

Physics 210B

L.II-2-b - Fokker-Planck II.

→ So far:

- derived equation for Pdf of process specified by Langevin Ezn.

$$\frac{d}{dt} \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} \psi = \frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial}{\partial t} P = - \frac{\partial}{\partial v} \left\{ \underbrace{\frac{\langle \Delta v \rangle}{\Delta t} P}_{\substack{\text{drift/drag} \\ \text{(deterministic)}}} - \frac{\partial}{\partial v} \underbrace{\frac{\langle \Delta v \Delta v \rangle}{2 \Delta t} P}_{\substack{\text{diffusion} \\ \text{(random)}}} \right\}$$

- restrictions $\Rightarrow \langle \Delta v \Delta v \rangle < \infty$

- simple example.

Now:

- motion of particle in specified potential

= Sedimentation

- Multiplicative Noise - Logistics (simple)

Now $\rightarrow$ Spatial Distribution / Pdf.

Consider $\left\{ \begin{array}{l} \text{Brownian Motion} \\ \text{random walk} \end{array} \right\}$ in potential $U(\underline{x})$, with noise

Seek $F(\underline{x}, t)$

$$\rightarrow \text{Seek } F(\underline{x}, t) = \int d\underline{v} P(\underline{x}, \underline{v}, t) \\ \equiv n(\underline{x}, t)$$

Now, $\frac{d\underline{x}}{dt} = \underline{v}$

particle orbits

$$\frac{d\underline{v}}{dt} = -\frac{\beta}{m} \underline{v} - \frac{1}{m} \underline{\nabla} U + \frac{\underline{F}}{m}$$

$\rightarrow$ Random Force

$\downarrow$
Stokes drag

$\downarrow$
Deterministic Force

Now, if (as appropriate) interested in long timescales, assume Brownian particle reaches terminal velocity,

$$t \gg (\beta/m)^{-1}$$

$$\frac{dx}{dt} = \underline{v}$$

$$\frac{d\underline{v}}{dt} = -\frac{\beta}{m}\underline{v} + \underline{a}_{ext} + \frac{\underline{\zeta}(t)}{m}$$

$$\underline{v} \equiv \frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = -\frac{\underline{\zeta}(t)}{\beta} + \frac{\underline{F}}{\beta}}$$

So now write F-P equation for $n(x, t)$,

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x} \left\{ \underbrace{\langle \underline{A}x \rangle \frac{\partial n}{\partial t}}_{\text{drift/drag}} - \frac{\partial}{\partial x} \cdot \underbrace{\langle \underline{A}x \underline{A}x \rangle \frac{n}{2\Delta t}}_{\text{spatial diffusion}} \right\}$$

$$\underline{S_0} \quad \langle \underline{\Delta x} \rangle = \underline{V} = - \frac{\underline{\partial U}}{\underline{\partial B}}$$

$$(\langle \tilde{F} \rangle = 0)$$

$$\begin{aligned} \langle \underline{\Delta x} \underline{\Delta x} \rangle &= \int dt' \int dt'' \langle \tilde{F}(t') \tilde{F}(t'') \rangle \\ &\equiv \int_0^t dt_+ \int_0^t dt_- \langle \tilde{F}(t_+) \tilde{F}(t_-) \rangle / B^2 \\ \langle \tilde{F}(t') \tilde{F}(t'') \rangle &\equiv |\tilde{F}|^2 \tilde{T}_{ac} \delta(t' - t'') \end{aligned}$$

and recall from FDT:

$$BT = \frac{|\tilde{F}|^2 \tilde{T}_{ac}}{2}$$

$$\leadsto \langle \underline{\Delta x} \underline{\Delta x} \rangle = 2 \frac{T}{B} + \left[\underline{\underline{I}} \right]$$

for forcing uncorrelated in
different directions

$$= 2D + \left[\underline{\underline{I}} \right]$$

$$\boxed{D = T/\gamma} \rightarrow \text{spatial diffusivity}$$

ie $D = \int_0^{\infty} du \langle \underline{\tilde{V}}(u) \underline{\tilde{V}}(0) \rangle$

then

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x} \left\{ -\frac{\partial u}{\partial x} n - \frac{\partial}{\partial x} \left(\frac{T}{\beta} \frac{\partial n}{\partial x} \right) \right\}$$

- Schmoluckowski Equation
(F-P Eqn.)

