Stochastic Magnetic Fields

PHYS 235 - Nonlinear Plasma Theory
Student Lecture Notes
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Contents

1 Prologue

The PHYS 235 course is about **transport** in disordered, random, and turbulent systems in different regimes and the system evolution due to such transport (i.e., relaxation). The topic of this write up will be on transport in stochastic magnetic fields. The discussion includes scattering & collisions as well as the regimes of $\mathrm{K_u} < 1$. This discussion focuses on the origin of irreversibility.

Before we discuss stochastic field lines, we pose the following question for the reader,"How many magnetic field lines are there in the universe?"

The answer to this question is probably **one**. There are perturbations to a magnetic field in the real world. This causes the field lines to wander and diffuse in space, forming a stochastic distribution and chaotic system. Therefore, a single field line can fill the volume of the entire universe. This phenomena leads to the perpendicular transport of charged particles and energy. Perpendicular transport is a crucial to the research of magnetic confinement in nuclear fusion.

2 Hamiltonian Chaos

To understand stochastic fields we need to review basics of Hamiltonian chaos. This includes starting from the basics of Hamiltonian systems. A brief crash course will be provided on Hamiltonian systems, but if the reader wants a more detail introduction on the topic, I highly recommend Professor Arovas Physics 200A course notes on Hamiltonian mechanics and chaos.

Hamiltonian systems are a set of dynamical systems that occur in a wide variety of circumstances. Examples of Hamiltonian dynamics include mechanical systems in the absence of friction, but most importantly, at least pertaining to this class is the paths followed by magnetic field lines in a plasma. The notes here will follow closely to that of the content in Ott, 2002.

2.1 Hamiltonian Systems

The dynamics of Hamiltonian systems are captured in the function known as the "Hamiltonian", $H(\mathbf{p}, \mathbf{q}, t)$. The state of the system is specified by the 'momentum' $\mathbf{p}$ and 'position' $\mathbf{q}$ coordinates. The vectors $\mathbf{p}$ and $\mathbf{q}$ have the same dimensions as N, which is the number of degrees of freedom of the system. The equations of motion that the system follows in the $2N$-dimensional phase space are given by

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial \mathbf{q}}, \tag{1}$$

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial \mathbf{p}}. \tag{2}$$

In the case that the Hamiltonian has no explicit time dependence, $H(\mathbf{p}, \mathbf{q})$, we can use Hamilton's equations of motion to show that the Hamiltonian remains a constant:

$$\frac{dH}{dt} = \frac{d\mathbf{q}}{dt}\frac{\partial H}{\partial \mathbf{q}} + \frac{d\mathbf{p}}{dt}\frac{\partial H}{\partial \mathbf{p}} = \frac{\partial H}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} = 0. \tag{3}$$

This implies a connection between the Hamiltonian and energy E of the system. Energy is conserved for time-independent systems, thus, $E = H(\mathbf{p}, \mathbf{q}) = \text{constant}$.

A basic property of Hamilton's equations is that the preserve $2N$-dimensional volumes in the phase space. This follows from the continuity equation for the phase space density,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\rho) = 0, \tag{4}$$

where $\mathbf{u} = (\dot{\mathbf{q}}, \dot{\mathbf{p}})$ is the velocity vector in phase space, and Hamilton's equations, which say that the phase flow is incompressible, i.e., $\nabla \cdot \mathbf{u} = 0$:

$$\nabla \cdot \mathbf{u} = \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} + \frac{\partial \dot{\mathbf{p}}}{\partial \mathbf{p}} \tag{5}$$

$$= \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial H}{\partial \mathbf{p}}\right) + \frac{\partial}{\partial \mathbf{p}}\left(-\frac{\partial H}{\partial \mathbf{q}}\right) = 0. \tag{6}$$

Thus, the convective derivative vanishes, which guarantees that the density remains constant in a frame moving with the flow. This incompressibility of phase space volumes for Hamiltonian systems is called Liouville's theorem. A consequence of conservation of phase space volumes for Hamiltonian systems is the **Poincaré recurrence theorem**. For a time-independent Hamiltonian where orbits are bounded **Poincaré recurrence theorem** states that the system will return to its original state if we wait long enough.

