Chirikov, Chaos and KAM → an OV

→ from small divisors → to Chirikov criterion

→ Development Chaos, (Standard Map)

→ KAM Theorem *

→ Aspects of Chaos. – Stat. Mech Relevant

Chirikov and Chaos and KAM → An OV.

- Recall:
  - defined action angle variables
  - addressed "perturbative integrability"
  - → defined resonant surface
  - → noted island formation at resonant surface due resonant perturbations

Some key observations:

- resonant surface defined by $\underline{n} \cdot \underline{\omega} = 0$

- averaging / secular P.T. recovers island with Ham:

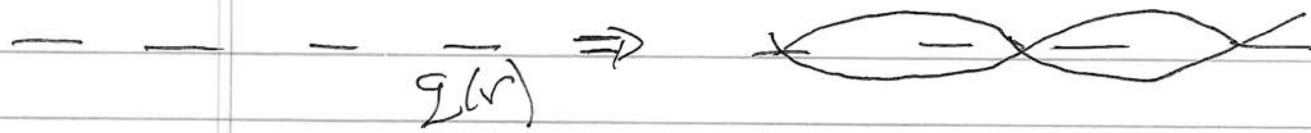
$$H = \underbrace{G}_{\sim \frac{\partial \omega}{\partial J}} \frac{\Delta J^2}{2} - \underbrace{F}_{\sim H_1} \cos \phi$$

- but secular P.T. really only works with one resonance / slow variable in region of averaging

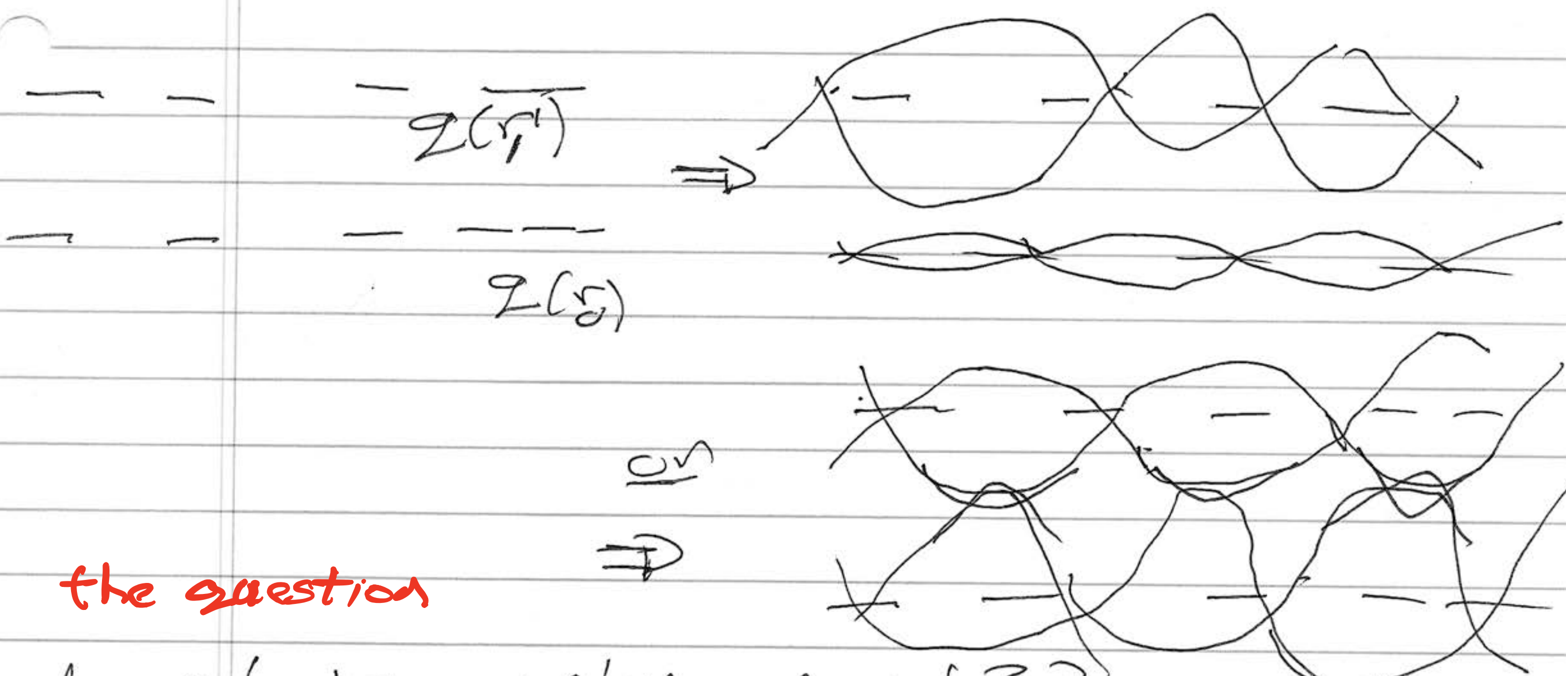
What of multiple perturbed resonances?

