

the Beyond Landau Problem

1.
Physics
2106

→ Recall Landau Problem:

- Linear response problem for
Vlasov - Poisson system

- n.b. → classic, well-posed example

* → example of collisionless
damping

$$\rightarrow \frac{ds}{dt} = 0 \quad \text{as } df/dt = 0$$

but $\tilde{n}, \tilde{E} \left(\int dV \tilde{F} \right)$
decay.

→ analytical continuation to
damped case, phase mixing
⇒ see notes.

→ Physics: Resonant particle heating
i.e. $\langle \underline{E} \cdot \underline{J} \rangle \neq 0$.

i.e. beam example.

2) Now,

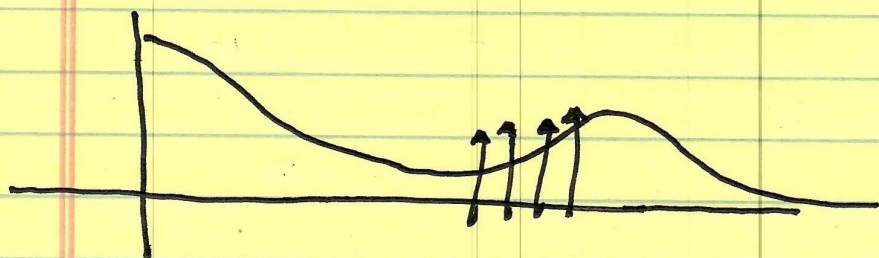
- Vlasov-Poisson is nonlinear

i.e. $E \frac{\partial f}{\partial v}$ with $-\partial_x^2 \phi = \int 4\pi n_0 f$

- So natural to ask of nonlinear evolution,

- observe can have $\left\{ \begin{array}{l} \text{instability} \\ \text{multiple modes} \end{array} \right.$

i.e. Bump-on-Tail (Plasma + Beam)

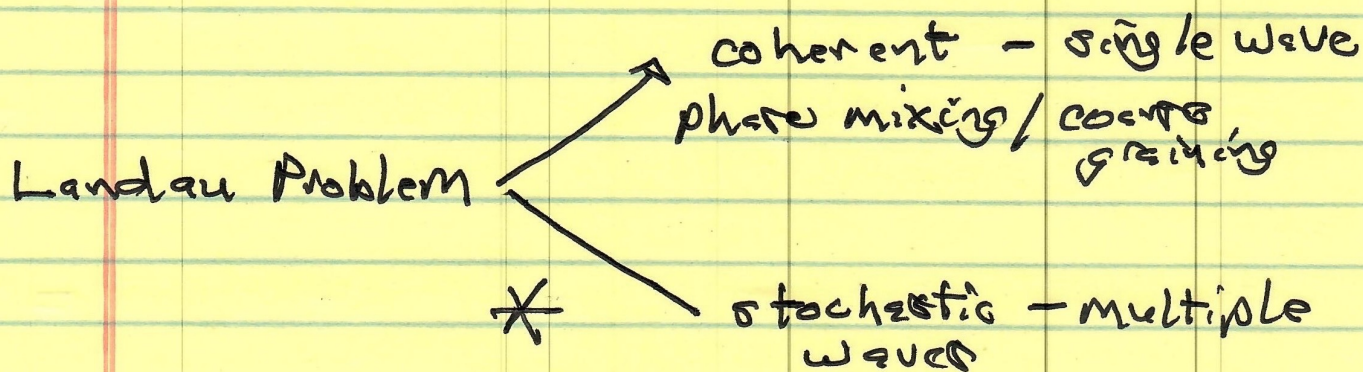


$$\frac{\partial \langle f \rangle}{\partial v} > 0 \Rightarrow \text{Im } \epsilon < 0$$

unstable

? How will system evolve?

3) Linear $\rightarrow$ Nonlinear story of Landau Problem?

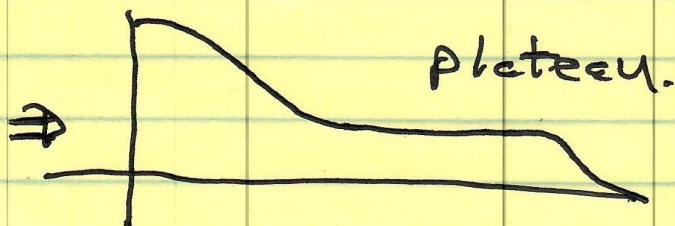
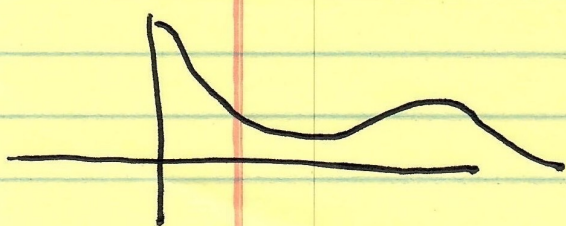


$\Rightarrow$ Quasilinear Theory

* Using linear response
in mean field theory

Q.L.T. answers question of how $\langle f \rangle$ evolves due to effects of multiple resonant waves.

