Physics 210b

Lecture II - 3 - b - Central Limit Theorem and Beyond 2

Recall:

→ Central Limit Theorem, and proof.

$\langle \Delta x^2 \rangle < \infty$, IID are key elements.

→ CLT distribution (Gaussian)

is a particular case of an <u>L- stable distribution</u>

→ CLT / Diffusion is <u>self-similar</u>.

So, ask if other self-similar Levy-stable distributions?

i.e. $P(c_1 z) P(c_2 z) = P(c z)$
condition.

Now, $\hat{P}_n(k) = [p(k)]^n$ — characteristic fctn

$$\begin{cases} Z_N = \text{[redacted]} \; x_N / a_N & \to \text{rescaling} \\ \text{Pdf}(Z_N) = F_N(x) / a_N & \end{cases}$$

$x = x_N / a_N.$

and:

— $\hat{F}_N(a_N k) = \hat{P}_N(k)$

(some fixed fctn. (attractor))

Now, want $F_n(k) \underset{n\to\infty}{\longrightarrow} F(k)$ [redacted].

let $\lim_{m\to\infty} \frac{a_{nm}}{a_m} = c_n$ (not sum)

~~Condition for function as~~ limiting case.

$$\boxed{\hat{F}(k\, \underset{\text{scale.}}{c_n}) = [\hat{F}(k)]^n} \quad \text{self-sim.}$$

so, need solve:

$$\boxed{\hat{F}(k\, u(\lambda)) = [\hat{F}(k)]^\lambda} \quad \text{fixed fctn. condition}$$

$$\Psi = \ln \hat{F}(k)$$

$$\Psi(k\,u(\lambda)) = \lambda\,\Psi(k)$$

$$k\left(\frac{du}{d\lambda}\right)\Psi'(k\,u(\lambda)) = \Psi(k)$$

$$u(\lambda = 1) = 1 \qquad \text{(must.)}$$

$$k\,u'\,\Psi' = \Psi(k)$$

$$\boxed{\frac{d\Psi}{dk} = \Psi \Big/ u'(1)\,k}$$

$$\frac{k}{\Psi}\frac{d\Psi}{dk} = 1/u$$

$$k\frac{d\ln\Psi}{dk} = 1/u$$

self-sim.

$$\Psi(k) = \begin{cases} v_1 |k|^{\alpha} & k > 0 \\ v_2 |k|^{\alpha} & k < 0 \end{cases}$$

$$\hat{F}(k) = \begin{cases} \exp(v_1 |k|^{\alpha}) & k > 0 \\ \exp(v_2 |k|^{\alpha}) & v < 0 \end{cases}$$

or, in more detail,

$$\boxed{\hat{F}(k) = \exp\left[-a|k|^{\alpha}\left(1 - i\beta \tan\frac{\alpha\pi}{2}\,\mathrm{sgn}(k)\right)\right]}$$

skewness

· Now, take $\beta = 0$ → Levy Distribution

$$\boxed{L_\alpha(a, k) = \hat{F}(k) = \exp(-a|k|^{\alpha})}$$

$\alpha = 2$ $\qquad \hat{F}(k) = \exp\left[-a k^2\right]$

→ Gaussian

$\alpha = 2$ is case of CLT

$$P(x) = \exp\left[-x^2/a\right]$$

$\alpha = 1$ $\qquad \hat{F}(k) = e^{-a|k|}$

→ Cauchy, Lorentzian

$$P(x) = 1/(a^2 + x^2).$$

Note $\alpha = 2$ is max, and only Levy stable distribution with 2nd moment finite.

→ Large x expansion: $0 < \alpha < 2$

$$L_\alpha(a, x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} e^{-a|k|^\alpha}\, dk$$

$$= \frac{1}{\pi}\int_0^\infty \cos(kx)\, e^{-a|k|^\alpha}\, dk$$

ibp

$$= \frac{a\alpha}{\pi}\int_0^\infty dk\, \frac{\sin k|x|}{|x|} \exp\left[-ak^\alpha\right] k^{\alpha-1}\, dk$$

$$= \frac{\alpha a}{\pi |x|^{1+\alpha}}\int_0^\infty \xi^{\alpha-1} \sin\xi \exp\left(\frac{-a\xi^\alpha}{|x|^\alpha}\right) d\xi$$

Now, $\xi = k|x|$

$d\xi = dk|x|$

$$L_\alpha(a, x) \underset{x\to\infty}{=} \frac{\alpha a}{\pi |x|^{1+\alpha}}\int_0^\infty \xi^{\alpha-1} \sin\xi\, d\xi$$

so

$$L_\alpha(a, x) \sim (\quad) / |x|^{1+\alpha}$$

$$(\;) \sim \frac{a\alpha}{\pi}\,\Gamma(\alpha)\,\sin\pi\alpha$$

Point: Levy Distributions $L_\alpha(a, x)$ have power law tails (fixed pt. fctn $\Leftrightarrow$ self-similarity)

$$L_\alpha(a, x) \simeq 1 \Big/ |x|^{1+\alpha}$$

$$\underline{0 < \alpha < 2}$$

obviously need $\alpha = 2$ for convergent 2nd moment.