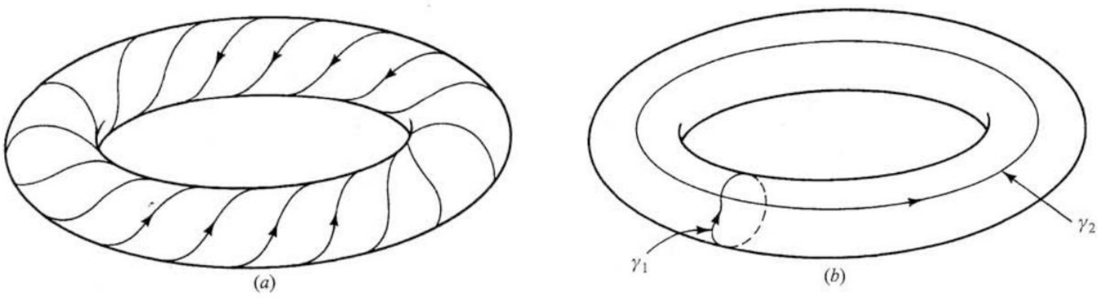

Figure 1: For the case $N = 2$, (a) Orbit on a 2-torus. (b) Two irreducible paths on a 2-torus.

2.2 Canonical Changes of Variables

A change of variable which preserves the Hamiltonian form of the equations are said to be **canonical**, and the momentum and position vectors that the describe the Hamiltonian are said to be **canonically conjugate**. What sorts of transformations are allowed? Well, if Hamilton's equations are to remain invariant, then

$$\dot{\bar{\mathbf{p}}} = -\frac{\partial \bar{H}}{\partial \dot{\bar{\mathbf{q}}}}, \tag{7}$$

$$\dot{\bar{\mathbf{q}}} = \frac{\partial \bar{H}}{\partial \dot{\bar{\mathbf{p}}}}, \tag{8}$$

where $\bar{H}$ is a new transformed Hamiltonian for the system. One way to specify a canonical change of variables is to introduce a generating function, $S(\bar{\mathbf{p}}, \mathbf{q}, t)$. Thus, we have the following,

$$\bar{\mathbf{q}} = \frac{\partial S}{\partial \bar{\mathbf{p}}}, \tag{9}$$

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}. \tag{10}$$

In terms of the generating function the new Hamiltonian is given by

$$\bar{H}(\bar{\mathbf{p}}, \bar{\mathbf{q}}, t) = H(\mathbf{p}, \mathbf{q}, t) + \frac{\partial S}{\partial t}. \tag{11}$$

2.3 Integrable Systems

For the case where the Hamiltonian has no explicit time dependence, $H = H(\mathbf{p}, \mathbf{q})$, this implies that the energy $E = H(\mathbf{p}, \mathbf{q})$ is a conserved quantity. A time-independent

Hamiltonian system is said to be integrable if it has N independent global constants of motion, where N is the degree of freedom. For a completely integrable system, one can transform canonically from $(\mathbf{q}, \mathbf{p})$ to a new coordinates $(\boldsymbol{\theta}, \mathbf{J})$ which specify a particular N-torus. These set of new coordinates are called the action-angle variables

$$(\bar{\mathbf{p}}, \bar{\mathbf{q}}) = (\mathbf{J}, \boldsymbol{\theta}), \tag{12}$$

where $\mathbf{J}$ is defined by

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p} \cdot d\mathbf{q}. \tag{13}$$

In eq. 13, the γ_i, where $i = 1, 2, \ldots, N$, denote N irreducible paths on the N-torus, each of which wrap around the torus in N angle directions that can be used to parameterize points on the torus (see Figure 1). The deformation of the paths γ_i on the torus do not change the values of the integrals in eq. 13 by the Poincarè-Cartan theorem.