Obviously, relieving $\partial \langle f \rangle / \partial v$ saturates instability. Rate? $\Rightarrow$ saturation amplitude



→ why?

→ Linear response

→ statistics (multi-wave)

→ mean field $\langle f \rangle$

on target!

c.f. - Vedenov, Velikov, Sagdeev
1960's ↑

- also Drummond, Pfister

- Pedagogical Treatments

- Sagdeev, Galeev "Nonlinear Plasma Theory"

- PD, Itoh, Itoh "Phys. Kin. of Plasmas"

210, Eur.

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see next

Christ Landau

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$$\Delta A P = 2\gamma \cdot W^2 \cdot \Delta A / 4$$

$$V_0 \Delta V' \rightarrow \gamma \Delta A P$$

$$\Delta V' \rightarrow \gamma \frac{\partial \Delta V}{\partial V}$$

Quasilinear Theory = Vlasov Plasma

i) Motivation and Overview

Linear theory determines instantaneous stability of plasma

[Landau Problem]

$$\text{ie. } \epsilon(k, \omega) = 1 + \frac{q_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

Via linear response

$$\Rightarrow \text{growth/damping rate } \gamma_k = \gamma_k[\langle f \rangle]$$

but $\langle f \rangle$ evolves... If $\langle f \rangle$ evolves slowly:

$$\text{ie. "slowly"} \Rightarrow \frac{1}{\langle f \rangle} \frac{\partial \langle f \rangle}{\partial t} < \gamma_k$$

can consider: $\gamma_k = \gamma_k[\langle f(p) \rangle] \rightarrow \begin{cases} \text{evolution driven} \\ \text{by instabilities} \end{cases}$

physics: mean distribution evolution...
 $\Rightarrow$ driven by relaxation.

$\Rightarrow$ quasilinear theory is concerned with describing and understanding the slow evolution of $\langle f \rangle$...

$\langle \rangle$ = space/time

To connect to Landau Theory:

in Landau theory, for weakly nonlinear evolution:

$$\partial_t A^2 = 2\gamma_0 A^2 - \sigma A^4$$

$$\gamma_0 \approx V_0' \quad (\text{i.e. shear inst.})$$

but

$$\partial_t \langle V \rangle = -\partial_x \langle \tilde{V}_x \tilde{V}_y \rangle$$

$$\text{so: } V_0' \rightarrow V_0' + \Delta V$$

$$\Delta V' \sim A^2, \text{ necessarily}$$

$$\text{write } \Delta V' \sim -\sigma' A^2$$

↳ PL-like feedback on driving profile.

$$\begin{aligned} \partial_t A^2 &= 2\gamma A^2 \\ &= 2(\gamma_0 + \Delta\gamma) A^2 \end{aligned}$$

but

$$\Delta\gamma \approx \Delta V' \approx -\sigma' A^2$$

$\Rightarrow$

$$\partial_t A^2 = 2\gamma_0 A^2 - \sigma A^4$$

$$= (2\gamma_0 - \sigma A^2) A^2$$

③ quasilinear theory is "mindless mean field theory", i.e.

$$\langle f \rangle = \langle f(y, t) \rangle \quad \text{where } \rightarrow \langle \rangle \text{ eliminates spatial dependence}$$

$\rightarrow t$ understood "slow"

so i.e.:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = 0$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} \left\langle \frac{q}{m} E \tilde{f} \right\rangle = 0$$

i.e. generic mean field equation (for $\langle f \rangle$)
for mean of conserved order parameter

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} \tilde{J}_v = 0 \quad \rightarrow \left[\begin{array}{l} \text{phase space} \\ \text{continuity equation} \end{array} \right] \quad (\langle f \rangle)$$

$$\tilde{J}_v = \tilde{J}_v = \left\langle \frac{q}{m} E f \right\rangle$$

$$= \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$\text{for: } E \approx \tilde{E}$$

$$f \approx \langle f \rangle + \tilde{f}$$

elementary closure problem
i.e. relate $\langle f \rangle$ to $\langle \tilde{E} \tilde{f} \rangle \rightarrow$
hierarchy!
How close?

simplest example of moment closure

then Q.L.T. simply takes form:

(f) $\tilde{f} \rightarrow \tilde{f}_{\text{linear}}$ (i.e. linear response of
— plug in linear response $\tilde{f}$)

i.e. $\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} + \frac{q}{m} E \frac{\partial \tilde{f}}{\partial v} = 0$ V/oscuv
Eqn.

