Phys 210B

→ a) Renormalization, cont'd

b) Long Time Tails and Mode-Mode Coupling Theory

a) Renormalization

- What is it? – Reduction or thinning of d-o-f. (c.f. Chain)

- An approach → Mori – Zwanzig Theory (in detail: Linear Chain)

$$(\partial_t + L)\rho = 0$$

$$L = \underbrace{L_a}_{\substack{\text{slow} \\ \downarrow \\ \text{explicit}}} + \underbrace{L_b}_{\substack{\text{fast} \\ \downarrow \\ \rho_{eq}(b)}} + \underbrace{L_i}_{\text{Couplings}}$$

Kernel defines renormalized Liouville

$$(\partial_t + L)\rho = 0 \rightarrow \left[\partial_t + L_a + \int ds\, \underline{\underline{T}}(t-s)\right]\rho = \text{Noise}$$

Quotes re: Renormalization

"In general, ordering the multitudes is just like ordering the few, in that it requires a division into units."

- Sun Zi
"Art of War", Chapt. 5
(translated by M. Nylan)

"The shell game that we play ..... is called renormalization. But no matter how clever the word, it is what I would call a dippy process!"

- R.P. Feynman
"QED: The Strange Theory of Light and Matter"

"I must say that I am very dissatisfied with the situation, because this so called "good theory" does involve neglecting infinities which appear in its equations neglecting them in an arbitrary way."

- P.A.M. Dirac
"Directions in Physics"

"I disagreed with Dirac and argued the point with him ... Taking account of the difference between the bare charge and the mass of an electron and their measured values is not merely a trick that is invented to get rid of infinities, it is something that we would have to do even if everything was finite. There is nothing arbitrary or ad-hoc about the procedure; it is simply a matter of correctly identifying what we are actually measuring in [the] laboratory ..."

— S. Weinberg
"Dreams of a Final Theory"

"In the renormalization group method, you take a structure you don't understand and convert it to another structure you don't understand. You keep doing it till you finally understand."

— Michael Berry

N.B. : Generically: Chain

$\rightarrow \underline{\underline{I}} \sim L_i \tau_c L_i \sim 1/\tau_{relax} \rightarrow \Delta\omega$

coupling opr — correlation time of fast — coupling opr.

$\sim \partial_p D \partial_p \rightarrow$ phase space diffusion

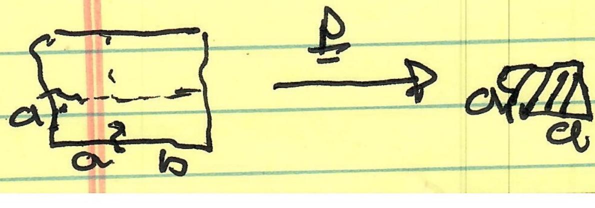
$D \leftrightarrow \int_0^\infty \langle \tilde{L}_i(0) \tilde{L}_i(\tau) \rangle d\tau$

→ Key element: coarse graining of fast d-o-fs

Chain ⟺ assume fast d-o-fs 'equilibrate' → known $P_{e\underline{q}}(b)$

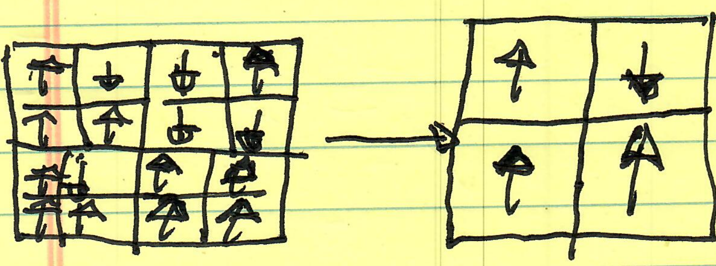
→ How is this like/different from other incarnations of renormalization?

ⓐ ~ Chain (2x2)



projects $(a+b)^2$ system onto $(a \times a)$

vs. Block spin — ($T \to T_c$, l_c diverges)



etc.

but no invariance argument...