i.e.

before:



now



the question

do islands overlap or not??

if no: can consider 2 isolated resonances with un-perturbed tori in between.

if yes: - orbits no longer are 'localized' to vicinity of a single resonance
- integrity of surfaces between resonances is violated ⇒ pass 1 to other
- orbit can pass from resonance-to-resonance ⇒ sample volume, not surface.

from ergodic to mixing

⇒
⇒ tori between 2 resonances are destroyed

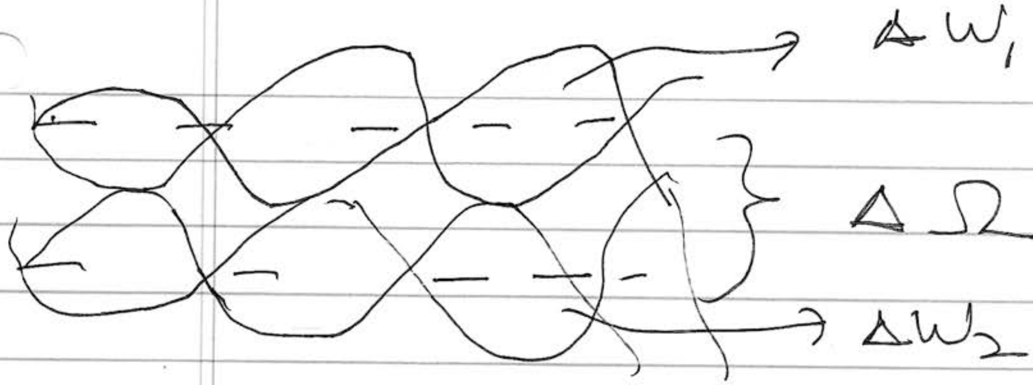
⇒ motion fills volume not surface

⇒
enter chaos !! ⟺ { breakdown of integrability for Hamiltonian system.

So

Working criterion for onset of Hamiltonian chaos is Chirikov Island Overlap Criterion

n.b. critical amplitude for onset chaos. $w_I \sim (\epsilon H_1 / \frac{\partial \omega}{\partial J})^{1/2}$



$\Delta\Omega \equiv$ spacing of resonances

$\Delta W_{1,2} \equiv$ 1/2 widths of resonances distortions (islands!) at neighboring resonant tori

$\Rightarrow$ $\boxed{\Delta W_1 + \Delta W_2 = \Delta\Omega}$

is Chirikov criterion for:
- overlap of islands at resonances
- destruction of surfaces, between

$\Rightarrow$
- onset chaos, mixing (in volume set by resonant helicities)