The new Hamiltonian in action-angle coordinates is independent of $\boldsymbol{\theta}$, and hence Hamilton's equations reduce to

$$\frac{d\mathbf{J}}{dt} = 0, \tag{14}$$

$$\frac{d\boldsymbol{\theta}}{dt} = \frac{\partial \bar{H}(\mathbf{J})}{\partial \mathbf{J}} \equiv \boldsymbol{\omega}(\mathbf{J}). \tag{15}$$

We can interpret $\boldsymbol{\omega}$ as an angular velocity vector specifying trajectories on the N-torus. The solution to these set of equations are

$$\mathbf{J}(t) = \mathbf{J}(0), \tag{16}$$

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \boldsymbol{\omega}(\mathbf{J})t. \tag{17}$$

A torus such that $\mathbf{m} = (0, 0, \ldots, 0)$ is not the only solution to

$$\mathbf{m} \cdot \boldsymbol{\omega} = 0, \tag{18}$$

where $m_i \in \mathbb{Z}$. This condition is called a "resonant" torus. The resonant tori are dense in phase space, so arbitrarily near to any non-resonant torus there exist resonant tori.

For a 2-torus ($N = 2$) system where $\boldsymbol{\omega} = (\omega_1, \omega_2)$ the resonant torus has

$$\frac{\omega_1}{\omega_2} = \frac{p}{q}, \tag{19}$$

where $p, q \in \mathbb{Z}$. This is called a "rational surface" and are closed trajectories. These rational surfaces define **natural** resonances of the system. If the fraction of p and q is an irrational number then this is a non-resonant torus, which has an ergodic trajectory.

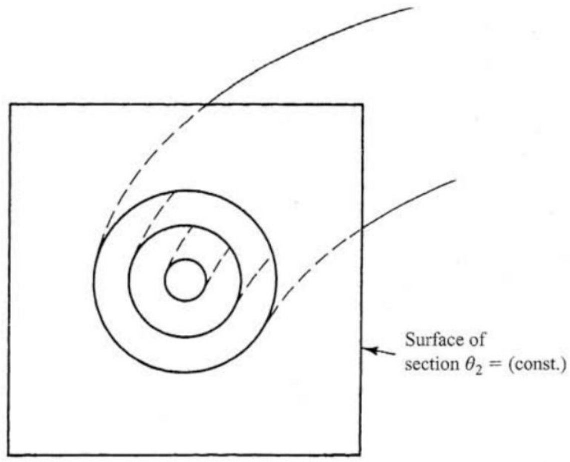

Figure 2: Surface of section for an integrable system.

2.4 Perturbation of Integrable Systems

A fundamental question concerning Hamiltonian systems is how prevalent is integrability? Here we are interested in determining whether a perturbation on the Hamiltonian has N-dimensional tori to which its orbits are restricted. Let's define the following new Hamiltonian as

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \epsilon H_1(\mathbf{J}, \boldsymbol{\theta}), \tag{20}$$

where H_0 is the unperturbed Hamiltonian of the integrable system, H_1 is the perturbation, and ϵ is a small number. If there are tori, there is a new set of action-angle variables $(\mathbf{J}', \boldsymbol{\theta}')$ such that

$$H(\mathbf{J}, \boldsymbol{\theta}) = H'(\mathbf{J}'), \tag{21}$$

where, in terms of the generating function S, we have

$$\mathbf{J} = \frac{\partial S(\mathbf{J}', \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \tag{22}$$

$$\boldsymbol{\theta}' = \frac{\partial S(\mathbf{J}', \boldsymbol{\theta})}{\partial \boldsymbol{J}'}. \tag{23}$$

The Hamilton-Jacobi equation for S is

$$H\left(\frac{\partial S}{\partial \boldsymbol{\theta}}, \boldsymbol{\theta}\right) = H'(\mathbf{J}'). \tag{24}$$

One approach to solving the Hamilton-Jacobi expression for S is to look for a solution in the form of a power series in ϵ,