$$\Rightarrow -i(\omega - kv) \tilde{f}_k = -\frac{q}{m} \tilde{E}_k \frac{\partial \langle f \rangle}{\partial v}$$

$$\text{so } J_v = -\frac{q^2}{m^2} \sum_{k \neq 0} |\tilde{E}_k|^2 \frac{1}{(\omega - kv)} \frac{\partial \langle f \rangle}{\partial v}$$

and with $\omega = \omega(k)$ only (i.e. spectrum of
eigenmodes, only)

i.e. contrast approach to
criticality in usual
phase transitions (2nd
order)

Q.L. equation is:

$$\begin{aligned} \rightarrow \frac{\partial \langle f \rangle}{\partial t} &= \frac{q}{m} D \frac{\partial \langle f \rangle}{\partial v} \\ D &= \frac{q^2}{m^2} \sum_k |\tilde{E}_k|^2 \frac{1}{\omega - kv + i\eta} \end{aligned}$$

Q.L. equation.

Diffusion
"well"
Fill in "as
expected"

here growth of order
parameter in broken symmet.
phase ... not necessarily
"not necessarily"

$$\begin{aligned} \rightarrow \text{with } \epsilon(k, \omega) &= 0 \\ \rightarrow \partial_t |\tilde{E}_k|^2 &= 2\gamma_k |\tilde{E}_k|^2 \end{aligned} \quad \rightarrow \text{advance fields.}$$

i.e. describes how mode spectrum
evolves mean $\langle f \rangle$.

But,

Surprisingly: Q.L.T. works quite well!

key issue: why?

N.B.: In contrast to critical phenomena, external noise ignored $\rightarrow$ instability driven

④ Some questions to keep in mind: deterministic.

$\rightarrow$ (i) why is Q.L. equation a diffusion equation? when is this valid?

$\Rightarrow$ nature of "irreversibility"

$\rightarrow$ (ii) can Q.L. equation be derived from Fokker-Planck theory?

$\Rightarrow$ also "irreversibility" related... what of drift?

$\rightarrow$ (iii) how does Q.L. equation balance the energy-momentum budgets?

$\rightarrow$ (iv) when does Q.L. theory fail? $\frac{\langle \delta^2 \rangle}{\langle \delta \rangle^2}$
how

$\Rightarrow$ related (i) ... What is "Ginzburg Criterion" for Q.L.T. Can such a criterion be formulated?

$\rightarrow$ (v) what is dynamics of quasilinear relaxation?

i.e. physics?

ii) Basic Scales / Regime Definition

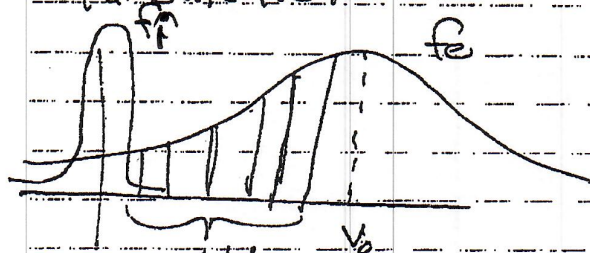
① → Generally, Q.L.T. concerned with

i) 'broad' spectrum of:

↳ how broad?

ii) unstable waves

ie for current-driven ion-acoustic (G.O.I.-A.) turbulence:



or B-O-T

②

→ In finite system, k quantized, i.e.

$$k_m = m\pi/L, \text{ etc.}$$

- so, have spectrum of phase velocities

$$\omega_m/k_m = \omega(k_m)/k_m = v_{ph,m}$$

- wave-particle resonance occurs when

$$V = v_{ph,m}$$

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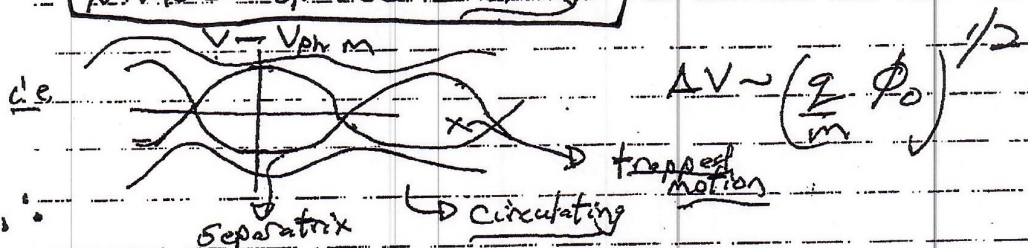
then $\sin$ is $\approx \cos \Rightarrow$

$$m\ddot{x} = \sum_m q E_m \cos(k_m x - \omega_m t) \quad \left\{ \begin{array}{l} \text{n.b.} \\ \text{deterministic,} \\ \text{no RPA} \end{array} \right.$$