~

(b) Chain vs. Self-Energy vs Viscosity

~ Chain

$$(\partial_t + L)\, P = 0$$

$$\rightarrow (\partial_t + L_i)\, P + \int \underset{\substack{\downarrow \\ \text{interaction with fast} \\ \text{d-o-fs}}}{\underline{\underline{T}}}\, P = 0$$

~ QED

$$\frac{1}{\not{p} - m_0} \rightarrow \frac{1}{\not{p} - \underbrace{m_0 + \Sigma}_{\text{renom mass.}}}$$

→ self energy

interaction with vacuum pol. cloud

~ Viscosity (today)

$$-i\omega + \nu_0 k^2 \Rightarrow -i\omega + (\nu_0 + \nu_T) k^2$$

interaction with turbulent spectrum

$$\nu_T = \sum_k |\tilde{v}_k|^2 \frac{\nu k^2}{\omega^2 + (\nu k)^2}$$

⇒ Fundamentally, all involve:

- relevant, irrelevant split
- some aspect of coarse graining and
~ equilibration of irrelevants.

→ basis for model reduction.

Ex.

→ Response fctn for $\boxed{\text{Noisy 'Burgerlence'}}$

$$\partial_t v + v\,\partial_x v - \nu\,\partial_x^2 v = \tilde{f}$$

Burgers (also KPZ)

↓ noise

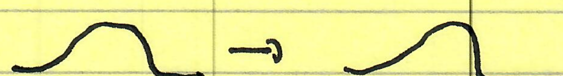
Seek $\delta v_k / \delta f$ → response

Burgerlence ⟺ Burgulence (Jeffman)

- 1D $p = 0$ Navier-Stokes

- shocks - $\dfrac{dv'}{dt} = -v'^2 + \ldots$

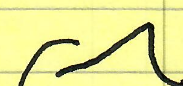
not captured in closure



- useful test.

⟺ Asymmetric Pdf v'

i.e.



→ shocks

ramps

→ not captured in low order closure.

- Point

$$v \partial_x v / \nu \partial_x^2 v \sim Re$$

so response dominated by nonlinearity

- Idea: NL coupling → (turbulent) mixing.

so seek $v \partial_x v \to -\nu_T \partial_x^2 v$

turbulent viscosity

key physics: space-time scales

i.e. $Re < 1$ (weak noise)

$$\partial_t v_k + \nu k^2 v_k + i k \sum_{k'} v_{k'} v_{k-k'} = f_k(t)$$

$$(-i\omega + \nu k^2) v_{k,\omega} = \tilde{f}_{k,\omega}$$

$$R_{k,\omega} = \delta v_{k,\omega} / \delta f_{k,\omega} = (-i\omega + \nu k^2)^{-1}$$

decay by viscosity

Now, for $Re \gg 1$ $\Rightarrow$ need faster
(strong noise) mixing rate.

$\rightarrow$ need extract effective time scale from nonlinearity

$\rightarrow$ physics is nonlinear scrambling time (analogous $\Delta\omega$)

so seek:

$$\partial_t V_k + \nu k^2 V_k + C_k V_k = f_k(t)$$

$\Downarrow$

seek response of test mode interacting with rest of turbulent spectrum

phase coherent with $f_k \Rightarrow (a e^{i\phi})_k$

$\Downarrow$

$C_k \sim |\tilde{V}|^2$ (no phase content)

To calculate C_k;

$$(-i\omega + \nu k^2) V_{\substack{k \\ \omega}} + ik \sum_{\substack{k+k' \\ \omega+\omega'}} V_{\substack{-k' \\ -\omega'}} V_{\substack{k+k' \\ \omega+\omega'}} = f_{\substack{k \\ \omega}}$$

$V_{k+k', \omega+\omega'} \to V^{(2)}_{k+k', \omega+\omega'}$ $\Rightarrow$ V driven by direct beat interaction of $V_k, V_{k'}$

so

$$(-i\omega + \nu k^2) V_{k,\omega} + ik \text{ [redacted] } \sum_{k',\omega'} V_{-k',-\omega'} V^{(2)}_{k+k',\omega+\omega'} = f_{k,\omega}$$

$$ik \text{ [redacted] } \sum_{k',\omega'} V_{-k',-\omega'} V_{k+k',\omega+\omega'} = C_{k,\omega} V_{k,\omega}$$

so/ when calculated

defines renormalized viscosity

$$\text{[redacted]} \quad \frac{\delta V_{k,\omega}}{\delta f_{k,\omega}} = 1 \Big/ -i\omega + \nu k^2 + C_{k,\omega}$$