- Deterministic Drift + Diffusion

- stationary state (1D)

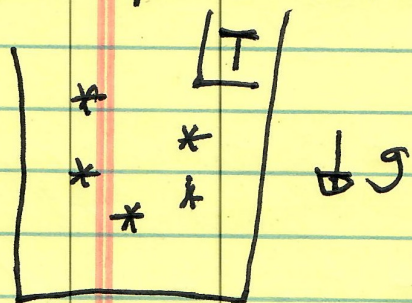
$$\frac{\partial n}{\partial t} = 0$$

$$\frac{\partial}{\partial x} \left(\frac{T}{\beta} \frac{\partial n}{\partial x} \right) = -\frac{\partial u}{\partial x} n$$

$\Rightarrow$

$$n = n_0 \exp \left[-\frac{u}{T} \right]$$

→ A specific Application - Sedimentation



Brownian Particles
in Fluid at T in
gravitational field.

- Spatial Distribution ? $t \rightarrow \infty$
How evolve ? $\sim \exp[-mgz/T]$

- Particles $\left\{ \begin{array}{l} - \text{drift, due gravity} \\ - \text{random walk} \end{array} \right.$

then, for particles ($m_p \gg \rho_{\text{fluid}} V_{\text{part.}}$)

$$m_p \frac{d\mathbf{v}}{dt} = -\beta \mathbf{v} - m_p g \hat{z} + \tilde{\mathbf{F}}$$

1D, $t \gg (\beta/m)^{-1}$

$$\frac{dz}{dt} = -\frac{g}{\beta} \hat{z} + \frac{\tilde{F}}{m_p}$$

$$\text{So, } \frac{dn(z,t)}{dt} = -\frac{\partial}{\partial z} \left\{ \frac{\Delta z \eta}{\Delta t} - \frac{\partial}{\partial z} \left(\frac{\langle \Delta z \Delta z \rangle \eta}{2\Delta t} \right) \right\}$$

From before:

$$\langle \Delta z \Delta z \rangle = \frac{1}{\beta} \frac{\partial}{\partial \Delta t}$$

and

$$\langle \Delta z \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \Delta t} \equiv \bar{V} = -\frac{1}{\beta} \frac{\partial}{\partial \Delta t}$$

Note: $\bar{V} = +\frac{1}{\beta} (mgz) = \frac{1}{\beta} F$
 drift velocity

$$\bar{V} = \mu E$$

$\mu = 1 / \text{resistance}$

Mobility $\rightarrow$ determine mobility from drift

$$\mu = 1 / \beta$$

here

$\rightarrow$ generic to friction - reduced terminal free-fall.

$$D = T \beta$$

$$= \mu T$$

$\rightarrow$ Einstein Relation

N.B. Mobility easy to calculate
From Diffusivity

→ generic to Brownian motion type
Langevin Equation and FDT

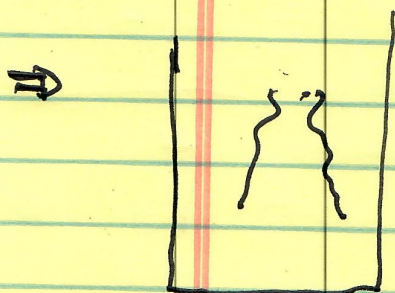
For steady state:

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial z} \left\{ \frac{-mg}{\beta} n - \frac{T}{\beta} \frac{\partial n}{\partial z} \right\} \rightarrow 0$$

⇒ $n = n_0 \exp \left[-\frac{mgz}{T} \right] \checkmark$

Particles end up in layer
 $\Delta z \sim T/mg$
at bottom.