and 1 resonance dominant $\Rightarrow$

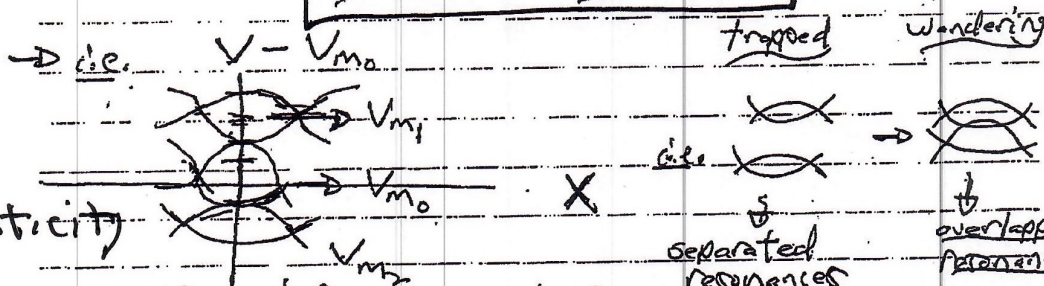
$$m\ddot{x} \approx q E_m \cos(k_m x + (k_m v - \omega_m) t)$$

$\Rightarrow$ each resonant velocity defines a phase space island



QLT is concerned with the case of:

$\rightarrow$ multiple overlapping resonances $\rightarrow$ $\left\{ \begin{array}{l} \text{separatrix proximity} \\ \text{destruction} \end{array} \right.$



Stochasticity

if overlap,

hop

particle can wander stochastically from resonance - to - resonance, i.e. hopping

$$\Rightarrow \text{diffusion in } V \quad D_v \approx \frac{(\Delta V)^2}{\tau_{ac}} \quad \left\{ \begin{array}{l} \text{AV resonance} \\ \text{with} \\ \tau_{ac} \text{ pattern} \\ \text{time} \end{array} \right.$$

ergodicity $\rightarrow$ mixing

(九州大学応用力学研究所)

$\rightarrow$ what is it?

- See Phys. 200B Notes 2014, 2015 - Google directly.
- On, Supplementary Material

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overlap $\Leftrightarrow \exists \mid h_n > 0$. (positive Lyapunov exp.) 184.

Overlap condition (B.V. Chirikov) :

$$\frac{1}{2} (\Delta V_m + \Delta V_{m+1}) \geq V_{ph, m+1} - V_{ph, m}$$

underlies
diffusion
eqn. - QL

• $\Rightarrow$ particle motion stochastic $\Rightarrow$ irreversibility

$\Rightarrow$ fundamental irreversibility $\Rightarrow$ orbit
stochasticity (not dissipation, Landau
damping) $\Rightarrow$ contrast critical phenomena.
 $\Rightarrow$ underpinning of diffusion equation. (noise)

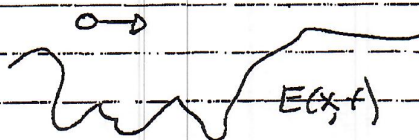
③ $\Rightarrow$ But, a swindle! $\Rightarrow$ [use of un-perturbed
orbit is estimated!
Why does linearization work?]

i.e. is $x \rightarrow x_0 + v t$ valid?

Consider: linear, un-perturbed orbit!

$$\text{have: } E(x, t) = \sum_k E_k \exp[i(kx - \omega_k t)]$$

• particle "sees" instantaneous pattern of
electric field, from modal superposition

i.e. 

relevant comparison is:

$T_L \rightarrow$ life time of 'instantaneous' pattern

$T_b \rightarrow$ 'bounce time' of particle in pattern

obviously, $\textcircled{1} T_L \ll T_b \rightarrow$ unperturbed orbit is satisfactory approximation
(pattern changes prior $\leftrightarrow$ bouncing)

$\textcircled{2} T_L \gg T_b \rightarrow$ particle bounces prior pattern changes
so must consider orbit perturbation...

ie 

vs. 

quasilinear theory relevant to evolution when:

- $\textcircled{1} \rightarrow$ orbits stochastic (Chirikov condition satisfied)
- $\textcircled{2} \rightarrow T_{\text{life}} < T_{\text{bounce}} \rightarrow$ unperturbed orbits $\textcircled{1}$ valid.

But, how relate $T_{lifetime}$, T_{bounce} to physical quantities?

Key point: Superposition patterns disperse!

$$E(x, t) \Rightarrow \sum_k E_k e^{i(kx - \omega_k t)} = \sum_k E_k \exp[i(k[x - \underbrace{\omega_k/k}_{v_{ph}(k)} t])]$$

$\Delta(\omega/k) =$ spread in phase velocities.
sets dispersal ~~rate~~ speed.

so dispersal rate is (time)⁻¹ to disperse by one wavelength $\rightarrow$ FOM

$$1/T_{life} = k \Delta(\omega/k) / \tau_{ac}$$

$$= k \left(\frac{d\omega_k}{dk} \frac{\Delta k}{k} - \frac{\omega_k}{k^2} \Delta k \right) /$$

$$\boxed{1/T_{life} = \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k} \right) \Delta k = (v_{gk} - v_{phk}) / \Delta k}$$

n.b. $T_{life} \rightarrow \infty$ for non-dispersive waves!