limits response $\Rightarrow$ reflects scrambling

Now, to calculate:

$$\left[-i(\omega+\omega') + \nu(k+k')^2 + C_{k+k',\omega+\omega'}\right] V^{(2)}_{k+k',\omega+\omega'}$$

$$= -\frac{i}{2}(k+k')\left(V_{k',\omega'} V_{k,\omega} + V_{k,\omega} V_{k',\omega'}\right)$$

$$= -i(k+k') V_{k'} V_{k}$$

N.B. Decomposition:

$$NLT = C_{\substack{k+k' \\ \omega+\omega'}} V^{(2)}_{\substack{k+k' \\ \omega+\omega'}} + i \underline{(k+k')} (V_{k'} V_k) \cancel{\chi}$$

→ all interactions other than beat of those selected are absorbed into C.

✱ → test field hypothesis: removal of 2 modes won't change C.

Now, define:

$$L^{-1}_{\substack{k+k' \\ \omega+\omega'}} = -i(\omega+\omega') + \nu(k+k')^2 + \overset{\substack{\text{NL interactions} \\ \downarrow}}{C_{\substack{k+k' \\ \omega+\omega'}}}$$

$L \equiv$ renormalized / dressed propagator

so $$V^{(2)}_{\substack{k+k' \\ \omega+\omega'}} = L_{\substack{k+k' \\ \omega+\omega'}} \left(-i(k+k')\right) V_{\substack{k' \\ \omega'}} V_{\substack{k \\ \omega}}$$

so, self-consistently:

$$C_{\underline{k},\omega} V_{\underline{k},\omega} = (i)k \sum_{\underline{k}',\omega'} V_{-\underline{k}',-\omega'} L_{\underline{k}+\underline{k}',\omega+\omega'} (-i)(k+k') V_{\underline{k}',\omega'} V_{\underline{k},\omega}$$

$$= k^2 \sum_{\underline{k}',\omega'} |V_{\underline{k}',\omega'}|^2 L_{\underline{k}+\underline{k}',\omega+\omega'} \left(1 + \frac{k'}{k}\right) V_{\underline{k},\omega}$$

so

$$\delta V_{\underline{k},\omega} / \delta f_{\underline{k},\omega} = 1 / \left[i\omega + \nu k^2 + C_{\underline{k},\omega} \right]$$

sym.

$$C_{\underline{k},\omega} = \nu_{\underline{k},\omega} k^2 = k^2 \sum_{\underline{k}',\omega'} |V_{\underline{k}',\omega'}|^2 L_{\underline{k}+\underline{k}',\omega+\omega'} \left(1 + \cancel{\frac{k'}{k}}\right)$$

"turbulent viscosity"

(n.b. k, ω dependence)

note recursive defn.

$\Rightarrow$ defines renormalized propagator

→ About $\nu_{k,\omega}$

- at long wavelength } $k < k'$
  low frequency } $\omega < \omega'$

⇒ Markovian limit
(Fokker–Planck)

$$\nu^T = \sum_{k',\omega'} |V_{k',\omega'}|^2 L_{k',\omega'} = \sum_{k',\omega'} |V_{k',\omega'}|^2 \frac{k'^2 \nu_{k',\omega'}}{\omega'^2 + (k'^2 \nu_{k',\omega'})^2}$$

effective transport coefficient → sets NL/turbulent time scale

$$\nu^T \sim \langle \tilde{v}^2 \rangle \tau_c \sim \tilde{v}_{rms}\, l_c$$

$$l_c \sim \tilde{v}\, \tau_c$$

- $k^2 \nu_T$ is emergent time scale/rate.

NL scrambling on ν sets itself time scale

⇓

Contrast $D = \dfrac{\langle \tilde{f}^2 \rangle \tau_{ac}}{B^2}$ in B.M.

$\langle \tilde{f}^2 \rangle \tau_{ac} = BT$; by F-D.T ;

or QLT.

- irreversibility from turbulent scrambling.

