Lecture 2c – More on Boltzmann Eqn.

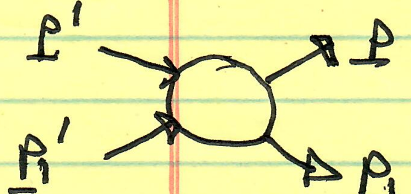
Recap:

- derived Boltzmann Equation from BBGKY Hierarchy

  $\Rightarrow$ $nd^3 \ll 1$ to truncate at $f(1,2)$ eqn.

  $\Rightarrow$ $f(1,2) = f(1)\, f(2)$ to complete

- $$C(f) = \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1'\; w(\underline{p}, \underline{p}_1; \underline{p}', \underline{p}_1') \,*\, \big(f(\underline{p}')\, f(\underline{p}_1') - f(\underline{p}_1)\, f(\underline{p}) \big)$$



- Proved H-Theorem, using Lemma and Stosszahlansatz

$$S = -\int d\underline{x} \int d\underline{p}\; f \ln f$$

$$\frac{dS}{dt} = -\int d\underline{x}\int d\underline{p}\; C(f)\ln f$$

$$\frac{dS}{dt} \geq 0.$$

Macroscopic irreversibility from microscopically reversible dynamics. $w = w^T$

- When is $\frac{dS}{dt} = 0$ $\Rightarrow$ f s/t

$C(f) = 0$?

$\Rightarrow$ identifies equilibrium distribution

Recall: $$\frac{dS}{dt} = \frac{1}{2}\int d\underline{x}\int d\underline{p}\; w f f_1\, x \ln x$$

$$x = f' f_1' / f f_1$$

so $\frac{dS}{dt} = 0$ at $x = 1$

$$f f_1 = f' f_1'$$

$$\ln f + \ln f_1 = \ln f' + \ln f_1'$$

so $\ln f + \ln f_1$ must be conserved in collision event

$\Rightarrow$

$$\ln f = c + \underset{\text{momentum}}{\underline{\alpha} \cdot \underline{p}} + \underset{\text{energy}}{\beta \epsilon}$$

Now, f normalizable; $\quad \epsilon = \dfrac{p^2}{2m}$

$$\ln f = c + \underline{\alpha} \cdot \underline{p} - \beta \epsilon$$

$$\beta > 0$$

And $\int d\underline{x}$ irrelevant, i.e. can specify $C(f_{eq}) = 0 \Rightarrow f_{eq} = f_{eq}(\underline{x})$, each $\underline{x}$

$c = c(\underline{x}) \rightarrow n(\underline{x})$

$\underline{\alpha} = \alpha(\underline{x}) \rightarrow \underline{V}(\underline{x})$

$\beta = \beta(\underline{x}) \rightarrow 1/T(\underline{x})$

normalization

$$f_{eq} = c\, n(\underline{x}) \exp\left[(\underline{p} \cdot \underline{V}(x) - \epsilon) / T(x) \right]$$

i.e. shifted Maxwellian, as expected!

- Local vs. Global?

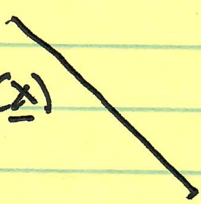
  → $\int d\underline{x}$ irrelevant

  i.e. $C(f_{eq}) = 0$ at each $\underline{x}$

  Entropy produced locally, on ν time scale $(w \sim \sigma V_{rel})$

  $(\nu = v_{th}/l_{mfp} \quad l_{mfp} = 1/n\sigma)$

  → But: What of inhomogeneity in thermodynamic parameters?

$T(\underline{x})$  $\Rightarrow \underline{Q}$ $\Rightarrow$ $\dfrac{T(\underline{x})}{}$

The point:

$f_{eq} = f_{eq}(T(\underline{x}), V(\underline{x}), n(\underline{x}))$ does

not satisfy Boltzmann Equation,

$$\partial_t f + \underline{v} \cdot \underline{\nabla} f = C(f)$$

$$\cancel{\partial_t f_{eq}} + \underline{v} \cdot \underline{\nabla} f_{eq} = \cancel{C(f_{eq})} \rightarrow 0$$

$$\underline{v} \cdot \nabla \underline{f_{eq}} \neq 0$$

!

so

$$f \rightarrow f_{eq} + \delta f \quad \rightarrow O(l_{mfp}/L)$$

Rate: $\sim \left(\frac{l_{mfp}}{L}\right)^2$ induces collisional fluxes, etc.

