c. for $\frac{dF^{(2)}}{dt} = 0$

$\frac{dF^{(2)}}{dt} = 0$, so $F^{(2)}$ always factorizes

- consistent with "Freely moving particles", "interacting only within $d \ll \bar{n}$."

- so $f^{(2)} \approx f(1) f(2)$ and

$$\frac{df}{dt} + \underline{v} \cdot \underline{\nabla} f = N \int d\Gamma_2 \frac{\partial V_{12}}{\partial \underline{r}_1} \cdot \frac{\partial}{\partial \underline{r}_2} [f(\underline{r}_1, t) f(\underline{r}_2, t)]$$

$\leadsto$ Boltzmann Equation.

Boltzmann Egn.

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F = C(F)$$

$$C(F) = \int d\Gamma_2 \frac{\partial U_{12}}{\partial \mathbf{r}_1} \cdot \frac{\partial}{\partial \mathbf{p}_1} [F(\mathbf{r}_1, \mathbf{p}_1) F(\mathbf{r}_2, \mathbf{p}_2)]$$

→ absorbed normalization

- $C(F) \equiv$ collision operator (integral)

- $C(F)$ nonlinear → 2 body collision

- have "test particles" ($F(\mathbf{r}_1)$) scattered by other, "field particles"
but "test", "field" the same.

⇒ nonlinearity

- $C(F) \sim v_{\text{coll}}$

$$C(F) \approx -\gamma [F - F_{\text{eq}}] \quad (\text{Langevin})$$

$$- \frac{df}{dt} = c(f)$$

Phase space density conserved along particle orbits, up to collisions.

- what if $c(f) \rightarrow 0$

have:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f + \frac{\underline{F}}{m} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$$\underline{F} = Z \underline{E}$$

$$\nabla_x E = 4\pi n_0 q \int f d\underline{v}$$

$$\left. \begin{array}{l} V/200V \\ \text{Jeans} \end{array} \right\} \text{Eqn.}$$

continuity eqn. for incompressible flow of phase space fluid.

$$- \text{Plasma: } \boxed{\bar{r} < \lambda_D < \ell_{msf} < L}$$

long range / glancing $\Rightarrow$ Fokker-Planck,

L2b - H Theorem.

→ To H-Theorem!

→ Recall:

- Hamiltonian system
- Non-dissipative interactions
- From Liouville (for $N \sim N_A \gg 1$)
to Boltzmann: BBGky

→ Key is diluteness ordering

$$\underbrace{d < \bar{n}^{-1/3} < \ell_{mfp} < L}$$

→ Molecular Chaos Assumption
Detailed Balance

→ f^N eqn. → $f^{(1)}$ eqn.

⇒ Boltzmann Equation.

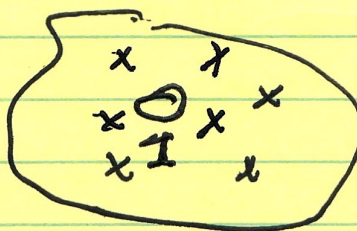
$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f = C(f)$$

↓
collision operator

$$C(f) = \int d\Gamma_2 \frac{\partial V_{12}}{\partial \underline{r}_1} \cdot \frac{\partial}{\partial \underline{p}_1} [f(\underline{r}_1, \underline{p}_1) f(\underline{r}_2, \underline{p}_2)]$$

↓
Nonlinear

in reality,
every particle (DOF)
is both a
test particle and
a field particle



$\bigcirc \rightarrow I$
test particle
 $x \rightarrow$ field
particles

→ H-Theorem:

- What does it say/mean?
- What makes it remarkable?
- ↕
 - What does it rest upon?
(Assumptions)
- How to prove?

