Physics 210B

L3 Beyond Boltzmann ...

So far: – derived Boltzmann equation

$$\partial_t f + \underline{v} \cdot \underline{\nabla} f = C(f)$$

$$\left[\partial_t f + \underline{v} \cdot \underline{\nabla} f + \underline{a} \cdot \underline{\nabla}_v f = C(f) \right]$$

$$\updownarrow$$

$$\frac{q}{m}\underline{E} + \frac{q}{mc}\underline{v} \times \underline{B}$$ – Vlasov for $C(f) \to 0$

– proved H-Thm. $\left\{ \begin{array}{l} \text{Phase space} \\ \text{Fluid continuity} \end{array} \right.$

$$\frac{dS}{dt} \geq 0$$

→ identified Eqbrm. Distribution f_{max}

→ $f \to f_{max}$ on ν_c time scale

→ local Maxwellian.

Now what ?

What to *do* with Boltzmann Equation ?

⇒ *real* problems:

- dynamics
- inhomogeneity

} combine collective and collisional processes

i.e. heat — conduction / convection.

⇒ two roads forward:

- low collisionality, work with 'modified' Vlasov equation

  ⇒ see Plasmas, Galactic Dynamics

$$d < n^{-1/3} < l_{mfp} < L$$

- Fluid Equations

i.e. $f \approx f_{Max}[T(\underline{x}), n(\underline{x}), \underline{V}(\underline{x})]$
here

⇒ evolution equations for:
$n(\underline{x})$, $\underline{V}(\underline{x})$, $T(\underline{x})$ etc.

⇒ macroscopic

⇒ Derived from Boltzmann . .

Related: Inhomogeneity - i.e. $T(\underline{x})$

⇒ Transport

$$\underline{Q} = -\chi\,\underline{\nabla}T$$ constitutive relation

thermal diffusivity
→ transport coefficient

$$\rightarrow f = f_{0\,Max.} + \delta f$$

i.e. distribution 'close to', but not equal to Maxwellian.
$l_{mfp}/L \ll 1$, but not $\rightarrow 0$

viscosity, heat conductivity..

- how compute transport coefficients
  ⇒ needed for real fluid equations
- point:

$$\partial_t f + \underline{v} \cdot \underline{\nabla} f = C(f)$$

For inhomogeneous f_{eq},

$$\cancel{\partial_t f_{eq}} + \underline{v} \cdot \underline{\nabla} f_{eq} = \cancel{C(f_{eq})}$$

- does not satisfy Boltzmann equation

- $f = f_{eq} + \underset{\substack{\Downarrow \\ \text{collisional flux}}}{\delta f}$

The Rest:

On Fluid Equations:

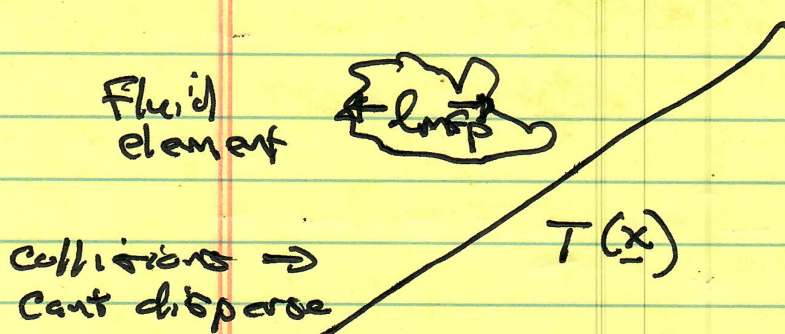
- Euler from Boltzmann ("ideal")
- Euler from macroscopics
- On Fluids (general thoughts).

Fluid Eqns.

- replace B.E. by set of equations which evolve thermodynamic parameters.

- local Eulerian description (lab frame) i.e $n(\underline{x}, t)$, $V(\underline{x}, t)$ etc

- describes 'blobs' of gas held together by collisions.



$L > \ell_{mfp}$

$\omega < \nu_{col}.$

- parametrizes dynamics in terms structure of distribution

$$f = \frac{n(\underline{x})}{(2\pi)^{3/2} v_{Th}(x)^3} \exp\left[-(\underline{v} - \underline{V}(x,t))^2 / v_{Th}(x,t)^2\right]$$

- works for slight deviation from equilibrium

i.e.

$f = f_{eq} + \delta f$ — will see.

$\downarrow$ $\rightarrow$ $\delta f \simeq - \frac{\underline{v} \cdot \underline{\nabla} f_{eq}}{\nu}$

local Maxwellian

$\downarrow$ $\simeq l_{mfp} \nabla f_{eq}$

Ideal Equations (Perfect fluid) $\sim \boxed{\frac{l_{mfp}}{L} f_{eq}}$

$\downarrow$ $\downarrow$

Euler — viscous, dissipative equations

- sacrifices higher moments of f.