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \dots \; . \tag{25}$$

For S_0 we use $S_0 = \mathbf{J}' \cdot \boldsymbol{\theta}$ which results in $\mathbf{J} = \mathbf{J}'$ and $\boldsymbol{\theta} = \boldsymbol{\theta}'$. We insert this into eq. 25 and expand for small ϵ and retaining first-order terms, we have

$$H_0(\mathbf{J}') + \epsilon \frac{\partial H_0}{\partial \mathbf{J}'} \cdot \frac{\partial S_1}{\partial \boldsymbol{\theta}} + \epsilon H_1(\mathbf{J}', \boldsymbol{\theta}) = H'(\mathbf{J}'). \tag{26}$$

Since the system has periodicity in $\boldsymbol{\theta}$ with 2π period, we can express the perturbation H_1 and S_1 as a Fourier series in the angle vector $\boldsymbol{\theta}$,

$$H_1 = \sum_{\mathbf{m}} H_{1,\mathbf{m}}(\mathbf{J}')\; e^{i\mathbf{m}\cdot\boldsymbol{\theta}}, \tag{27}$$

$$S_1 = \sum_{\mathbf{m}} S_{1,\mathbf{m}}(\mathbf{J}')\; e^{i\mathbf{m}\cdot\boldsymbol{\theta}}, \tag{28}$$

where $\mathbf{m}$ is an N-component vector of integers. Substituting these Fourier series in eq. 26, we obtain

$$S_1 = i \sum_{\mathbf{m}} \frac{H_{1,\mathbf{m}}(\mathbf{J}')}{\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{J}')}\; e^{i\mathbf{m}\cdot\boldsymbol{\theta}} \tag{29}$$

where $\boldsymbol{\omega}_0(\mathbf{J}) \equiv \partial H_0(\mathbf{J})/\partial \mathbf{J}$ is the unperturbed N-dimensional frequency vector for the torus corresponding to action $\mathbf{J}$.

Note that when $\mathbf{m} \cdot \boldsymbol{\omega} = 0$ we have the 'problem of small denominator'. The 'small denominator' problem is responsible for creating orbit resonance islands where trajectories braid from one X point to the other. Around these islands, orbits become less perturbed as they get further away from the O point. If two or more islands are present close to each other, chaos fills in the separating volume by non-ergodic turbulent mixing.

In particular the expression given above does not work for values of $\mathbf{J}$ for which $\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{J}) = 0$ for some value of $\mathbf{m}$. These $\mathbf{J}$ define *resonant tori* of the unperturbed system. Here it is emphasized that the resonant tori are dense in the phase space of the unperturbed Hamiltonian. The question of what happens is a central issue in chaos theory. The small denominator problem is akin to Landau resonance where for a 2-tori system we have

$$m\omega_1 + n\omega_2 = 0, \tag{30}$$

$$\frac{m}{n} = -\frac{\omega_2}{\omega_1} = -\frac{q}{p}, \tag{31}$$

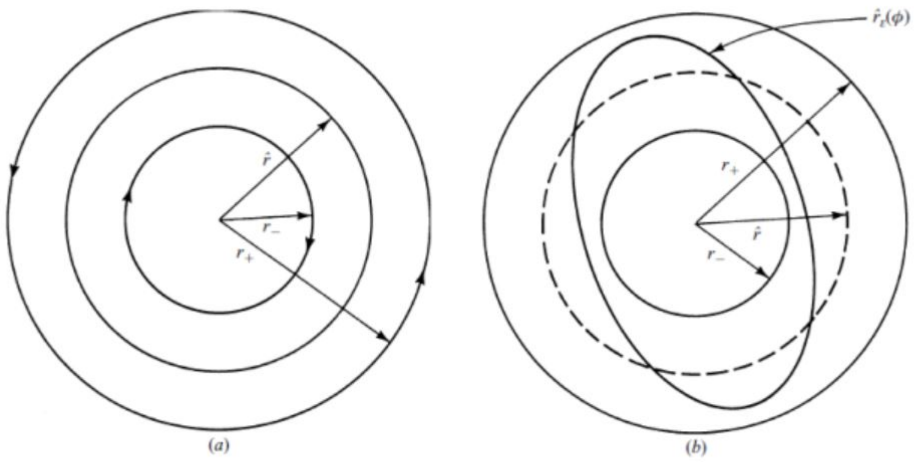

Figure 3: (a) Three invariant circles of the unperturbed map. (b) The curve $r = \hat{r}_\epsilon(\phi)$.

where m/n is the pitch of perturbation and q/p is the pitch of the trajectory.

