So Far: – Noisey Dynamics
– Fokker–Planck & Applications

Physics 210 B

Central Limit Theorem and Beyond I

Contents

→ CLT ↔ Conventional wisdom on random processes, in depth

→ Beyond: – Gaussian, from CLT, as special case of Levy Stable Distribution

– Levy Distribution, an Introduction

– Levy Process – generalizing random walk.

Central Limit Theorem: (DeMoivre, Laplace, Gauss)

– Consider a sum of n-independent random variables, increments ..

$$\Delta x_1, \Delta x_2 \;\ldots\; \Delta x_n \qquad (n \gg 1)$$

Let sum $x_n = \sum_i^n \Delta x_i$

Steps "IID"
"Identically, independently distributed"

- Each Δx_i; $\langle \Delta x_i \rangle = 0$

$\langle \Delta x_j^2 \rangle \equiv \sigma_i^2$

i.e. Variance of step distribution converges $\langle \Delta x_j^2 \rangle < \infty$.
Only variance required.

- then $\sigma_n^2 \equiv \sum_i \sigma_i^2$

CLT $\Rightarrow$

$$\text{Pdf}(x_n) \cong \frac{1}{(2\pi\sigma_n)^{1/2}} \exp(-x_n^2/2\sigma_n^2)$$

$\boxed{n \gg 1}$

i.e. Pdf sum $\rightarrow$ Gaussian

- Key Points / Buried Bodies

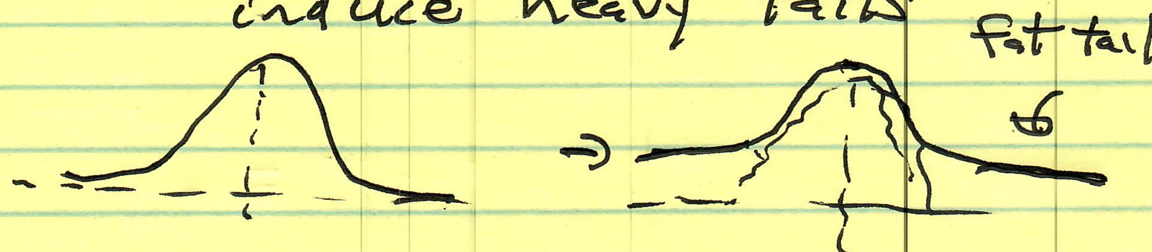
a1.) Δx_i alike, sum not dominated by few (if so, "intermittency")

a2.) finite variance of step increment $\langle \Delta x_j^2 \rangle < \infty$

b1.) What of higher moments ?

i.e. $\langle \Delta x_i^2 \rangle < \infty \not\Rightarrow \langle \Delta x_i^4 \rangle < \infty$

$\Rightarrow$ large kurtosis can induce heavy tails



b2) Quantify ?

Observation:

- CLT states (effectively) that (given conditions satisfied),

  $x_i \rightarrow$ statistically distributed, consistent with CLT

  then if x_i, y_i are series to be summed, which follow CLT conditions, then $(a x_i + b) + (a' y_i + b')$

$$= a'' z_i + b''$$

is also Gaussian distributed, i.e. follows CLT ("L-stability")

here: a, b ; a', b' all > 0, and not stochastic

In simple terms:

$\Rightarrow$ Adding two Gaussian distributed series yields a sum which is Gaussian distributed.

$\Rightarrow$ CLT $\Rightarrow$ "Gaussian, modulo conditions, is an attractor in function space"

More generally: A class of distributions exist which are

L-stable (L for Paul Levy)

i.e. have property that if two series distributed, sum is also distributed similarly.

The Message:

$$\boxed{\text{The beloved Gaussian of CLT is merely one particular case of an L-stable distribution, and the only one with finite variance.}}$$

⇒ Many elements in class of L-stable
⇒ 'attractors on function space'

⇒ $\boxed{\text{Family of allowed distributions is larger than you thought...}}$

To understand: First re-visit CLT!

→ Proving the C.L.T.

General ideas:

- Markov process → Chapman - Kolmogorov Eqn.

⇒

- Convolutions

- Convolution → Product of F.T.

→ "Generating" or "characteristic" Function

Point: Fourier Transform of step probability is more significant (and useful) than probability.

① So C-K Eqn: takes $x-y \to x$

$$P_N(x) = \int dy\, P_{N-1}(y)\, P_N(x \mid x-y)$$

don't expand ----

then,

if F.T., and noting F.T. (convolution) $= \prod_i$ F.T.$_i$ i.e. Fourier transform of convolution = product of functions convolved.

then, N identical step C-K:

$$P_N(k) = \tilde{P}_1(k)\,\tilde{P}_2(k) \cdots \hat{P}_N(k)$$
$$= \sum_{n=1}^{N} \hat{P}_n(k)$$

so

$$\boxed{P_N(x) = \int dk\, e^{ikx} \prod_{n=1}^{N} \hat{P}_n(k)}$$

applies for identical steps.