- To estimate

$$\nu \sim |\tilde{\omega}|^2 / \gamma k^2$$

$$\nu \sim \frac{V_{rms}}{k_{rms}} \sim \underline{\ell \tilde{V}}$$

- $\nu^T_{k,\omega}$ vs ν^T

$k, \omega \to 0$ if $k < k'$, $\omega < \omega'$

no memory ! — a la' F.-P. Eqn.

k, ω $\quad$ $\approx k'$ $\quad$ $\approx \omega'$ $\quad$ $\sim$

→ interaction behaves as memory-less kick, as in walk for $\omega < \omega'$, $\Delta\omega'$

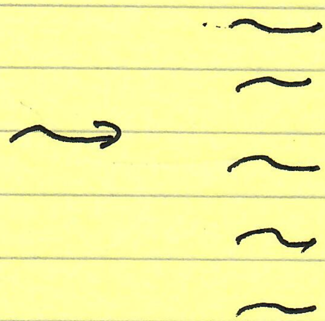
⇒ Markovian ($\overset{\text{slosh}}{\to}$ kick) $\quad$ $k < k'$, $\Delta k'$

If not, feel time history of sloshing

⇒ Non-Markovian

- Approximate Resp. is exact for what system?

→ Oscillators with Random Couplings
Ensemble



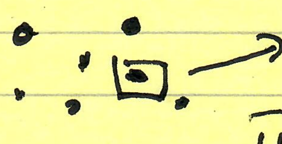
[n.b. not random phase!]

see Kraichnan '61.

→ **Long Time Tails and Mode-Mode Coupling**

– Long Time Tails (Alder & Wainwright 1968 → ; et seq.)

→ molecular dynamics (i.e. particles) simulation of fluid. Few ν_{coll}^{-1} ....

→  'tag' a particle, pulse it, measure its diffusion, correlation

"self-diffusion"

→ expect, for velocity correlation

$$\langle \underline{V}(0)\,\underline{V}(t)\rangle \cong |\underline{V}(0)|^2 e^{-t/\tau_{ac}}$$

so

$$D = \int_0^\infty \langle V(0)V(t)\rangle\, dt \to |V(0)|^2 \tau_{ac}$$

but

Surprise!

→ Actually → long time tail
(power law)

3D (hard spheres)

$$\langle \tilde{v}(0)\,\tilde{v}(t) \rangle \simeq t^{-3/2}$$

2D (disks)

$$\langle \tilde{v}(0)\,\tilde{v}(t) \rangle \simeq t^{-1}$$

$$\underbrace{D_{self} \to \infty}$$

dimension

$$\Downarrow$$

$$\sim t^{-d/2}$$

$$\Downarrow$$

long time tail in correlation

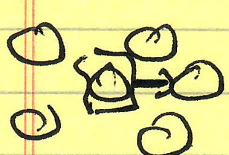
Why?

How treat, theoretically?

→ Heuristics (see Pomeau + Resibois)

What happens

i)

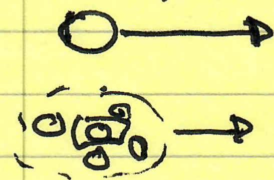


deliver impulse to tagged particle!

ii.) shortly, tagged and impulsed

particle shares its momentum with neighbors. 'Quasi particle' moves . .

particle



Quasi-particle

$$\tilde{V}(t) = \frac{\tilde{V}(0)}{n\, V_n}$$

Velocity drops as size of neighbors grows/expands

n → density

V_n → volume of neighbors

iii.) what sets V_n ?

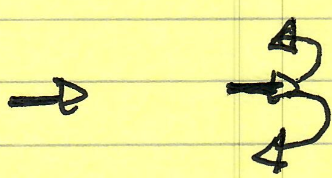
- response of system / fluid

- candidates (modes)

→ acoustic compression (sound)

(short time) $\omega \approx k c_s$

shear viscosity

→  vortex pair (ring in 3D) → expands

⇒ shear, $\omega \approx -i\nu k^2$

$$V_n \approx (R_s)^d$$

where $R_s \sim (\nu t)^{1/2}$

$\rightarrow$ so $\boxed{\bar{V}_n \sim R_g^{\,d} \sim (\nu t)^{d/2}}$

quasi-particle volume grows in time

(vi) $\tilde{v}(t) \cong \tilde{v}(0) / (\nu t)^{d/2}$

$$\langle \tilde{v}(0)\tilde{v}(t)\rangle \simeq (\tilde{v}(0))^2 / (\nu t)^{d/2}$$

long time tail

Technically:

$$\langle \tilde{v}(0)\tilde{v}(t)\rangle \cong \tilde{v}(0)^2 / [(\nu + D)t]^{d/2}$$

self-wandering
cf.: aggregation

modes: $\omega = -i\nu q^2$

Key here:

- long time collective dynamics of system ('fluid')
- particle $\leftrightarrow$ collective coupling, has flavor of renormalization

→ Now, how approach systematically ?