Can obtain relaxation to equilibrium
profile. See Chandrasekhar



Full evolution
→ diffusion spreading
+ sedimentation.

$$\langle x^2 \rangle = 2Dt$$

Can write:

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial z^2} + c \frac{\partial n}{\partial z}$$

$$D = \tau/B$$

$$n \rightarrow \delta(z-z_0) \quad t \rightarrow 0$$

$$c = \frac{mg}{B}$$

$$D \frac{\partial n}{\partial z} + cn = \Gamma = 0, \quad z=0 \quad t > 0$$

$\Rightarrow$

$$n(t, z, z_0) = \frac{1}{2(\pi Dt)^{1/2}} \left\{ \exp\left[-(z-z_0)^2/4Dt\right] \right.$$

$$+ \exp\left[-(z+z_0)^2/4Dt\right] \Bigg\}$$

$$+ \frac{c}{D\sqrt{\pi}} e^{-cz/D} \int_0^\infty e^{-x^2} dx$$

$$\frac{z+z_0-ct}{2(Dt)^{1/2}}$$

$t \rightarrow \infty$

$$\rightarrow \frac{c}{D\sqrt{\pi}} e^{-cz/D}$$

Aside: In principle could obtain
F-~~A~~ eqn. for $F(x, v, t)$

$$\frac{dx}{dt} = \underline{v}$$

$$\frac{dv}{dt} = -\frac{\beta}{m} \underline{v} - \frac{1}{m} \underline{\nabla} U + \frac{\underline{f}}{m}$$

i.e.

$$\frac{\partial F}{\partial t} + \underline{v} \cdot \underline{\nabla} F = - \frac{\partial}{\partial \underline{v}} \cdot \left\{ \left(-\frac{\beta}{m} \underline{v} - \frac{1}{m} \underline{\nabla} U \right) F - \frac{\partial}{\partial \underline{v}} \cdot \underline{D}_v F \right\}$$

U indep. $\underline{v}$. $\Rightarrow$

$$\begin{aligned} \frac{\partial F}{\partial t} + \underline{v} \cdot \underline{\nabla} F - \frac{\underline{\nabla} U}{m} \cdot \frac{\partial F}{\partial \underline{v}} &= \frac{dF}{dt} \Big|_{H. \text{ orb.}} \\ &= - \frac{\partial}{\partial \underline{v}} \cdot \left\{ -\frac{\beta}{m} \underline{v} F - \frac{\partial}{\partial \underline{v}} \cdot \underline{D}_v F \right\} \end{aligned}$$

and extract Schmoluchowski in
 $t \gg (\beta/m)^{-1}$ limit

$$D_v = \frac{\beta}{m} v_{th}^2 \sim \frac{|\tilde{f}_0|^2}{m^2} T_{ec}$$

- Multiplicative Noise (Sample)

Consider Logistic Eqn $\rightarrow$ Population

$$\frac{dN}{dt} = N(k - N)$$

$\downarrow$ $\rightarrow$ saturation by competition $N \equiv \#$ of population
 Malthusian growth (exponential) $\sim N^2$

$$x_{n+1} = r x_n (1 - x_n)$$

Logistic Map

$N=0$, $N=k$ are fixed pts

Now, could consider variability in k , and treat as stochastic variable

$$\frac{dN}{dt} = N \left(k_0 + \tilde{\gamma}(t) - N \right) + \tilde{\chi}(t)$$

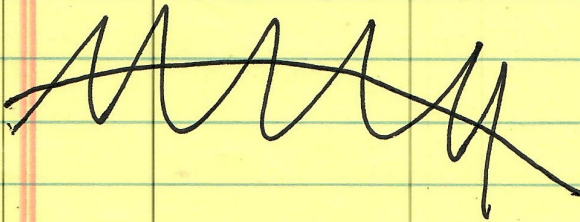
$\downarrow$
 variability in resources $\Rightarrow$ multiplicative noise
 $\Rightarrow$ rate

$\downarrow$
 external input variability $\Rightarrow$ additive noise

ie additive:

$$\overbrace{f + \text{noise}}^{\text{noise on top of deterministic base}} = \text{noise}$$

Multiplicative:



multiplies by fast, random quantity

How treat?