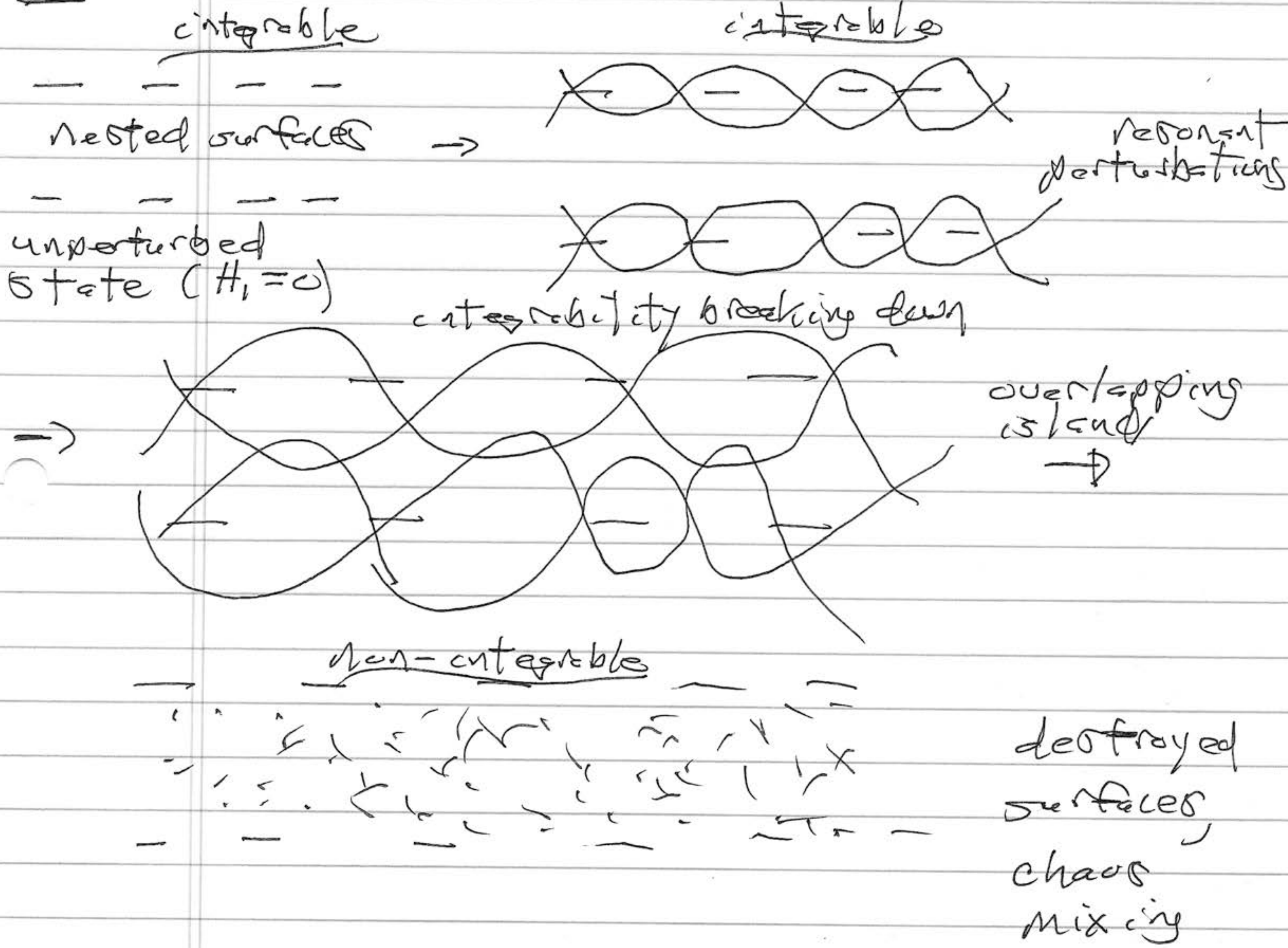
i.e.:

→ global criterion (i.e. region not point).

Contrast Lyapunov exponent

→ end state of resonance distortions:

i.e.



→ Why Believe this Story ?

- for numerical studies convenient to work with maps instead ODE's.

c.f. random #'s for Monte Carlo ⇒ Logistic Map. (Feigenbaum) (Rank, Sun)

∴ enter the Standard Map, i.e.