Generally; QLT / weak turbulence encounters trouble for $\left\{ \begin{array}{l} \text{non dispersive} \\ \text{weakly dispersive} \end{array} \right.$ waves.
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$$1/T_L \sim 1/T_{ac} \text{ for resonant particle.}$$

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How systematize? — E-field correlation Fctn

$$\text{Consider: } \langle E(x_1, t_1) E(x_2, t_2) \rangle_{x, t} = C$$

electric field correlation function

$$C = C(x, T) \text{ , for } \left\{ \begin{array}{l} \text{homogeneous} \\ \text{stationary} \end{array} \right\} \text{ fluctuations}$$

relative coords, space/time

$$x_1 = x_+ + x_-$$

$$t_1 = t_+ + t_-$$

$$x_2 = x_+ - x_-$$

$$t_2 = t_+ - t_-$$

$$\langle \rangle_{x, t} = \langle \rangle_{x_+, t_+}$$

what the bracket means

so

$$C(x, T) = \left\langle \sum_{k, k'} E_k E_{k'} e^{i(k+k')x_+} e^{-i(\omega_k + \omega_{k'})t_+} + e^{i(k-k')x_-} e^{-i(\omega_k - \omega_{k'})t_-} \right\rangle_{x_+, t_+}$$

$$x_+, t_+ \text{ average } \Rightarrow k = -k' \quad \omega_k = -\omega_{k'}$$

so

$$C(x, T) = \sum_k |E_k|^2 e^{ikx} e^{-i\omega_k t}$$

Now:

→ assume continuous spectrum - i.e. part-overlap

→ for simplicity, take model

$$|E_k|^2 = E_0^2 / \left[\left(\frac{k - k_0}{\Delta k} \right)^2 + 1 \right] \quad (\text{trial form})$$

↪ width

→ evaluate on u.p.o.

$$x_- = x_0 + v\tilde{t}$$

$$\langle E^2 \rangle = \int dk \frac{E_0^2}{\left[\left(\frac{k - k_0}{\Delta k} \right)^2 + 1 \right]} e^{ikx_0} e^{i(kv - \omega_k)\tilde{t}}$$

integrating:

phase int. - irrelevant

$$\sim E_0^2 e^{ik_0 x_0} e^{-|\Delta k x_0|} *$$

$$e^{i(k_0 v - \omega_{k_0})\tilde{t}} e^{-|\Delta(kv - \omega_k)|\tilde{t}}$$

↓
oscillation
(→ on resonance)

↳ correlation decay
due to dispersion
and its interplay
with resonance.

note: note that spread is doppler-shifted
ω is critical

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$$\text{now } A(kV - \omega_k) = V \Delta k - V_{gr} \Delta k$$

$$= |(V - V_{gr}) \Delta k|$$

$$V_{gr} = \frac{d\omega}{dk}$$

$$\text{So } \langle F^2 \rangle = C(x, y)$$

$$= E_0^2 e^{i k_0 x} e^{i(k_0 V - \omega_{k_0}) T} e^{-| \Delta k x_0 |}$$

$$* \exp[-|(V - V_{gr}) \Delta k| T]$$

sets lifetime

$$\left\{ \frac{1}{T_L} = |(V - V_{gr}(k)) \Delta k| \equiv (\text{Autocorrelation Time}) \right.$$

$$\left. \equiv 1/T_{ac} \right.$$

Note:

for resonant particles, $V = \omega_k/k$

$$\frac{1}{T_L} = |(V_{ph} - V_{gr}) \Delta k| \rightarrow \text{recovers earlier!}$$

can think: $|V \Delta k| \rightarrow 1/T_{ac}^{\text{wave-particle}}$

$|V_{gr} \Delta k| \rightarrow 1/T_{ac}^{\text{wave}}$ packet dispersed!

generally, shorter time dominates,
except for non-dispersive waves.

So, can enumerate key time scales

$$\rightarrow \tau_{ac} = [\Delta k (v_{ph} - v_{gr})]^{-1}$$

$\equiv$ persistence of E pattern (E^2)
autocorrelation) for resonant
particles

$$\rightarrow \gamma^{-1} = \text{growth/damping time}$$

$$\rightarrow \tilde{\tau}_{Tr} = (k \sqrt{2\phi/m})^{-1} \equiv \text{trapping time}$$

$$\rightarrow \tilde{\tau}_{relax} = \left(\frac{1}{\langle F \rangle} \frac{\partial \langle F \rangle}{\partial t} \right)^{-1} \equiv \text{avg. distribution}$$

relaxation time

So

$$\tau_{ac} < \tilde{\tau}_{Tr} \rightarrow \text{u.p.o. valid}$$

$$\left[\begin{array}{l} \tau_{ac} < \tilde{\tau}_{relax} \rightarrow \langle F \rangle \text{ closure} \\ \gamma^{-1} \text{ meaningful} \end{array} \right]$$