3 Can also define moments:

$$m_n = \int dx \; x^n \, P(x)$$

$$\langle x_1 \rangle = m_1$$
$$\langle x^2 \rangle = m_2$$
$$\vdots$$
$$\langle x^n \rangle = m_n$$

so

$$\hat{P}(k) = \sum_{n=0}^{\infty} (-i)^n \frac{k^n}{n!} m_n$$

$$= \int e^{-ikx} dx \; P(x)$$

$$= \int dx \left(1 - ikx + \frac{(ikx)^2}{2} + \dots\right) P(x)$$

$$= \int dx \sum_{n=0}^{\infty} (-i)^n \frac{k^n}{n!} x^n P(x) \quad \checkmark$$

$$\boxed{m_n = i^n \frac{\partial^n \hat{P}}{\partial k^n}} \rightarrow \text{from FT}$$

Useful identity: n^{th} moment $\leftrightarrow$
n^{th} derivative of generating fctn.

so $\hat{P}(k) = 1 - i m_1 k - \frac{1}{2} k^2 m_2 + \cdots$

easily generalized to higher dimensions.

③ Cumulants

– i.e. nonlinear combinations of moments

$$\psi(k) \equiv \ln \hat{P}(k)$$

$$P(x) = \int e^{ikx} \frac{dk}{(2\pi)} \hat{P}(k)$$

$$= \int e^{i[kx + \psi(k)]} \frac{dk}{2\pi}$$

expand:

$$\boxed{\psi(k) \equiv -i c_1 k - \frac{1}{2} c_2 k^2 + \cdots}$$

(series of cumulants)

$$\left.\begin{aligned} c_1 &= m_1 \\ c_2 &= m_2 - m_1^2 = \sigma^2 \end{aligned}\right\} \text{cumulants}$$

etc.

if exist, one has from moments:

$$i^n C_n^{(j_1, j_2 \ldots)} = \frac{\partial^n \psi}{\partial k_{j_1} \partial k_{j_2} \cdots \partial k_{j_n}}$$

Now, assuming identical independently distributed (IID) steps, cumulants additive:

IID is an important restriction!

$$P_N(x) = \int e^{ikx} \frac{dk}{2\pi} P_N(k)$$

$$= \int e^{ikx} \frac{dk}{2\pi} (p(k))^N$$

$$= \int \frac{dk}{2\pi} e^{ikx} \left(e^{\psi(k)}\right)^N$$

$$= \int \frac{dk}{2\pi} e^{ikx} e^{N\psi(x)}$$

so $\boxed{\psi_N(x) = N\psi(x)}$

So, for C.L.T.: Consider $N \to \infty$ (asymptotic !)

$$P_N(x) = \int_{-\infty}^{+\infty} e^{ikx} \prod_{n=1}^{N} \hat{P}_n(k) \frac{dk}{(2\pi)}$$

$$= \int_{-\infty}^{+\infty} e^{ikx} \frac{dk}{(2\pi)} (\hat{P}(k))^N$$

$$= \int_{-\infty}^{+\infty} e^{ikx} \frac{dk}{(2\pi)} e^{N\psi(k)}$$

i.e. additivity: $\psi_N = N\psi(k)$

$$\psi(k) = -ick - \frac{k^2}{2}c_2$$

$$P_N(x) = \int \frac{dk}{2\pi} e^{ikx} e^{N\psi(k)}$$

For $N \to \infty$, only the region 'near' $k = 0$ contributes (Laplace's Method)

$\Rightarrow$ only low order cumulants contribute/determine $P_N(x)$

[N.B. Fundamental reasoning for truncating Kramers - Moyal]

So

$$P_N(x) = \int_{|k|<\epsilon} e^{ikx} e^{N\psi(x)} \frac{dk}{2\pi}$$

$$= \int_{|k|<\epsilon} \frac{dk}{2\pi} e^{ikx} \exp\left[-iNC_1 k - \frac{Nk^2}{2} C_2\right]$$

and can:

- $C_1 \to 0$ (simplicity)
- $N \to \infty$, limits $\to \infty$

$$P_N(x) = \int \frac{dk}{2\pi} e^{ikx} e^{-NC_2 k^2/2}$$

$$\boxed{P_N(x) = \frac{1}{\sqrt{2\pi}\,(NC_2)^{1/2}} \exp\left[-x^2/NC_2\right]}$$

$$NC_2 = N\langle\sigma^2\rangle \to \frac{t}{\tau}\langle\sigma^2\rangle \to Dt$$

$$P(x,t) \cong \frac{1}{(Dt)^{1/2}} \exp\left[-x^2/Dt\right]$$

etc.