⇒ Mode - Mode Coupling Theory
c.f. Kadanoff + Swift '68 *
Zwanzig (book)
Pomeau + Resibois (review)
DeGennes (after Kirkwood) (book)

– built on Hilbert space picture

⇒ complete orthonormal set $\varphi_j(X)$ of position X of system in phase space

$$\langle \varphi_j | \varphi_k \rangle = \int dX \, \varphi_j(X) \, \varphi_k^*(X) \, f_{eq}(X) = \delta_{j,k}$$

can recast Liouville Eqn as matrix equation

so

$$D(t) = \frac{1}{d} \langle \underline{V}(t) \cdot \underline{V} \rangle$$

exploits state vector approach

$$= \frac{1}{d} \, \mathrm{tr} \sum_{j,k} \langle \underline{V} | \varphi_j \rangle \langle e^{tL} \varphi_j | \varphi_k \rangle \langle \varphi_k | \underline{V} \rangle$$

- What is $e^{tL}|\mathcal{C}_j\rangle$? → evolution of state vector.

→ tractable if $\mathcal{C}_j$ constructed from slow variables

Slow ⇒ conserved concentration

$$\partial_t C = -\underline{\nabla}\cdot\underline{J} = D\nabla^2 C$$

$\sim q^2$

$\omega \to 0$

$$\partial_t \underline{v} = -\underline{\nabla}\cdot\underline{\underline{\Pi}}$$

$\omega \to 0$

decay very slowly at large scale

Fast $\partial_t \alpha = -\sigma\alpha$ not conserved

const/decay

→ Examples of Slow Variables:

$$C(\underline{r},t) = \delta(R_0(t) - r)$$ tagged

$$C(\underline{r},t) = \sum_q C_q(t)\, e^{i\underline{q}\cdot\underline{r}}$$

$$C_q(t) = e^{tL} C_q = e^{-Dq^2 t} C_q$$

diffusive decay

Likewise: $\underline{V}_{\ell}$ → fluid velocity modes
(long time → incompressible)

→ Now recall:

$$D(t) = \frac{1}{d} \, tr \sum_{j,k} \langle \underline{V} | \varphi_j \rangle \langle e^{tL} \varphi_j | \varphi_k \rangle \langle \varphi_k | \underline{V} \rangle$$

to calculate D, need find $|\varphi_j\rangle$ s/t

$\langle V | \varphi_j \rangle \neq 0$; i.e. seek project the particle velocity onto a system mode

→ For long time behavior, natural system mode is a slow mode

→ But $\langle \underline{V} | \begin{smallmatrix} \delta\rho \\ \underline{V}_2 \end{smallmatrix} \rangle = 0$ → projection won't work

indep. position (tag is translationally invariant)

depends on position

→ $\mathcal{L}$ consider product of two slow variables $\{V_q C_{-q}\}$ s/t product best is translationally invariant

Product best of slow modes ⇔ "mode-mode coupling"

so with normalization formalities, have mode coupled state vectors

$$\boxed{\varphi_j \rightarrow \underset{\text{label}}{\varphi_q} = \left(m/NT\right)^{1/2} \underline{V_q C_{-q}}}$$

→ state vector.