$$\tau_{ac} < \gamma^{-1} < \tilde{\tau}_{relax} \rightarrow \text{QL.T. valid.}$$

iii) Energy - Momentum Budgets

→ Key Point: There are two ways of implementing the book-keeping and accounting

ie $\left\{ \begin{array}{l} \text{resonant particles} \\ \text{or} \\ \text{particles} \end{array} \right.$ vs. $\left\{ \begin{array}{l} \text{'waves'} \\ \text{or} \\ \text{fields} \end{array} \right.$ / $\left\{ \begin{array}{l} \text{Quasi-} \\ \text{particles} \end{array} \right.$

keep in mind: Wave = Field + Non-resonant particles

ie for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial \omega} (\omega \epsilon) \frac{|E|^2}{8\pi}$$

$$= \omega \frac{\partial \epsilon}{\partial \omega} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant
particle

(show!)

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→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f \rangle}{\partial t} = - \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} f \rangle$$

$$\begin{aligned} \frac{\partial}{\partial t} \int dv \frac{mv^2}{2} \langle f \rangle &= - \int dv \frac{mv^2}{2} \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} f \rangle \\ &= \int dv mv \frac{q}{m} \langle \tilde{E} f \rangle \end{aligned}$$

plugging in $\tilde{f}_k^{\text{linear}}$ for $\tilde{f}$!

Resonant

$$\frac{\partial}{\partial t} \Sigma_{kin} = - i \int dv \frac{v q^2}{m} \sum_k |E_k|^2 \left(\frac{1}{\omega_k v} - \pi \delta(\omega_k - kv) \right) \frac{\partial \langle f \rangle}{\partial v}$$

resonant
only

$$\frac{\partial}{\partial t} \Sigma_{kin} = - \int dv \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \underbrace{\delta(\omega/k - v)}_{\text{resonant only}} \frac{\partial \langle f \rangle}{\partial v} |E_k|^2$$

$$= - \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \frac{\partial \langle f \rangle}{\partial v} \bigg|_{\omega/k} |E_k|^2$$

As resonant particles stabilize/destabilize waves, expect resonant particles conserve energy against waves.

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For wave energy evolution:

Recall: $\epsilon = 1 + \frac{\omega_p^2}{k} \int dV \frac{\partial f / \partial v}{\omega - kv}$

$$\epsilon'(\omega_n + i\gamma_n) + i\epsilon^{IM} = 0$$

$$i\gamma_n = -\frac{\epsilon^{IM}}{\partial \epsilon^n / \partial \omega}$$

$$i\gamma_n = -\frac{\epsilon^{IM}}{\partial \epsilon^n / \partial \omega} = -\epsilon^{IM} / \partial \epsilon^n / \partial \omega$$

Now, $W \equiv$ Wave Energy Density action density

$$W = \sum_k \frac{\partial(\omega \epsilon)}{\partial \omega} \frac{|E_k|^2}{8\pi} = \sum_k \omega_k N_k$$

$$= \sum_k \underbrace{\omega_k \frac{\partial \epsilon^n}{\partial \omega}}_{\omega_k} \frac{|E_k|^2}{8\pi}$$

energy particle density

$$\frac{\partial W}{\partial t} = \sum_k 2\gamma_k \omega_k \frac{\partial \epsilon^n}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$|E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$$

$$= \sum_k 2 \left(-\frac{\epsilon^{IM}}{\partial \epsilon^n / \partial \omega} \right) \omega_k \frac{\partial \epsilon^n}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$= \sum_k -\epsilon^{IM}(k, \omega_k) \omega_k \left(\frac{|E_k|^2}{4\pi} \right)$$

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$$i E_{IM} = \frac{\omega_p^2}{k} \frac{\partial \langle F \rangle}{\partial v} \bigg|_{\omega/k} \frac{(-i\pi)}{|k|}$$

$$(n_0 = 1)$$

$$\begin{aligned} \therefore \frac{\partial W}{\partial t} &= \sum_{\underline{k}} \pi \frac{q^2}{m} \frac{\omega_{\underline{k}}}{k|k|} \frac{\partial \langle F \rangle}{\partial v} \bigg|_{\omega/k} \frac{|E_{\underline{k}}|^2}{|k|} \\ &= + \pi \frac{q^2}{m} \sum_{\underline{k}} \frac{\omega}{k|k|} \frac{\partial \langle F \rangle}{\partial v} \bigg|_{\omega/k} |E_{\underline{k}}|^2 \end{aligned}$$

$$\equiv \boxed{\frac{\partial E_{\text{resonant}}}{\partial t} + \frac{\partial W}{\partial t} = 0} \quad \text{it works } \checkmark \quad \frac{\partial}{\partial t} \left[\sum_{\underline{k}} \omega_{\underline{k}} N_{\underline{k}} + \frac{E_{\text{res}}}{k} \right] = 0$$

$$\frac{\partial}{\partial t} \text{field energy} = - \frac{\partial \langle E \cdot F \rangle}{\partial t}$$