$\downarrow$

Navier-Stokes

Ideal Equations:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f = C(f)$$

demand:

$$\int d^3v \, C(f) = 0$$

$$\int d^3v \, m v \, C(f) = 0$$

$$\int d^3v \, \epsilon \, C(f) = 0$$

[easily shown. Collision operator conserves mass, energy momentum.

So natural to define:

$$n = \int d^3v\, f \quad \rightarrow \text{density}$$

$$\underline{V} = \underline{V}(\underline{x}, t) = \frac{1}{n}\int d^3v\, \underline{v}\, f \quad \rightarrow \text{fluid / momentum.}$$

$$\bar{\varepsilon} = \frac{1}{n}\int d^3v\, \varepsilon f \rightarrow \text{energy density}$$

Now,

$$\boxed{\partial_t f + \partial_{x_i}(v_i f) =} \text{ [redacted] } \boxed{C(f)}$$

Conservative Form

Taking moments:

$\int d^3 \underline{v}$ *

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n \underline{V}) = 0}$$

Continuity, # Cons.

number density flux

cons.

$$\int d^3v \; m V_\alpha \; *$$

$\Rightarrow$

$\partial_t (m$ [redacted] $V_\alpha) + \dfrac{\partial}{\partial x_\beta} \Pi_{\alpha,\beta} = 0$

$\Downarrow$

mom. cons

$$\Pi_{\alpha\beta} = \int d^3v \; m V_\alpha V_\beta \; f$$

$\Downarrow$

momentum flux

and $\int d^3v \; \epsilon \; *$

$\Rightarrow$

$$\partial_t (n \bar{\epsilon}) + \underline{\nabla} \cdot \underline{q} = 0$$

$$\underline{q} = \int d^3v \; \underline{v} \, \epsilon f$$

Note: $\left.\begin{matrix} n \\ \underline{V} \\ \bar{\epsilon} \end{matrix}\right\}$ equations have the form:

① $\partial_t(\text{Stuff}) + \underline{\nabla}\cdot(\text{Flux of Stuff}) = 0$

i.e. of form:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0$$ ← macroscopic conservation

② all rest upon conservation properties of the collision operator.

③ Key is constitutive relation, i.e.

relating $\underline{J}$ to something useful.

i.e Fick's Law : $\underline{J} = -D\,\underline{\nabla}\rho$

$$\Rightarrow \frac{\partial \rho}{\partial t} - D\,\nabla^2\rho = 0$$

usually not so simple ....

→ closure problem.

(4) Essence of fluid equation construction is calculation of fluxes

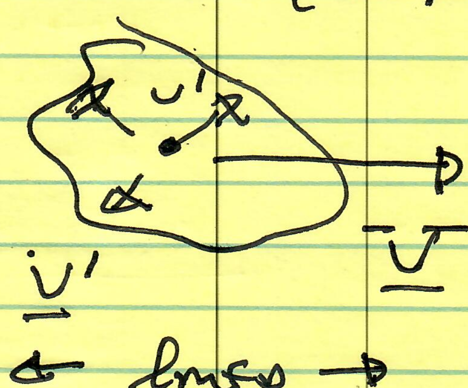
Further simplify:

$$\underline{v} = \underline{V}(\underline{x},t) + \underline{v}'$$

$\underline{v}$ ↓ particle velocity (micro)

$\underline{V}(\underline{x},t)$ ↓ mean bulk flow (macro) - linked to body forces

$\underline{v}'$ ⟶ thermal fluctuation abt mean $(\sim\sqrt{T/m})$



Realistically, $|\underline{V}| < |\underline{v}|$

but $\underline{v}'$'s cancel ⟹ random

so

$$\Pi_{\alpha,\beta} = \int d^3v\; m\left(V_\alpha(\underline{x},t) + v'_\alpha\right)\left(V_\beta(\underline{x},t) + v'_\beta\right) f$$

Now here,

$$f = f_{eq} + \cancel{\delta f} \quad \Rightarrow \underline{\underline{\text{ideal}}} \text{ fluid}$$

$\downarrow$ local Maxwellian

↳ would have ∇f dependence.

and taking out $\underline{V}$, f_{eq} is $\underline{\text{even}}$ in v'.