- [redacted] width, for N steps, $\sim N^{1/\alpha}$
- $\alpha > 2 \rightarrow$ super-diffusive.

→ Now, obvious analogy question arises:

CLT : Diffusion :: Levy Distribution : ?

Gaussian $\qquad\qquad\qquad\qquad L_\alpha(a, x)$

→? Levy Process !

see Zaslovsky, Chapt. 15

- analogue / generalization of diffn.
  → Levy Flights

- time dependent

- Levy dist. at infinitesimal time

Now, C-K Eqn:

(t understood)

$$P(x_0, t_0 \mid x_N, t_N) = \int dx_1 \int dx_2 \cdots \int dx_{N-1}$$

$$* \left[P(x_0, t_0; x_1, t_1)\, P(x_1, t_1; x_2, t_2) \cdots P(x_{N-1}, t_{N-1}; x_N, t_N) \right]$$

now $\quad t_{j+1} - t_j = \Delta t$

$\quad t_N - t_0 = N \Delta t \qquad N \gg 1$

and assume process uniform in space and stationary in time, so:

so

$$P(x_j, t_j ; x_{j+1}, t_{j+1}) = P(x_{j+1}-x_j, \Delta t)$$

$\Rightarrow$

$$P(x_N - x_0 ; N\Delta t) = \int dy_1 \cdots \int dy_N \, P(y_1, \Delta t) \cdots P(y_N, \Delta t)$$

Now,

$$P(q) = \int dy_j \, e^{iqy_j} P(y_j, \Delta t)$$

$$\begin{cases} P_N(q) = \int dy^N \cdots \, e^{iqy^N} P(y^N, N\Delta t) \\ y^N = \sum_{i=1}^{N} y_j = y_N - y_0 \end{cases}$$

so

$$P_N(q) = \left[P(q)\right]^N \quad \text{, as before.}$$

(identical steps)

Now, take generating fctn. to be <u>Levy Distribution.</u>

Now,

$$P(q) = P(q \mid \alpha, c)$$

here:

$$P(q) = P_\alpha(q, \Delta c)$$

$$= L_\alpha(q, \Delta c)$$

and need

$$P_N(q) = P_\alpha(q, c_N)$$

above consistent if:

N increments

$$c_N = N \Delta c = N \Delta t \frac{\Delta c}{\Delta t}$$

$$= (N \Delta t)\, \overset{\text{const}}{c}$$

$$= tc = ct$$

$$\boxed{P_N \to P_\alpha(q, ct) = \exp\left[-cN\Delta t\, |q|^\alpha\right] = \exp\left[-ct\, |q|^\alpha\right]}$$

characteristic fctn of <u>Levy Process</u>

Characteristic fctn:

$$\boxed{P_\alpha(\xi, t) = \exp\left(-ct\,|\xi|^\alpha\right)}$$

[redacted] $$\boxed{P_\alpha(x, t) = \int \frac{d\xi}{2\pi}\, e^{i\xi x}\, e^{-ct|\xi|^\alpha}}$$

- $\alpha = 2 \Rightarrow$ diffusion ✓
- $\langle x^2 \rangle \to \infty$, for $\alpha < 2$, at any t.

for $x \to \infty$

$$\boxed{P_\alpha(x, t) \sim t \,/\, |\Delta x|^{\alpha+1}}$$

- 'accelerating tail' distribution
  'expanding tail'
- i.e. P equal at $t > t'$
  $\Rightarrow (\Delta x) > (\Delta x)'$.

Next:

- Physics: Weeks, Swinney experiment
- how extend Fokker-Planck Theory?

  ~ CTRW

  ~ Fractional Kinetics.