(Taylor, Chirikov, early 60's)

winding ~~rate~~

$$\theta_{n+1} = \theta_n + p_n \quad \text{mod } 2\pi$$

perturbation

$$p_{n+1} = p_n + k \sin\theta_{n+1} \quad \text{mod } 2\pi$$

perturbed winding rate. strength (field amplitude)

$\theta \rightarrow$ position
$p \rightarrow$ momentum

→ 2D, 2 degs freedom

→ phase space a (toroidal) surface
(1 dim angle, 1 dim radius/action)

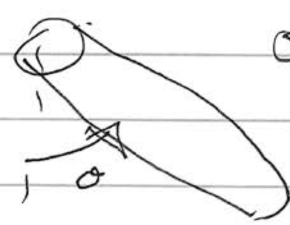
$$\rightarrow \det \begin{vmatrix} \frac{\partial\theta_{n+1}}{\partial\theta_n}, & \frac{\partial\theta_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial\theta_n}, & \frac{\partial p_{n+1}}{\partial p_n} \end{vmatrix} = \begin{vmatrix} 1, & 1 \\ k\cos\theta_{n+1}, & 1+k\cos\theta_{n+1} \end{vmatrix}$$

area preserving ✓. $= 1$

→ physics: "kicked rotor"

$$H(p_\theta, \theta, t) = \frac{p_\theta^2}{2I} + k\cos\theta \sum_n \delta(t - n\tau)$$

so



$\sigma = 0$

vertical impulsive force, at τ period

$$\frac{dp_\theta}{dt} = K \sin\theta \sum_n \delta(t - nT)$$

$$\frac{d\theta}{dt} = p_\theta / \bar{I}$$

integrating and $T/\bar{I} = 1 \Rightarrow$ standard map.
$\rightsquigarrow$ lab animal for studies of stochasticity.

$\Rightarrow$ so:

① $(\Delta p)_{max} = \underline{2K^{1/2}}$ → island size $\sim \sqrt{K}$ (dyn. system) $(m=1)$

$\Delta_{res} = 2\pi$

$\Rightarrow K_{crit} \cong 2.47$ → [destruction of surfaces for $m=1$ overlap]

② if interaction bet. period 1, period 2

$\Rightarrow K_{crit} \cong 1.46$ → overlap criterion.

∴ if examine OH fig 7.3, pg. 275
see:

Details:

a.) $K_1 = .5$

- surfaces preserved, except for stochastic layer near separatrixes.
- ~~[illegible]~~ period 1, ~~[illegible]~~ period 2 islands clearly preserved

b.) $K_1 = 1$

- onset stochasticity
- surface between period 1, 2 islands destroyed

c.) $K_1 = 2.5$

- stochasticity, strong
- ~~[illegible]~~ period unstable → islands only residual

d.) $K_1 = 4$
- ~~[illegible]~~ period 1 going unstable.

Point:

- Chirikov is slight under-estimate of stochasticity onset

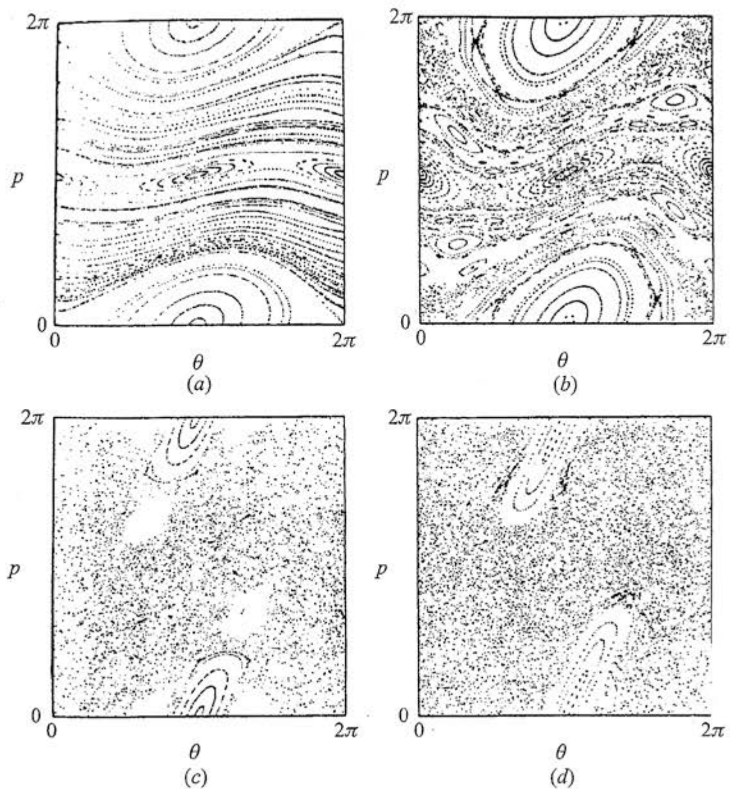


Figure 7.13 Plots of p modulo 2π for four values of K: (*a*) $K = 0.5$; (*b*) $K = 1.0$; (*c*) $K = 2.5$; (*d*) $K = 4.0$. (This figure courtesy of Y. Du.)

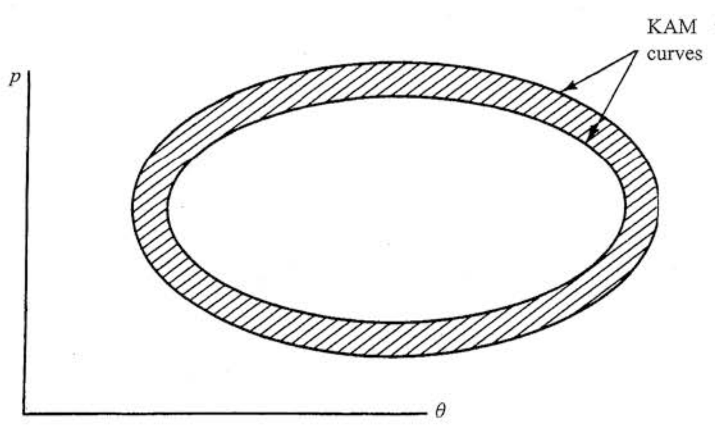


Figure 7.14 Two KAM curves bounding an annular region.

KAM curves (as, for example, in the island structures surrounding elliptic orbits), these chaotic orbits are necessarily restricted to lie between the bounding KAM curves. (As we shall discuss later, this picutre is fundamentally different for systems of higher dimensionality.)

- ignores secondary islands, stochastic layers etc.

⇒ Resonance overlap leads to breakdown of integrability, destruction of tori, onset of chaos, and mixing.

⇒ Prototype of mechanism for onset of deterministic, Hamiltonian chaos.

Now:

- story presented is 'tip of very large iceberg'
  → see Lichtenberg & Lieberman, and literature for details
- non-Hamiltonian chaos is fundamentally different ⇔ attractors.

Some key Questions:

(i)
- is there a theorem? ⇒ can we prove the story? → KAM theorem

(2) – how characterize chaos ? → dynamical entropy ?

(3) – how calculate in chaotic regime → stat mech. (this course)

Very Abbreviated story:

→ KAM Theorem (Kolmogorov, Arnold, Moser)

– how resolve the "small divisor" problem, rigorously ?

↓

i.e. can we integrate the system perturbatively ?!

– Thm:

For $H = H_0 + \epsilon \overset{\text{pert.}}{H_1}$, if H_1 is small enough, then for almost all frequencies $\underline{\omega}^*$, there exists an invariant torus $T(\underline{\omega}^*)$ of perturbed system that $T(\underline{\omega}^*)$ is close to $T_0(\underline{\omega}^*)$.

T_0 → torus at irrational surface, unperturbed

T → perturbed surface.

Translation :

$\boxed{\text{" A sufficiently non-resonant irrational torus can survive a sufficiently weak perturbation"}}$.

↝ threshold for chaos!

Some clarification :

- What is an irrational or non-resonant torus ?

$$|\underline{m} \cdot \underline{\omega}| > K(\underline{\omega})\, |m|^{-(N+1)}$$

N dofs

$$|\underline{m}| = |m_1| + |m_2| + \cdots |m_N|$$

$K(\omega)$ indep. $\underline{m}$

↝ set of N dim. vectors not satisfying above ⟺ set $\mathcal{M} \to 0$

↝ irrational tori are 'common' rational tori are 'unusual'.

rationals vs irrationals

- What does "survive" mean ?

torus of perturbed system has frequency $\underline{\omega}(\epsilon) = K(\epsilon)\, \underline{\omega_0}$ s/t $K(\epsilon) \to 1$ as $\epsilon \to 0$.