Notes:

— this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\frac{\partial W}{\partial t} + \nabla \cdot \underline{S} + Q = 0$$

$\downarrow$ wave energy $\downarrow$ divergence of wave energy density flux $\downarrow$ $\langle \underline{E} \cdot \underline{J} \rangle$ coupling $\downarrow$ resonant particle heating

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$$\frac{\partial}{\partial t} \int (W) + \frac{\partial}{\partial t} \text{RPKE} = 0$$

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For homogeneous system: $\nabla \cdot \mathbf{S} = 0$

$$\frac{\partial W}{\partial t} + Q = 0$$

$\int \langle \mathbf{E} \cdot \mathbf{J} \rangle$ mediated by
resonant particles
(DC field)

$$\Leftrightarrow \frac{\partial W}{\partial t} + \frac{\partial (\text{RPKE})}{\partial t} = 0$$

$\int$
resonant
particle kinetic
energy density

Energy Thm I

Waves and
Resonant particles
conserve energy!

? What is the fate
of RPKE for saturated
waves. What must
happen ??

→ Now can observe:

$$W = \text{NR.PKE} + \text{FED}$$

$\int$
non-resonant
particle kinetic
energy density

$\int$
field energy
density

so, simply re-grouping terms:

$$\frac{\partial (\text{FED})}{\partial t} + \frac{\partial (\text{RPKE} + \text{NR.PKE})}{\partial t} = 0$$

$\int$
PKE → total

So $\frac{\partial}{\partial t} F(E) + \frac{\partial}{\partial t} (P K E) = 0$ Energy Thm.
II

fields and particles conserve energy.

What is the physics of all this?

$D = \sum_k \frac{q^2}{m^2} |E_k|^2 (i/\omega - kv)$
 §
 PL diffusion for generally weakly non-stationary state....

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left(|X_k| \sqrt{(\omega - kv)^2 + \gamma_k^2} \right)$$

n.b. causality $\Rightarrow$
 no negative
 diffusion for
 damped waves

$$D \approx \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \underbrace{\pi \delta(\omega - kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|X_k|}{\omega_k^2}}_{\text{non-resonant diffusion}} \right\}$$

Resonant Diffusion $\rightarrow$ irreversible - resonance
overlap is underpinning

$\rightarrow$ rooted in particle stochasticity

- Relation to Fokker-Planck?

$$D_{QL}(v) \equiv \underbrace{Res.}_{D_R} + \underbrace{Non-Res.}_{D_{NR}}$$

$D_R \Rightarrow$ stochasticity / chaos
 - irreversible
 - short τ_{ac}

$\Rightarrow F-P.$

$D_{NR} \Rightarrow$ No. (sloshing)

Now, as Langevin Egn. here is:

$$\frac{dv}{dt} = \frac{1}{m} \tilde{E}, \quad \frac{dx}{dt} = v$$

(Hamiltonian / no drag)

then $F-P$ is:

(for res.)

$$\partial_t \langle F \rangle = - \frac{\partial}{\partial v} \left[\underbrace{F \langle F \rangle}_{\text{drift/friction}} + \frac{1}{2} \frac{\partial^2}{\partial v^2} \underbrace{D \langle F \rangle}_{\text{diffusion}} \right]$$

but

$$\partial_t \langle F \rangle = \partial_v D(v) \partial_v \langle F \rangle$$

order of
derivatives!

To resonate, for resonant particles:

$$F = \frac{1}{2} \frac{\partial D(v)}{\partial v}$$

(cf Landau '37)

- drift, diffusion partially cancel for Hamiltonian systems

To show:

$$V(t + \Delta t) = V(t) + \dot{V} \Delta t + \frac{a}{2} \ddot{V} (\Delta t)^2$$

$$\ddot{V} = -\partial^2 H / \partial x^2 \quad (m=1)$$

$$\ddot{V} = -\frac{\partial^2 H}{\partial x^2} \dot{x} - \frac{\partial^2 H}{\partial x \partial v} \dot{v} - \frac{\partial^2 H}{\partial x \partial t}$$

$$= -\frac{\partial^2 H}{\partial x^2} \frac{\partial H}{\partial v} + \left(\frac{\partial^2 H}{\partial x \partial v} \right) \frac{\partial H}{\partial x} - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial t} \right) \quad \text{partial cancel.}$$

$$= -\frac{\partial}{\partial x} \left(\frac{\partial H}{\partial x} \frac{\partial H}{\partial v} \right) + \frac{\partial}{\partial v} \left(\frac{\partial H}{\partial x} \right)^2 - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial t} \right)$$

then

$$\Delta V = -\frac{\partial H}{\partial x} \Delta t + \frac{a}{2} (\Delta t)^2 \left[\frac{\partial}{\partial v} \left(\frac{\partial H}{\partial x} \right)^2 - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial x} \frac{\partial H}{\partial v} + \frac{\partial H}{\partial t} \right) \right]$$