$\rightarrow$ C.L.T

A few points:

- no questions asked about higher moments, for $\underbrace{N \rightarrow \infty}$.

- These need not be well behaved, and can induce Fat tails

i.e. $P(x) = 1/_{1+x^2}$

has $\langle x^2 \rangle \rightarrow \infty$, so C.L.T not apply

but $P(x) = 2/\sqrt{\pi}\,(1+x^4)$

$\langle x^2 \rangle < \infty$

meets C.L.T. criteria, but kurtosis diverges

$\Rightarrow$ Fat Tail

- Can show,
  - Gaussian eroded (fat tail, + probability conserved $\Rightarrow$ erode central Gaussian)
  - large x (how large is "large"?)

$$P_N(x) \sim NA/x^4$$

(power Law, not Gaussian)

N.B. - Refs: Chandrasekhar Review; Kubo et. al.; Hughes, B.D. or any book..

- M.I.T. OCW 18.366 ("Random Walks and Diffusion")

- Physics 235, Spring '19 (Note write-ups, Supplementary)

NB Issue of Fat Tail behavior within CLT is good paper topic.

→ Levy Distributions

- observe: A property of diffusion → Self-Similarity

$$D = \frac{\langle \delta x^2 \rangle}{\delta t}$$

$\delta x \to \alpha\, \delta x'$,
$\delta t \to \beta\, \delta t'$

$$D' = \frac{\alpha^{-2} \langle \delta x^2 \rangle}{\beta^{-1}\, \delta t} = D$$

if: $\beta = \alpha^2$

- What is the class of self-similar distributions which are L-stable and normalizable?

Now, x_i → random variables

$$x_N = \sum_{i=1}^{N} x_n$$

so $\left\{ \begin{matrix} \text{generating} \\ \text{characteristic} \end{matrix} \right\}$ function:

$$\hat{P}_n(k) = [p(k)]^n$$

Rescale:
$$\begin{cases} Z_N = x_N/a_N \\ \mathrm{pdf}(Z_N) = F_N(x)/a_N \\ x = x_N/a_N \end{cases}$$

$$\boxed{P(c_1 z)\,P(c_2 z) = P(c z)}$$

$$\hat{F}_N(\underset{\text{scale}}{a_N} k) = \hat{P}_N(k)$$

Now seek 'attractors' in function space, so:

$$F_n(k) \underset{n\to\infty}{\longrightarrow} \hat{F}(k)$$

Let $\lim_{m\to\infty} \dfrac{a_{nm}}{a_m} = c_n$. (coeffs of stability)

then have condition for function $\hat{F}(k)$ as limiting case

$$\boxed{\hat{F}(k\,\underset{\text{scale}}{c_n}) = [\hat{F}(k)]^n}$$

self-similarity

So need solve

$$\boxed{\hat{F}(k\,\mu(\lambda)) = \left(\hat{F}(k)\right)^{\lambda}}$$

fixed pt./recursion condition

$\psi = \ln \hat{F}(k)$

$$\boxed{\psi(k\,\mu(\lambda)) = \lambda\,\psi(k)}$$

with $\mu(\lambda=1) = 1$.

So, $\frac{d}{d\lambda}$

$$k\frac{d\mu}{d\lambda}\psi'(k\,\mu(\lambda)) = \psi(k)$$

$$k\,\mu'\,\psi' = \psi$$

$$\boxed{\frac{d\psi}{dk} = \psi / \mu'\, k}$$

i.e. $\rightarrow$

$$k\, d\psi/dk = 1/\mu' \quad (\text{self-sim.})$$

Power Law for ψ!

$$\psi(k) = \begin{cases} \nu_1 |k|^{\alpha}, & k>0 \\ \nu_2 |k|^{\alpha}, & k<0 \end{cases}$$

Can show in more detail: (Hughes)

$$\hat{F}(k) = \exp\left[-\varepsilon |k|^{\alpha}\left(1 - i\beta \tan\left(\frac{\alpha\pi}{2}\right) \operatorname{sgn}(k)\right)\right]$$

skewness

take $\beta = 0 \rightarrow$ Levy Distribution

$$\boxed{L_{\alpha}(a, k) = \hat{F}(k) = \exp(-a|k|^{\alpha})}$$

$\alpha = 2 \rightarrow \hat{F}(k) = \exp(-ak^2) \rightarrow L_2$

Gaussian.

$\alpha = 2$ is self-similar attractor)

L-stable with normalizable

2nd moment (C.L.T. case) (only!)

Can show $\alpha = 2$ is max. α.

$\alpha = 1 \rightarrow \hat{F}(k) = e^{-a|k|} \rightarrow$ Cauchy Lorentzian.

$$P(x) = ?/(a^2 + x^2)$$