$\langle \varphi_q | \varphi_{q'} \rangle = \delta_{q,q'}$ etc.

$$\langle V_q C_q | V_q C_{-q} \rangle = \langle V_q V_{-q} \rangle \langle C_{-q} C_q \rangle$$

$$= \frac{NT}{m} I$$

→ Then

$\langle \underline{V}(0)\underline{V}(t)\rangle \equiv D(t)$ correlation in slow disc basis

$$D(t) = \underset{\text{other modes}}{\underline{\frac{D_{fast}(t)}{d}}} + \frac{1}{d}\,\mathrm{tr}\underset{\substack{\text{sum over}\\\text{slow modes}}}{\sum_{q}}\left\langle e^{tL}V_q C_{-q} \middle| V_q C_{-q}\right\rangle$$

[large scale key's]

$$e^{tL}\underline{V}_q C_{-q} = \left(e^{tL}\underline{V}_q\right)\left(e^{tL}C_{-q}\right)$$

$$\cong \underset{?}{\left(e^{tL}\underline{V}_q\right)}\, e^{-Dq^2 t}\, C_{-q}$$

$\underline{V}_q = \underset{\text{longitudinal}}{\underline{V}_{q\ell}} + \underset{\text{transverse}}{\underline{V}_{q\perp}}$ $\quad V_{q\ell} = \frac{\underline{q}\underline{q}}{q^2}\cdot\underline{V}_q$

$\underline{\nabla}\cdot\underline{V} = 0$ $\qquad V_{q\perp} = \left(\underline{\underline{I}} - \frac{\underline{q}\underline{q}}{q^2}\right)\cdot\underline{V}_q$

$\Rightarrow \underline{V} = \underline{V}_{q\perp}$ $\quad \nabla_\perp\cdot\underline{V}_{q\perp} = 0$

and [redacted] $\underline{V}_{q\perp}$ decays due shear viscosity, only.

so

$$\partial_t \underline{v}_{q_\perp} = -\nu q^2 \underline{v}_{q_\perp} + \text{noise}$$

$$\underline{v}_{q_\perp} = e^{tL} \underline{v}_{q_\perp} \equiv e^{-\nu q^2 t} \underline{v}_{q_\perp}$$

and so:

$\nabla \cdot \underline{v} = 0$

$$\langle e^{tL} v_{q_\perp} | v_{q_\perp} \rangle = \frac{NT}{m} \left(\underline{\underline{I}} - \frac{\underline{\underline{q\,q}}}{q^2} \right) e^{-\nu q^2 t}$$

so

→ Finally:

$$\boxed{D(t)} = D_{fast}(t) + \frac{1}{d} \operatorname{tr} \frac{T}{mN} \sum_{\underline{q}} e^{-(D+\nu) q^2 t} \left(\underline{\underline{I}} - \frac{\underline{q}\,\underline{q}}{q^2} \right)$$

$$= D_f + \frac{d-1}{d} \frac{T}{mN} \sum_q e^{-(D+\nu) q^2 t}$$

N.B.: In mode coupling time decay $D(t)$ set by the faster of the slow modes (i.e. viscous or diffusion).

and

$$\sum_{q} \rightarrow \left(\frac{L}{2\pi}\right)^d \int d^d q$$

$$\rho = mN/L^d$$

and integrating over q:

$$\boxed{D(t) = D_0(t) + \frac{d-1}{d}\,\frac{T}{\rho}\left[(1)/\left[4\pi(D+\nu)t\right]^{d/2}\right]}$$

long time tail

N.B.

- long time tail results from slow, diffusive decay of (slow) modes.

  → symptom of conservation

- large scales ⟺ slowest of the slow modes.

  Obviously result sensitive to b.c.'s macro-structure.

→ $d = 2$, $D_{self} \to \infty$ as $\int_0^\infty \langle v\, v \rangle \to \infty$

Consistent with Stokes Paradox →
hydro friction or Stokes drag
does not exist in 2D.

3D

$\underline{v} \sim 1/r^3$ dipole

2D

$\phi \sim \ln r$

$v \sim 1/r$

insufficient fall-off.

→ For stress (flux) → M.C. $\underline{V}_2 \underline{V}_2$ basis

$\Rightarrow \eta$

so stress correlation + thus $\eta \Rightarrow$

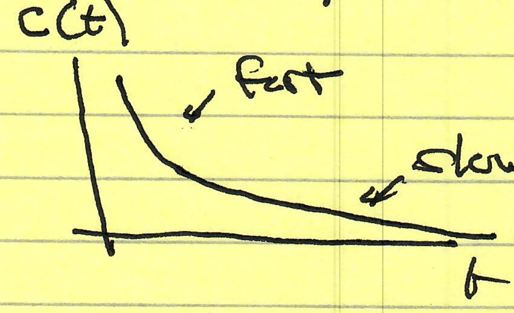
$\sim 1/t^{3/2}$ tail.