$f(N, t) \rightarrow$ population pdf
↓

→ Fokker-Planck Equation for $f(N)$

→ here $\langle \tilde{y}(t) \tilde{x}(t') \rangle = \tilde{\gamma}_0 \tilde{\Gamma}_{ac} \delta(t-t')$

Delta correlated for simplicity.

N.B. This is a "textbook model".

→ additive, as usual

$$\langle \tilde{x} \tilde{\delta} \rangle = 0$$

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Then : $\frac{dN}{dt} = N(k_0 + \tilde{\delta}(t) - N) + \tilde{x}$

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$$\frac{d}{dt} F(N, t) = -\frac{\partial}{\partial N} \left[(k_0 N - N^2) F(N, t) \right] - \frac{\partial}{\partial N} [D F(N)]$$

For D :

$$\langle \Delta N \Delta N \rangle = \int_{-t}^t dt' \int_{-t}^t dt'' \langle \tilde{\delta}(t') \tilde{\delta}(t'') \rangle N^2 + \int_{-t}^t dt' \int_{-t}^t dt'' \langle x(t') x(t'') \rangle$$

$$= |k_0|^2 \tau_{ac} N^2 + \underbrace{|x_0|^2 \tau_{ac}}_{\text{Nonlinearity in } D}$$

→ one trademark feature of multiplicative noise

→ Note: $N \rightarrow \infty \Rightarrow D \rightarrow 0$

Rate variation $\Rightarrow$ Adf spread
in proportion to population.

→ Additive correction significant at
low N .

Now, ignoring additive correction,

$$\partial_t F(N) = - \frac{\partial}{\partial N} \left\{ (k_0 N - N^2) F(N, t) - \frac{\partial}{\partial N} \left(\frac{1}{2} k_0^2 T_{ac} N^2 F(N, t) \right) \right\}$$

is Fokker-Planck Equation

and stationarity:

$$N (k_0 - N) F(N) = \frac{\partial}{\partial N} \left(\frac{1}{2} k_0^2 T_{ac} N^2 F(N) \right)$$

so Norm

$$FCN = \frac{1}{C \cdot n} [2(k_0/\sigma^2) - 2] e^{-2N/\sigma^2}$$

$\sigma^2 = |\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}$
↑
Power
↑
exponential tail

Need $k_0^2 > (\sigma^2/2)^2 \Leftrightarrow f > 1/n$

i.e. $\left\{ k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2} \right.$

$n \rightarrow 0$
to avoid log. singularity

Physics of $k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2}$?

Convenient to linearize around
Fixed point:

$$\frac{dN}{dt} = (k + \tilde{\gamma} - N)N$$

Validity?

$$N = k_0 + \tilde{n}$$

$$\begin{aligned} \frac{d\tilde{n}}{dt} &= (k_0 + \tilde{n})(k_0 + \tilde{\gamma} - k_0 - \tilde{n}) \\ &\approx k_0 \tilde{\gamma} - k_0 \tilde{n} + O(n^2) \end{aligned}$$

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$$\partial_t F(n) = -\frac{\partial}{\partial n} \left[-k_0 n F(n) - \frac{\partial}{\partial n} \left(\frac{k_0^2 \tau^2}{2} F(n) \right) \right]$$

linearize abt fixed pt.

$$= -\frac{\partial}{\partial n} \left[-k_0 n F(n) - \frac{\partial}{\partial n} \left(\frac{k_0^2 \tau^2}{2} F(n) \right) \right]$$

⇒ Zero flux / stationarity:

$$F(n) = \sum_{const} C \exp \left[-n^2 / k_0 \tau^2 \right]$$

Valid for: $\langle (\tilde{n}/N_0)^2 \rangle = \langle (\tilde{n}/k_0)^2 \rangle < 1$

Now $\langle \tilde{n}^2 \rangle = \frac{\tau^2 k_0}{2}$

$$\sigma_0 \langle (\tilde{n}/k_0)^2 \rangle < 1 \Rightarrow \left\{ \frac{\tau^2}{2k_0} < 1 \right\}$$

→ again ; $\sigma^2 < 2k_0$

i.e. fluctuations small compared
logistic growth.

N.B.:

- can determine time evolution
- can get moments
- spatio-temporal dynamics,