The KAM theorem states that under very general conditions for small ϵ 'most' of the tori of the unperturbed integrable Hamiltonian survive. We say that a torus of the unperturbed system with frequency vector $\boldsymbol{\omega}_0$ 'survives' perturbation if there exists a torus of the perturbed ($\epsilon \neq 0$) system. Such perturbed toroidal surface with frequency $\boldsymbol{\omega}(\epsilon)$ goes continuously to the unperturbed torus as $\epsilon \to 0$.

Since the resonant tori on which $\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{J}) = 0$ are dense, we expect that, arbitrarily near surviving tori of the perturbed system, there are regions of phase space where the orbits are not on surviving tori. These regions are occupied by chaotic orbits as well as new tori and elliptic and hyperbolic periodic orbits are created by the perturbation.

2.5 The Fate of Resonant Tori

Most tori survive small perturbation. The resonant tori, however, does not. The questions remains on what happens to them? Let's consider the case of a Hamiltonian system described by a two-dimensional area preserving map. The tori of the integrable system (see figure 2) intersect the surface of section in a family of nested closed curves. We can take these curves to be concentric circules represented by polar coordinates.

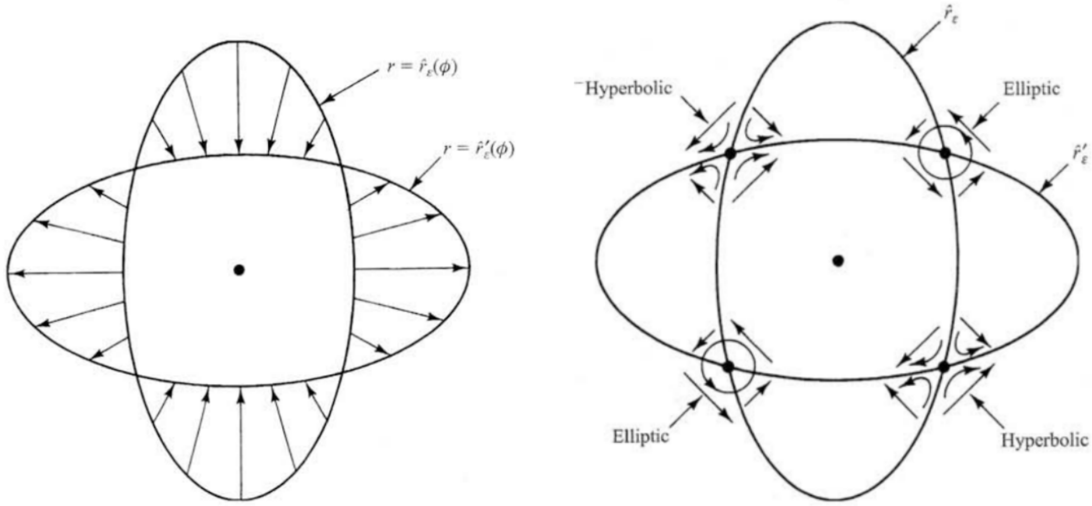

Figure 4: (a) Points on the curve $\hat{r}_\epsilon(\phi)$ map under $\mathbf{M}_\epsilon^{\tilde{q}}$ purely radially to the curve $\hat{r}'_\epsilon(\phi)$. (b) The elliptic and the hyperbolic points.