  Later

2/06

Notes [redacted] — Physics of Levy Flights [CTRW and Fractional Kinetics.]

cont'd

Recently:
- introduced L-stable distributions
- ex: Gaussians $\Leftrightarrow$ C.L.T.
- extended range of L-stable, not require finite variance.
- used self-similarity

$$\Rightarrow \quad L_\alpha(a, k) = \exp\left[-a|k|^\alpha\right]$$

$\Downarrow$

Levy Distrib $\qquad 0 < \alpha \leq 2$

$\alpha \to$ power law

$a \to$ ~~skin~~ extended strength parameter (i.e. D)

of course: $$P_\alpha(x, a) = \int \frac{dk}{2\pi} e^{ikx} L_\alpha(a, k)$$

Properties: $\quad L_\alpha > 0 \qquad \left(\Psi(k\mu(\lambda)) = \lambda\Psi\right)$

$$\int dx\, P_\alpha(a, x) < \infty$$

obviously: $$P_\alpha(x, a) \xrightarrow[x \to \infty]{} 1/|x|^{\alpha+1}$$

$\alpha \to \mu$

$\alpha = 2 \quad \to \quad$ Gaussian.

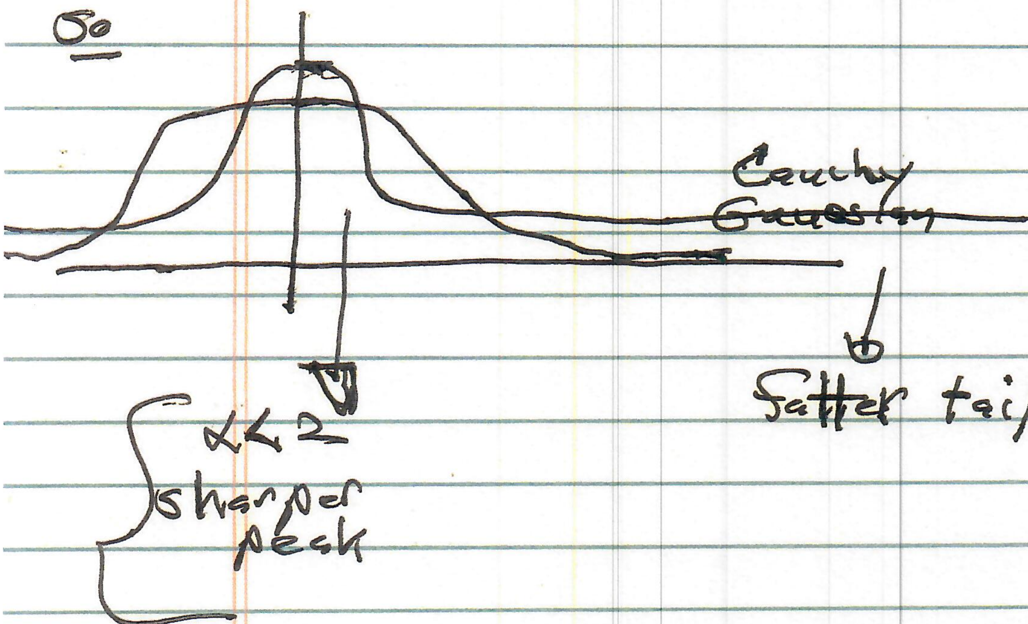
$$\frac{d\Psi}{d|k|} = \Psi / \mu'(1) k$$

B. D. Hughes

$\alpha = 1 \rightarrow$ Cauchy
(Lorentzian)

$$P_1(x) = a/\pi(x^2+a^2)$$

So



And can consider time-dependent evolution, analogous to diffusion

- first, if discretize:

$$P_N(x) = \frac{1}{N^{1/\alpha}} L_\alpha\left(a, x/N^{1/\alpha}\right)$$

width $\sim N^{1/\alpha}$

$\alpha < 2$, $N^{1/\alpha} \gg N^{1/2} \rightarrow$ super-diffusive

– making time explicit:

$$P_\alpha(k, ct) = \exp\left[-ct|k|^\alpha\right]$$

$\alpha \to$ index

$C \Leftrightarrow a \to$ strength

characteristic fctn.

$$P_\alpha(x,t) \xrightarrow[x\to\infty]{} t \Big/ |x|^{\alpha+1}$$

N.B. Can see description of Levy process involves fractional calculus.

i.e.

$$P_\alpha(x,t) = \int \frac{e^{ikx}}{\sqrt{2\pi}} P_\alpha(k,t)$$

so $\partial_t P_\alpha(x,t) =$

$$\int \frac{e^{ikx}}{\sqrt{2\pi}} \left(-c|k|^\alpha\right) \exp\left[-ct|k|^\alpha\right]$$

$\Downarrow$

Fractional derivative for $\alpha \neq 2$.