→ $\boxed{\text{smooth distortion}}$.

- KAM theorem says for small ϵ, the perturbed system's phase volume not occupied by surviving tori is small, and $\to 0$ with ϵ.

Key Mathematical Points:

- rationals are set of $\mu = 0$ in numbers.
- number theoretic arguments for 'sufficiently non-resonant'
- exponential decay of $H_1(m) \sim e^{-|m|}$

Implication:

$\Rightarrow$ Resonant, rational tori are key to Hamiltonian chaos, integrability breakdown.

N.B: ① Poincare – Birkhoff theorem provides rigorous underpinning for resonant tori and their distortion by perturbations.

Details

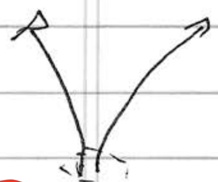
② Can improve on Chirikov by Greene calculation of when KAM torus penetrated by perturbation.

→ $\boxed{\text{Characterizing Chaos}}$

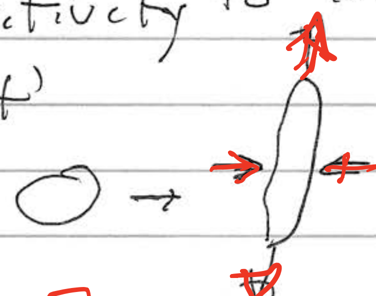
① → how *locally* diagnose chaos?

② → how characterize *strength* of chaos.

① Chaos → exponential divergence of neighboring trajectories
⇒ 'instability' — sensitivity to i.c.
— 'Butterfly effect'



$$\ell \sim \ell_0 e^{ht}$$



$$\boxed{h \equiv \text{Lyapunov exponent}}$$

stretching + volume preservation

N.B. :

→ Lyapunov exponent ⇔ *direction*
→ chaos ⇔ stretching
→ volume *preservation* (Hamiltonian)

$$\boxed{\sum_i h_i = 0}$$

i.e. must have negative h corresponding to direction of *shrinking*.

→ Lyapunov exponent is local — pt./ball
Chirikov is global.

② Strength → Metric or K-S
(Kolmogorov – Sinai) Entropy

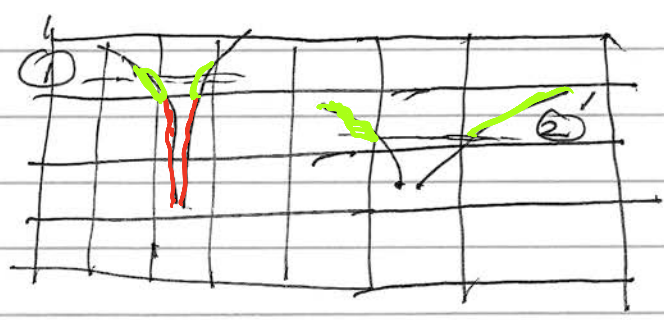
K-S Entropy: Rate of creation of information as system evolves

→ phase space is partitioned
i.e. — resolution scale
— coarse graining

Note dynamics — info. connection

→ rate of info creation ⟺ rate at which two sub-grid sep. orbits become separated by > grid scale ⇒ need keep track of these as distinct orbits.

i.e.



① ②

①' ②' } when "info created"

② has greater metric entropy.

i.e. { Key pt. is info created when separation exceeds the partition scale

$$\Rightarrow \boxed{h(\mu) = \sum_{h_i > 0} h_i}\,,$$

K-S entropy

dynamical entropy

for Hamiltonian systems

∴ $\boxed{\text{positive Lyapunov exponents characterize dynamical entropy of system}}$

③ How calculate in chaotic regime?

— This course.