Onward :

- Can also write Boltzmann Equation as change in occupation of 'state' $\leftrightarrow P$

i.e. [consider as hard sphere collisions]

$\frac{df(p)}{dt} =$ rate of change of f due collisions (interactions)

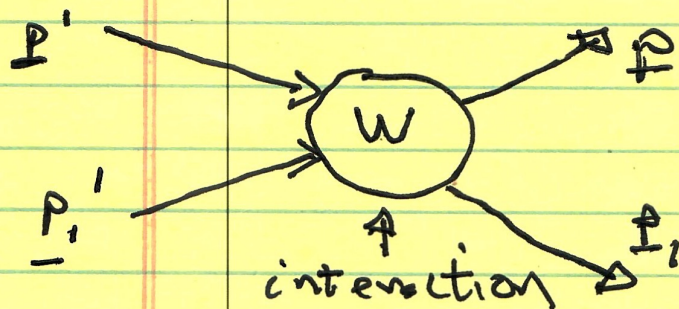
i.e.

$\frac{df(p)}{dt} =$ rate scattering in

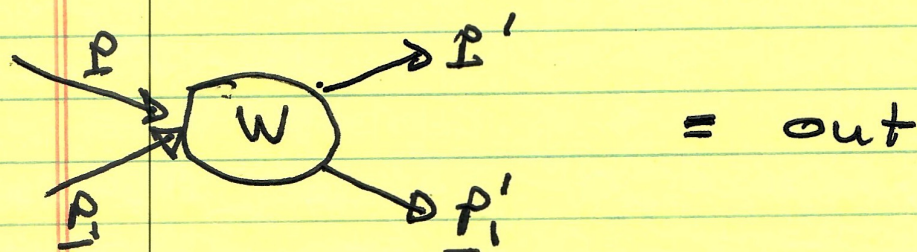
- rate scattering out

(anticipates Master Equation)

in, out ? :



$=$ in



$$in = \int d\underline{P}' \int d\underline{P}'_1 \int d\underline{P}_1 F(\underline{P}') F(\underline{P}'_1) W(\underline{P}', \underline{P}'_1; \underline{P}, \underline{P}_1)$$

$$out = \int d\underline{P}_1 \int d\underline{P}' \int d\underline{P}'_1 F(\underline{P}) F(\underline{P}_1) W(\underline{P}, \underline{P}_1; \underline{P}', \underline{P}'_1)$$

as $W = W^T$ (reversible micro-dynamics)

$$\left\{ \frac{dF(\underline{P})}{dt} = \int d\underline{P}_1 \int d\underline{P}' \int d\underline{P}'_1 W(\underline{P}, \underline{P}_1; \underline{P}', \underline{P}'_1) * \right. \\ \left. (F(\underline{P}') F(\underline{P}'_1) - F(\underline{P}_1) F(\underline{P})) \right\}$$

N.B. - $\underline{P} + \underline{P}_1 = \underline{P}' + \underline{P}'_1$

so detailed Balance applies.

$$- F = f_0 = C \exp \left[- \frac{(\epsilon + \underline{P} \cdot \underline{V})}{T} \right]$$

So $C(f_0) = 0$ conservation $\begin{cases} \text{energy} \\ \text{momentum} \end{cases}$

and will show Maxwellian renders $dS/dt = 0$.

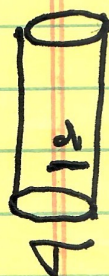
N.B. - What about $V_{1,2}(\underline{x})$?

$\Rightarrow$ Stoßzahlansatz

L. Boltzmann

total # $(\underline{v}_1, \underline{v})$ collisions taking place
in dt

= (Volume of $\underline{v}_1, \underline{v}$ collision cylinder)
* (# of particles with $\underline{v}_1$, per volume)



$$|\underline{v}_1 - \underline{v}| \sigma$$

in collision cylinder

$$= \sigma v_{rel} f(\underline{v}_1) d^3 v_1$$

$$\sigma \sim d^2$$

and:

- dilute: non-overlapping cylinders
- collisions as 'point events' (*)

$$d < \bar{r} < l_{\text{mfp}} \quad \text{ordering!}$$

Also: $\left[\int_{\underline{v}_1, \underline{v}_2} d(\underline{v}_1 - \underline{v}_2) \right]$
+ integrates.

$$- w d^3 \underline{v}' d^3 \underline{v}_i = v_{\text{rel}} dT$$

relates transition (probability) to familiar items like cross-section.

$$- l_{\text{mfp}}, l_{\text{mfp}} = 1/nT$$

Onward: $\Rightarrow$ H-Theorem.