$$\text{so } \langle \Delta V \rangle = \frac{1}{2} (\Delta t)^2 \frac{\partial}{\partial V} \left\langle \left(\frac{\partial H}{\partial x} \right)^2 \right\rangle \quad (1)$$

$$\langle \cdot \rangle = \int \frac{dx}{L}$$

$$\text{and } \Delta V \Delta V = \dot{u}^2 (\Delta t)^2 = \left(\frac{\partial H}{\partial x} \right)^2 (\Delta t)^2$$

$$\langle \Delta V \Delta V \rangle = \left\langle \left(\frac{\partial H}{\partial x} \right)^2 \right\rangle \Delta t^2 \quad (2)$$

Comparing (1), (2)

$$\frac{1}{2} \frac{\partial}{\partial V} D = F \quad \checkmark$$

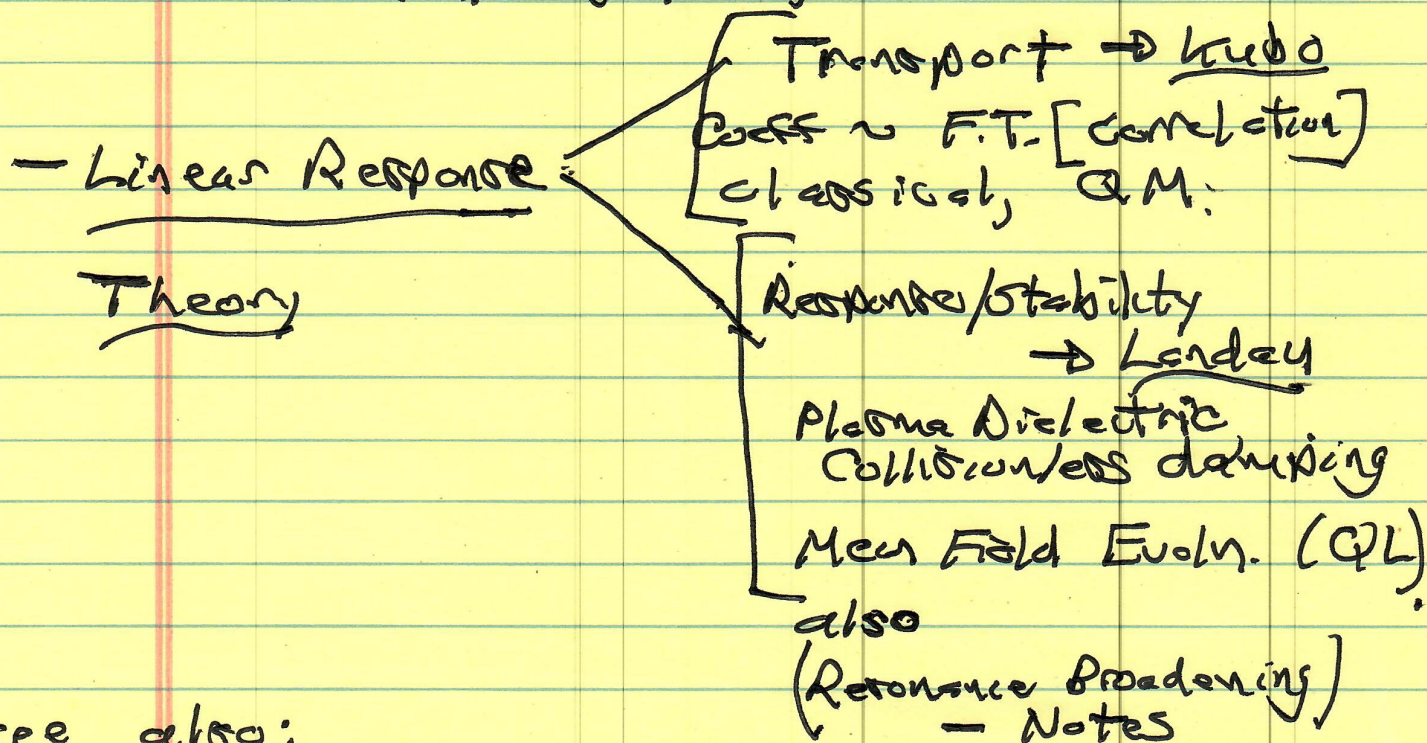
Drag - Diffusion cancellation for
1D QL $\leftrightarrow$ Hamiltonian

① Beware order of derivatives!

② Relevant QL diffusion - Fokker-Planck

→ Linear Response - Review

- Linear response fundamental to transport, collective response
- Lin. resp th. → formulated at fundamental kin Egn. (conservative)
↔ Liouville (Vlasov)



see also:

Kadanoff + Martin, posted
"Hydrodynamic Equations and
Correlation Functions"