if $\alpha = 2$

$c \Rightarrow D$

$$\partial_t P_{\alpha = 2}(x,t) = \int \frac{e^{ikx}}{\sqrt{2\pi}} (-Dk^2) \exp\left[-Dtk^2\right]$$

$$= D \frac{\partial^2}{\partial x^2} \int \frac{e^{ikx}}{\sqrt{2\pi}} \exp\left[-Dtk^2\right]$$

$$= D \frac{\partial^2}{\partial x^2} P_2(x,t)$$

i.e. $\frac{\partial^2}{\partial x^2} \Leftrightarrow -k^2$

$\partial/\partial x \Leftrightarrow ik \Rightarrow (\partial/\partial x)^\alpha \Leftrightarrow (ik)^\alpha$

so $(ik)^\delta \Rightarrow (\partial/\partial x)^\delta$

meaning of fractional calculus
(i.e. integral transform defines
fractional derivative)

→ What does it all mean??

— Levy Walks self-similar

i.e. even diffn → $\frac{x^2}{t} \to \frac{\lambda^2 x^2}{\delta t}$

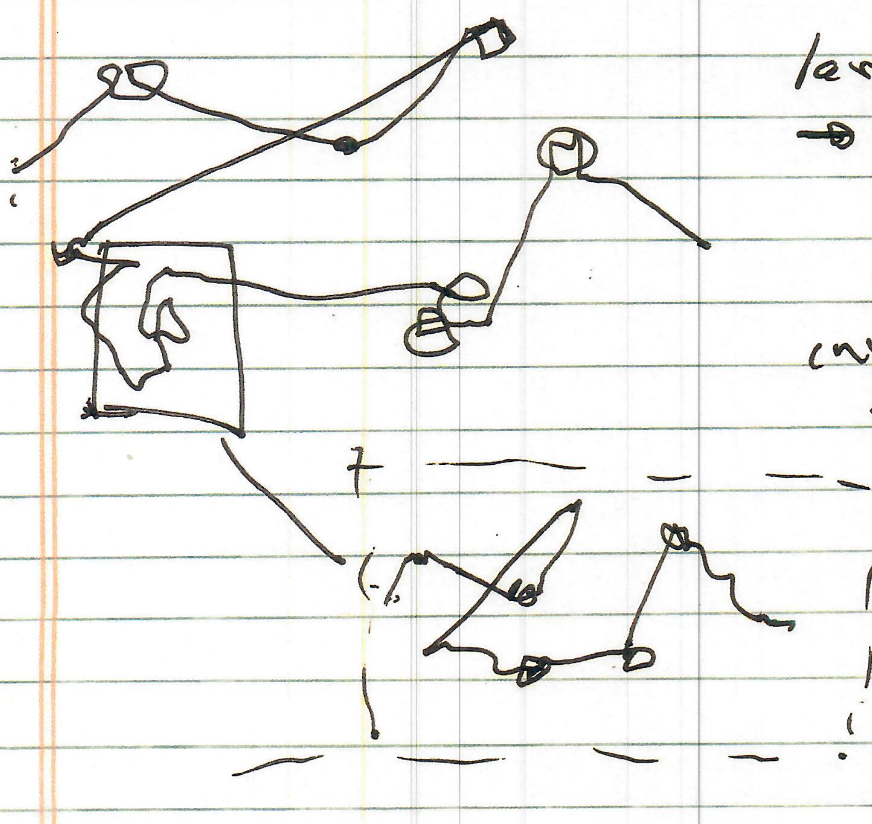
invariant under $\delta \sim \lambda^2$.

— some very large excursions (contrast standard random walk)

⇒ fat tail, large events ✓

weighted more

i.e.



large jumps

→ flights

invariant under zoom.

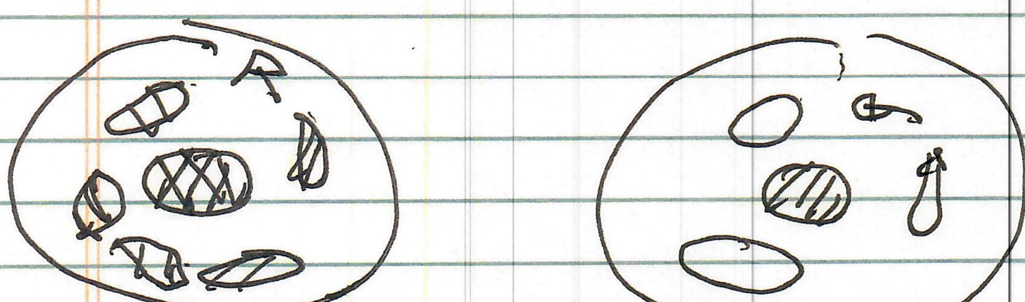
- a bit of physics:

Solomon,
Weeks, ... Swinney et al. PRL '93
a must

→ rotating tank, water pumped in, out via bottom.