- a gas left alone, will evolve to an equilibrium of maximal entropy.
- evolution accompanied by entropy production

ie. $\frac{dS}{dt} \geq 0$

- evolution is to uniform Maxwellian
- Ideal gas:

$$S^* = \int dx \int dp f \ln(e/f)$$

(discussed later)

$$= - \int dx \int dp f \ln f$$

$$\frac{dS}{dt} = - \int d\Gamma \left[\frac{df}{dt} \ln f + f \frac{1}{f} \frac{df}{dt} \right]$$

$$= - \int d\Gamma \left[C(f) \ln f + C(f) \right]$$

$$\int d\Gamma C(f) = 0 \quad \text{show!}$$

$$\frac{dS}{dt} = - \int d\Gamma C(f) \ln f$$

$$\frac{ds}{dt} = - \int dx \int dp \int dp_i \int dp'_i \int dp'_i' (\ln f) w * \\ (f(p') f(p'_i) - f(p) f(p_i))$$

Lemma

$$\int d^4p \, U(p) C(p) = \frac{1}{2} \int d^4p \, (U + U - U' - U'_i) * \\ w f' f'_i$$

explicitly:

$$\int d^4p \, U(p) C(p) = \quad (1)$$

$$\int U w(p, p_i; p'_i, p'_i) f' f'_i d^4p$$

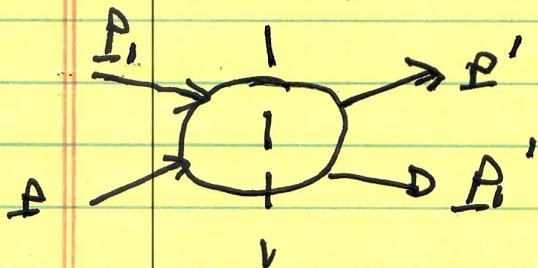
(2)

$$- \int U w(p', p'_i; p, p_i) f f_i d^4p$$

To show:

Now, on (2):

→ interchange $p, p_i \leftrightarrow p'_i, p'_i$



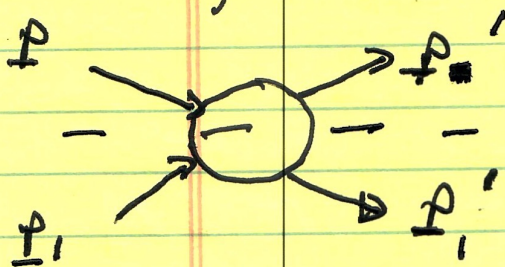
{ flip about ;
use $w = w^T$
micro-reversibility

so

$$\int dP \mathcal{L} \text{CCA}$$

$$= \int d^4 P \left\{ (\mathcal{L}(P) - \mathcal{L}(P')) w(P, P'; P', P') * F' F' \right\}$$

→ now, consider:



and interchange about

i.e. P, P' with P_1, P'_1

n.b.: up-down symmetry
equivalent
(no reason $\leftrightarrow$)

so

$$\int dP \text{CCA} \mathcal{L} = \frac{1}{2} \int d^4 P \left\{ (\mathcal{L}(P) - \mathcal{L}(P')) + (\mathcal{L}(P_1) - \mathcal{L}(P'_1)) w F' F' \right\}$$

Proves Lemma 1

Now, let $\varphi = \ln f$

so, From Lemma:

$$\begin{aligned} \frac{dS}{dt} &= -\frac{1}{2} \int d\underline{x} \int d^4 p \left(\ln f + \ln f_i - \ln f' \right. \\ &\quad \left. - \ln f'_i \right) w f' f'_i \\ &= \frac{1}{2} \int d\underline{x} \int d^4 p \ w f' f'_i \ln (f' f'_i / f f_i) \end{aligned}$$

define $x = f' f'_i / f f_i$

$$\frac{dS}{dt} = \frac{1}{2} \int d\underline{x} \int d^4 p \ w f f_i \ x \ln x$$

Since: $\int C(f) d^D p = 0$

have $\int w f f_i (x - 1) d^4 p \ d\underline{x} = 0$

→ a case of writing zero in a complicated way!

so adding 0 to ds/dt expression:

$$\frac{ds}{dt} = \frac{1}{2} \int d^4p \int dx \, w \, ff_i [x \ln x - x + 1]$$

→ entropy production rate.