→ Cynic: Other couplings ?
- Hard to see.

⇒ nonlinear Langevin Eqns

→ time scale separation in correlation

i.e.



⇝

Is there another way ?

→ See Pomeau & Resibois ! (many..)

also Bedeaux + Mazur, et. seq.

→ Renormalized continuity equation for tagged density, given thermal velocity field.

↔ Method of fluctuating hydrodynamics

→ i.e. derive total diffusivity in thermal flow field.

⇒ seek non-Markovian D. (long time)

so, number density, tagged

$$\frac{\partial n_1}{\partial t} = -\underline{\nabla}\cdot\underline{J}$$

(*)

$$\underline{J} = -D_0 \underline{\nabla} n_1 + \underline{V}(x,t)\, n_1 + \underline{J}_R$$

$\underline{V}(x,t)$: ambient velocity field at T

$\underline{J}_R$: thermal fluctuations

(*) → the point → fluid convection adds to self-diffusion

→ take fluid as ambient, thermal

→ passive scalar (tagged) in thermal flow.

drop J_R, from continuity:

$$(i\omega + D_0 q^2)\, \tilde{n}_{1_{q,\omega}} = i\underline{q}\cdot\widetilde{\underline{A}\, n_1}_{q,\omega} + \tilde{n}_{1q}(t=0)$$

$i\underline{q}\cdot\widetilde{\underline{A}\, n_1}_{q,\omega}$: effects advection

$\tilde{n}_{1q}(t=0)$: initial part.

where

conventions ! → cumbersome

$$\underline{A}\, \psi_{q,\omega} = \frac{1}{(2\pi)^{4}} \int dq' \int d\omega'\, \underline{v}_{q-q'}\, \psi_{q',\omega'}$$

advection operator, arbitrary $\psi_{q,\omega}$

$$(i\omega + Dq^2)(\tilde{n}^{0}_{q,\omega} + \tilde{n}^{1}_{q,\omega})$$

$$= i\underline{q}\cdot\underline{A}\,(\tilde{n}^{0}_{q,\omega} + \tilde{n}^{1}_{q,\omega}) + \tilde{n}_{q}(t=0)$$

$$\tilde{n}^{0}_{q,\omega} = i\, G_0\, \tilde{n}_q(t=0)$$

$$G_0 = (i\omega + Dq^2)^{-1}$$

$\Rightarrow$

$$(G_0^{-1} - i\underline{q}\cdot\underline{\underline{A}})\,\tilde{n}^{1}_{q,\omega} = -\,n_{q,\omega}\, G_0^{-1}\,\tilde{n}^{0}_{q,\omega}$$

$$\boxed{n^{1}_{q,\omega} = (1 + iG_0\underline{A}\cdot\underline{q})^{-1}\, n^{0}_{1}}$$

$\Rightarrow$

$$\boxed{\underline{J}_{q,\omega} = (\underline{A} + i\underline{q}D_0)(1 + iG\underline{q}\cdot\underline{A})\, n_{q,\omega}}$$

and $\quad \underline{J} = -D_{tot}\, \underline{\nabla} n \quad \Rightarrow$

dressing for transport coefficient

$$\boxed{D = D_0 + \frac{1}{q^2}\langle \underline{q}\cdot\underline{A}\, G_0\, \underline{q}\cdot\underline{A}\rangle} + \text{h.o.t.}$$

$D_{\text{renorm}} \sim \dfrac{|\tilde{V}|^2}{(i\omega + D_0 q^2)}$

and

$$|\tilde{V}_{q,\omega}|^2 = \left(\underline{\underline{I}} - \frac{\underline{q}\,\underline{q}}{q^2}\right)\frac{T}{\rho}\left[\frac{1}{i\omega + \nu q^2}\right]$$

↳ corner.

$$\delta D = D - D_0$$

and expanding δD at low ω:

$$\delta D \sim \omega^{1/2} \sim t^{-3/2}$$

(n.b. integrate over q, leaving ω dependence)

TBC

→ Point is that long-time tail emerges from collective fluid effects entering effective self-diffusion.