→ Stat. Mech.! → i.e. pdf

→ in particular, mean field theory
i.e. coarse grained probability most practical.

$$\theta_{n+1} = \theta_n + p_n$$

$$p_{n+1} = p_n + K \sin \theta_{n+1}$$

(relax mode)

if θ random (chaos!) $\Rightarrow$ $K > K_{crit}$ for stoch. assumed

$$p_{n+1} - p_n = \Delta p_n = K \sin \theta_{n+1}$$

$\langle\ \rangle$ = avg over steps (time)

$$\langle \Delta p_n^2 \rangle = K^2 \langle \sin^2 \theta_{n+1} \rangle \cong \frac{K^2}{2}$$

i.e. $\langle \Delta p_n^2 \rangle$ diffusive ~ random walk

$$\langle p^2/2 \rangle \simeq Dn. \qquad (7.44)$$

The quasilinear result (7.42) is valid for very large K. For moderately large, but not very large, values of K, neglected correlation effects can significantly alter the diffusion coefficient from the quasilinear value. These effects have been analytically calculated by Rechester and White (1980) (see also Rechester *et al.* (1981), Karney *et al.* (1981) and Carey *et al.* (1981)). Figure 7.17 shows a plot of the diffusion coefficient D normalized to D_{QL} as a function of K from the paper of Rechester and White. The solid curve is their theory, and the dots are obtained by numerically calculating the spreading of a cloud of points and obtaining D from Eq. (7.44). Note the decaying oscillations about the quasilinear value as K increases.[7]

7.3.4 Other examples

So far in this section we have dealt exclusively with the standard map. We now discuss some other examples, also reducible to two-dimensional maps, where similar phenomena are observed.

We first consider a time-independent two-degree-of-freedom system investigated by Schmidt and Chen (1994). This system, depicted in Figure 7.18, consists of two masses, a large mass M connected to a linearly behaving spring of spring constant k_s and a small mass m which elastically

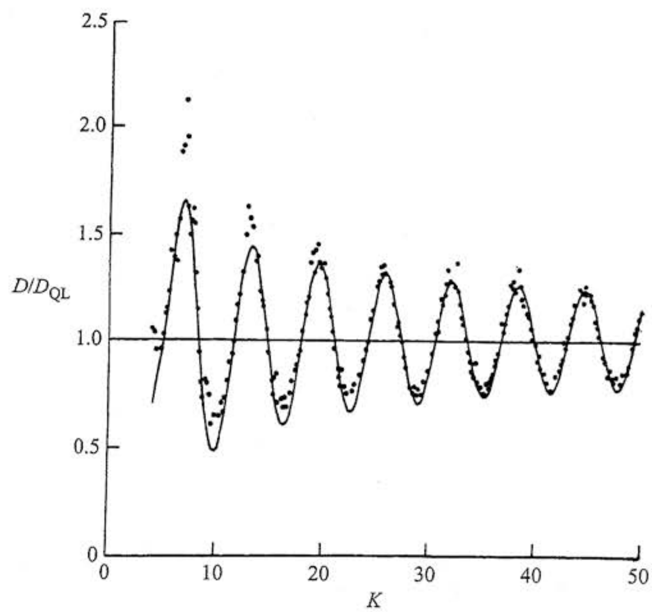


Figure 7.17 D/D_{QL} versus K for the standard map (Rechester and White, 1980).

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \epsilon \sin x \, f(t)$$

$$\frac{\sin[\pi t(2N+1)]}{\sin \pi t}$$

Now, as θ random, φ diffuses so:

$$\langle (\Delta p_n)^2 \rangle = 2 D n$$

diffusion coeff in p (momentum)

$\Rightarrow$

$$\boxed{D = \frac{1}{4} K^2}$$

~> Quasi-linear Diffusion coefficient

~> QL works rather well -> see OTT pg. 281

For systematics: Fokker-Planck Theory

$f \to$ dist.

schematically

$$f(t+\tau, p)_{NL} = \int d\Delta p \; f(t, p-\Delta p)\, T(\Delta p, \tau)$$

~> Chapman-Kolmogorov Eqn.

~> T is transition probability, for step Δp in τ

~> expansion -> Fokker-Planck Theory / Eqn. and can calculate divergence. etc.