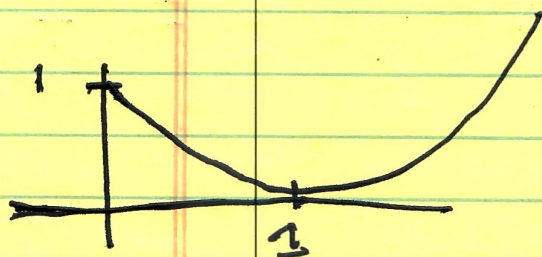
Then

$$F(x) = x \ln x - x + 1$$

$$F' = 1 + \ln x - 1$$

$$F(1) = 1$$

$$F(1) = 0$$



so $\boxed{\frac{ds}{dt} \geq 0}$! → H-Theorem!

Now, $\left. \frac{dS}{dt} = 0 \right\} \text{ for } x=1$

$$F F_i = F' f_i'$$

$$\ln f + \ln f_i = \ln f' + \ln f_i'$$

$\Rightarrow \ln f + \ln f_i$ conserved
in
collision

$\Leftrightarrow \ln f = c + p \cdot \underline{V} - \alpha \epsilon$

$\frac{dS}{dt} = 0 \Rightarrow \text{Maxwellian Distribution}$

Note:

1.) keys: $W = W^T \rightarrow \text{Detailed Balance}$

$F(1,2) = F(1)F(2)$ factorization.
(Molecular chaos)

2.) $\frac{dS}{dt} = 0 \Leftrightarrow CCF = 0$

collisions drive system to equilibrium.

3) dx irrelevant!

Entropy produced locally.

i.e. f relaxes to local Maxwellian,
then to uniform Maxwellian

(i.e. transport: $T(x) \rightarrow \overline{T(x)}$)

Essence of H-Theorem:

Macroscopic irreversibility from
microscopically reversible dynamics

* Molecular chaos ("micro-chaos").

→ Some observations:

①

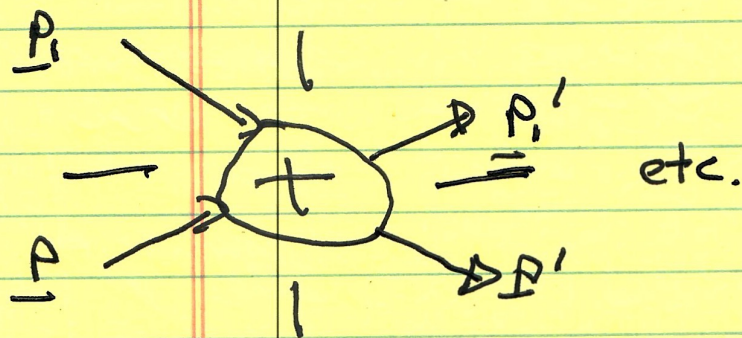
- If no a-priori concept/idea of equilibrium distribution, how derive it?

Recall: $\frac{dS}{dt} = 0$, for $x=1$

$$x=1 \Rightarrow f'f_i' = ff_i$$

$$\ln f' + \ln f_i' = \ln f + \ln f_i$$

as labels in collision arbitrary, i.e.



so $\ln f + \ln f_i = \text{const.}$

sum logs
conserved

What is conserved? (Dynamically):

- energy (kinetic energy / particle)
- momentum
- number

so

$\ln F$ can be expressed as a linear combination of conserved quantities

$$\ln F = a + \underline{b} \cdot \underline{p} + c \frac{p^2}{2m}$$

$c < 0$ for normalizability

N.B.: Angular momenta not independent as collision event at 1 position.

$\Rightarrow$

$$F = c' \left[\frac{-p^2}{2mT} + \frac{\underline{p} \cdot \underline{V}}{T} \right]$$

$$c' \sim n$$

n, T, V can be $n(\underline{x}), T(\underline{x}), V(\underline{x})$ can all be functions of $\underline{x}$ for $\lim_{x \rightarrow L}$

Thus, derived equilibrium distribution function from H-Thm.