$\underline{\text{so}}$

$$\Pi_{\alpha,\beta} = mn \left(V_\alpha(\underline{x},t)\, V_\beta(x,t) + \langle v'_\alpha v'_\beta \rangle \right)$$

$$f = \frac{n(x)}{(2\pi)^{3/2}[v_{th}(x)]^3} \exp\left[-\frac{v'^2}{v_{th}(x)^2} \right]$$

$$v' = \underline{v} - \underline{V}(\underline{x},t)$$

l.o. in $\frac{l_{mfp}}{L}$

$\underline{\text{so}}$

$$\langle v'_\alpha v'_\beta \rangle = 1/3\, v'^2 \delta_{\alpha,\beta}$$ (isotropic f_{eq})

$$\langle v'^2 \rangle = 3T/m.$$

3×3

so, can define: $\underline{\underline{P}} = [\quad]$

$\underline{\underline{P}} = mn \langle v_\alpha' v_\beta' \rangle$ → stress tensor (pressure)

$= \frac{1}{3} mn \langle v'^2 \rangle \delta_{\alpha,\beta}$ → diagonal for ideal fluid

⇒ 1st moment:

$$\partial_t (n\underline{V}) + \underline{\nabla} \cdot (\underbrace{n\underline{V}\,\underline{V}}_{\text{Reynolds stress tensor}} + \underline{\underline{I}} P) = 0 \qquad * \quad (\text{Euler})$$

recall: $\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n\underline{V}) = 0 \qquad **$

and subtract ** from *:

$$\boxed{n\left(\frac{\partial}{\partial t}\underline{V} + \underline{V} \cdot \underline{\nabla}\,\underline{V}\right) = -\underline{\nabla} P}$$

"Euler Eqn."

→ microscopically, Euler Eqn.

corresponds to $l_{mfp}/L \to 0$.

→ $\underline{\underline{\Pi}} = n \underline{V}\,\underline{V} + P \underline{\underline{I}}$ is constitutive relation for

$$\partial_t (n\underline{V}) + \underline{\nabla} \cdot \underline{\underline{\Pi}} = 0$$

→ Note C.R. contains no derivatives

⇒ ideal.

Contrast: Viscous fluid:

viscosity

$$\underline{\underline{\Pi}} = n \underline{V}\,\underline{V} + P \underline{\underline{I}} - \eta \underline{\nabla}\underline{V}$$

viscous stress

Viscous fluid

⇒ Navier Stokes

→ What is p ?

$\underline{\nabla}\cdot$ Euler $=0$

for $\underline{\nabla}\cdot\underline{V}=0$ → incompressible

$\underline{\nabla}\cdot\underline{V}\neq 0$ ⇒ Equation of state

Similarly;

$$\epsilon = \underbrace{\tfrac{1}{2} m v^2}_{KE} + \underbrace{\epsilon'}_{\text{internal dof}}$$

$$= \frac{1}{2} m \left(\underline{V}(\underline{x},t) + \underline{v}'\right)^2 + \epsilon'$$

$$q = \int \epsilon\, \underline{v} f \, d^3v$$

$$\cong \int \epsilon v f_{eq} \, d^3v$$

So

$$\underline{q} = \int d^3v \, \left(\underline{V}(\underline{x},t) + \underline{v}'\right)\left(\epsilon' + \tfrac{1}{2}\left(\underline{V}(\underline{x},t) + \underline{v}'\right)^2\right) f$$

$f = f_{eq.}$ $\quad$ PV work

$$\underline{q} = \underline{V}(\underline{x},t)\left(\frac{1}{2} n m V^2 + \underbrace{P + n\bar{\epsilon}'}_{\text{enthalpy}}\right)$$

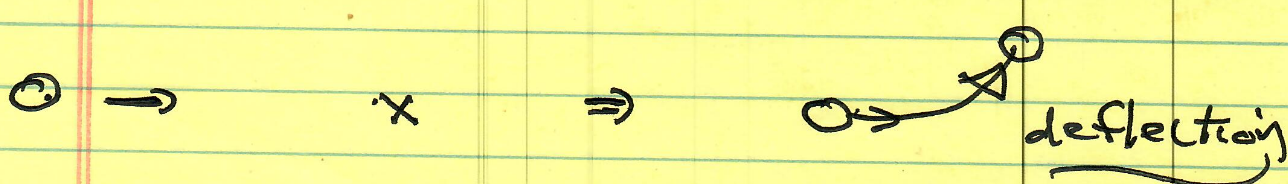
<u>so</u>

$$\boxed{\partial_t (n\bar{\epsilon}) + \nabla \cdot \left[\underline{V}(\underline{x},t)\left(\frac{1}{2} n m V^2 + P + n\epsilon'\right)\right] = 0}$$

can simplify, as for momentum.

N.B:

- angular momentum <u>not</u> conserved by $C(f)$



→ Most truncations stop at 3rd moment.