- How reconcile:
  - reversible Hamiltonian dynamics
  - $dS/dt \geq 0$ Macro irreversibility

→ Coarse Graining ! ⇒ Partition

(recall Lyapunov Exponents

→ partition

$\Delta q \Delta p$ – partition cell

sets resolution scale

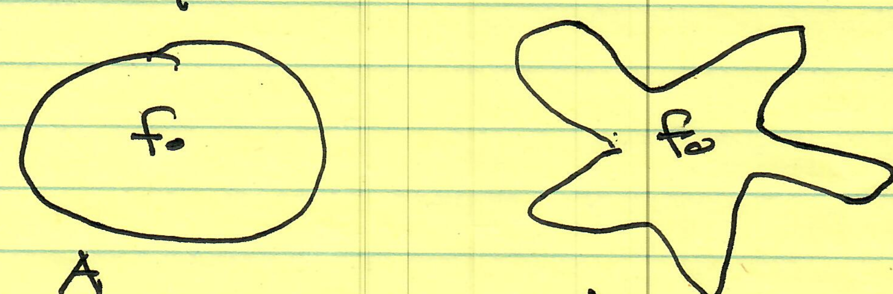
S is integrated quantity

Why significant?

→ partition kills small details in phase volume evolution.

$t=0$ exact

i.e

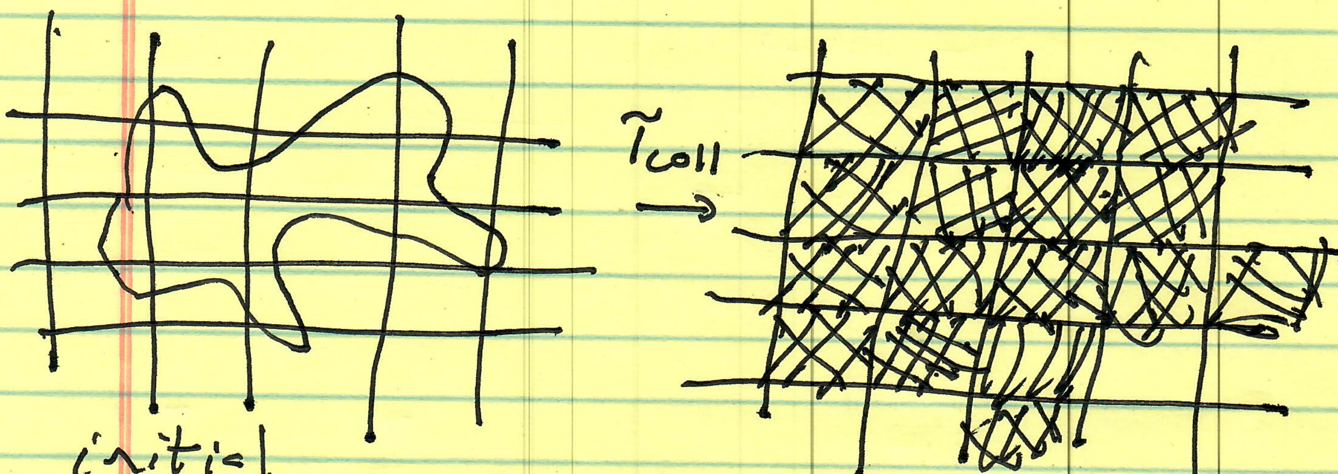


A A

via Hamiltonian evolution

2 time scales — step — Hamiltonian mapping
— τ_c

with coarse graining:



initial

coarse grained area

{ local phase space density modified by coarse graining

~~so~~

$$f_0 A_0 = A_{CG} \bar{f} \qquad A_0 < A_{CG}$$

$$\bar{f} = f_0 A_0 / A_{CG} \qquad \bar{f} < f_0 .$$

$\bar{f}$ → coarse grained phase space density

N.B. Prediction of close recurrence impossible as partition sets resolution limit

What Next ?

→ We have the Boltzmann Equation and H-Theorem. (Yay!)

→ What do we do with them?

N.B.: "The solution of the above equation [B.E.], as we will see shortly, is truly a gruesome task"

– Stewart Harris, in "An Introduction to the Theory of the Boltzmann Equation".

So ?

Hint: Consider time scales ...