② How reconcile?

- reversible, Hamiltonian dynamics
- $dS/dt \geq 0$.

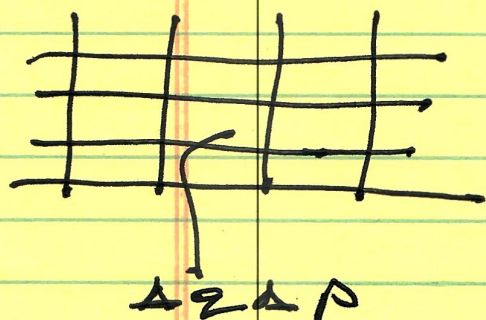
Related: What happened to Poincaré Recurrence?

Poinc:

- statistical description: $f(x, p, t)$

- Coarse Graining:

(recall Lyapunov exponents)



→ partition

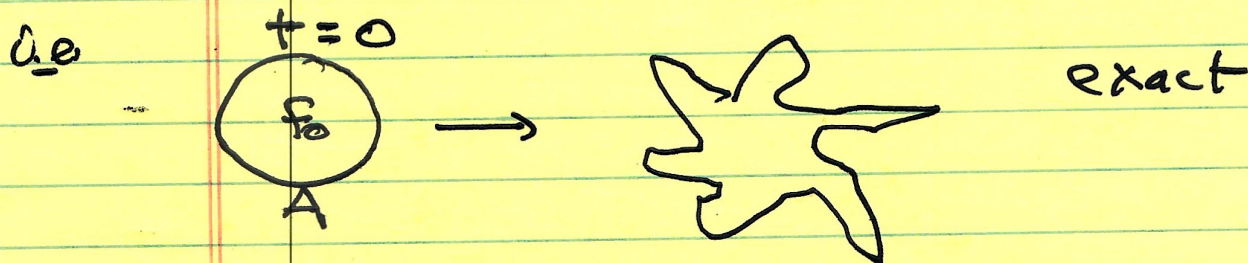
$\Delta x \Delta p$

→ sets resolution scale.

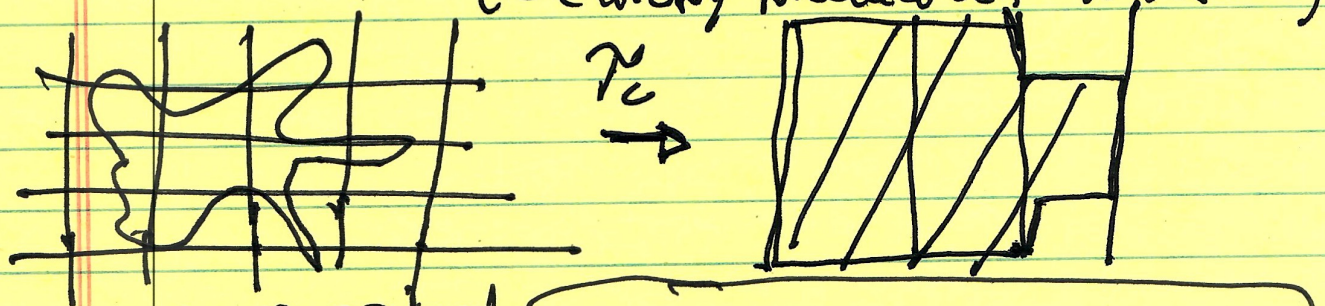
(S is integrated quantity)

Why Significant?

→ partition kills small details in phase volume evolution



With coarse graining (smearing)
(~ entropy production timescale)



$$f_0 A_0 = A_{CG} \bar{f}$$

$$\bar{f} = f_0 A_0 / A_{CG}$$

→ coarse grained distribution

$$\bar{f} < f_0 \quad \text{as} \quad A_0 < A_{CG}$$

So

- prediction of close recurrences impossible, as partition sets resolution limit.

N.B.: $l_{\text{mfp}} = 1/n\sigma$

etc

$$\gamma_c^{-1} = \nu_c \cong \nu_m / l_{\text{mfp}}$$