→ sheared, counter-rotating azimuthal jet.

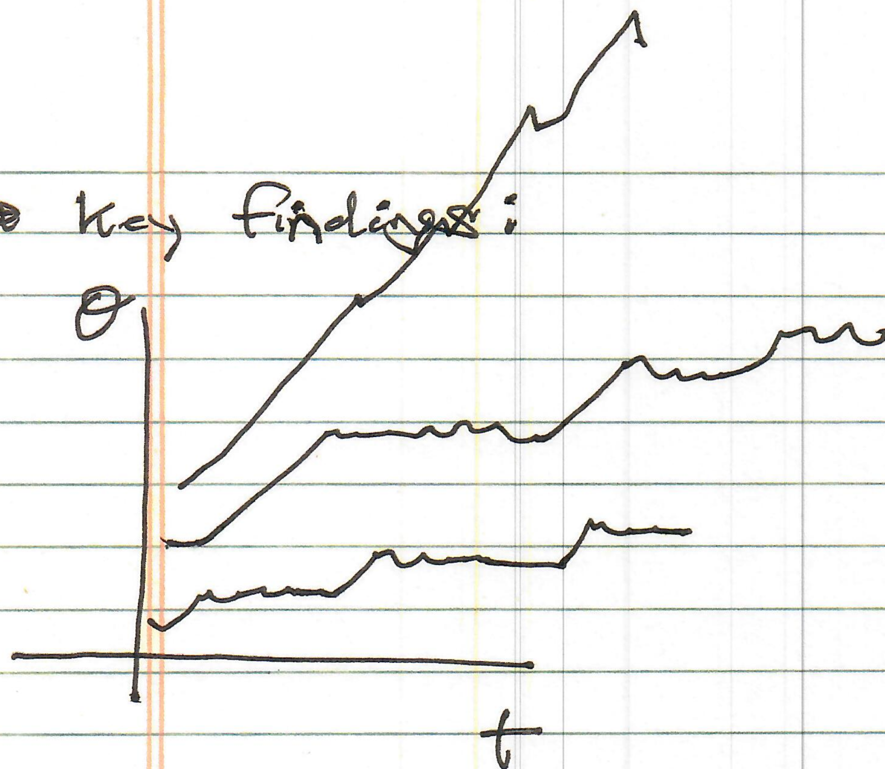
→ passive tracers injected



→ azimuthal vortex chain. (6)

→ follow tracer trajectories, which are chaotic
but no "turbulence" (flow chaotic, not-turbulent)

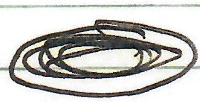
→ key findings:



obviously:

→ are circulations in individual vortex.

tracer 'stuck' in individual vortex



→ are jumps, or "flights" between vortices.

These are the large excursions of the Levy walk.

→ Can compute $\langle (\Theta - \Theta_0)^2 \rangle$

and $\langle (\Theta - \Theta_0)^2 \rangle \sim t^{1.6}$

→ Levy Flight →

→ super-diffusive
$\frac{1}{2} < H < 1$ (not calculated)

→ anomalous diffusion

— — — — — —

N.B. → Anomalous Transport — Confinement Application
$\Rightarrow D > D_{coll.}$

Anomalous Diffusion

$\langle \delta x^2 \rangle \sim t^{\alpha}$

$\alpha \neq 1$

→ Analogy → marker particles
in - island chain (positrons)
(chaos)
- Zonal flow + vortex
(novelty ?)

→ How Live and Calculate in Levy World
— A Short Introduction?
— Coming...

— how calculate anything for diffusion processes, noise, etc.?

⇒ Fokker-Planck Theory → yields ~~this~~

— What is F-P Eqn?

~ recipe for turning (micro) or step pdf into eqn.

for distribution

~ can calculate D, V, $P(x,t)$

$P(x,t)$
$\cancel{t \to \infty}$

of course:

~~first~~ 1 step ago.

$$P(x, t+\Delta t) = \int d(\Delta x)\, T(\underline{x}, \underline{\Delta x}, \Delta t)\, P(\underline{x} - \underline{\Delta x}, t)$$

transition probability

etc