In this case we obtain a map $(r_{n+1}, \phi_{n+1}) = \mathbf{M}_0(r_n, \phi_n)$,

$$r_{n+1} = r_n,$$
$$\phi_{n+1} = [\phi_n + 2\pi R(r_n)] \text{ modulo } 2\pi.$$

Here $R(r)$ is the ratio of the frequencies ω_1/ω_2. On a resonant torus the rotation number $R(r)$ is rational:

$$R = \frac{\omega_1}{\omega_2} = \frac{\tilde{p}}{\tilde{q}}, \tag{32}$$

$$\tilde{q}\omega_1 - \tilde{p}\omega_2 = 0, \tag{33}$$

where $\tilde{p}$ and $\tilde{q}$ are integers which do not have a common factor. On the intersection $r = \hat{r}(\tilde{p}/\tilde{q})$ of the resonant torus with the surface of section, every point is a fixed point of $\mathbf{M}_0^{\tilde{q}}$ defined as

$$\mathbf{M}_0^{\tilde{q}}(r, \phi) = (r, \phi). \tag{34}$$

Let's now inquire, what happens to this circle when we add the terms proportional to ϵ. Assume $R(r)$ is a smoothly increasing function of r in the vicinity of $r = \hat{r}$. For the unperturbed map we choose a circle at $r = r_+ > \hat{r}(\tilde{p}/\tilde{q})$ which is rotated by $\mathbf{M}_0^{\tilde{q}}$ in counterclockwise direction and a circle at $r = r_- < \hat{r}(\tilde{p}/\tilde{q})$ which is rotated by $\mathbf{M}_0^{\tilde{q}}$ in the clockwise direction. Due to the intermediate-value theorem, the circle $r = \hat{r}(\tilde{p}/\tilde{q})$

is not rotated at all. If ϵ is sufficiently small, then $\mathbf{M}_0^{\tilde{q}}$ still maps all the points initially on the circle $r = r_-$to new positions whose ϕ coordinate is clockwise displaced from its initial position. The same applies for all points on r_+. For any given fixed value of ϕ, as r increases from r_- to r_+, the value of the angle that the point (r, ϕ) maps to increases from below ϕ to above ϕ. For the perturbed map, there is a closed curve, $r = \hat{r}_\epsilon(\phi)$, lying between $r_+ \geq r \geq r_-$ and close to $r = \hat{r}(\tilde{p}, \tilde{q})$, on which points are mapped by $\mathbf{M}_0^{\tilde{q}}$ purely in the radial direction (see figure 3 (b)).

Next, we apply the map $\mathbf{M}_0^{\tilde{q}}$ to the curve $r = \hat{r}_\epsilon$ and obtain a new curve $r = \hat{r}'_\epsilon$. This causes some sections of the contour to move outward, others to move inward, and some countable fixed points. These fixed points are elliptic and hyperbolic as illustrated in figure 4. Poincaré Birkhoff theorem states that there are the same number of elliptic and hyperbolic points.

The rotation of points around an elliptic point forms KAM curves surrounding it (this is also referred to as an "O point"). Between the surrounding curves is the destroyed resonant region, which corresponds to chaotic resonant orbits. A hyperbolic point tends to be a heteroclinic intersection, thus also called an "X point". Therefore, we have the formation of an island chain. The island width ΔJ is given by

$$\Delta J \approx \sqrt{\frac{\epsilon H_1}{\partial \omega_0 / \partial J}}, \tag{35}$$

where ϵH_1 represents the strength of the perturbation in the Hamiltonian and $\partial \omega_0 / \partial J$ represents the unperturbed shear. The derivation of this result is given in Rosenbluth et al., 1966 for magnetic field lines in a torus by Fourier series expansion of the field fluctuation. For an illustration of this, see figure 5.

The result given above can be derived through simple arguments as well. Recall the definition of the generating function S:

$$\begin{aligned} H &= H_0 + \frac{\partial S}{\partial t} \\ \implies H_0 + \epsilon H_1 &= H_0 + \frac{\partial S}{\partial t} \\ \implies \frac{\partial S}{\partial t} &= \epsilon H_1, \end{aligned}$$

but S also satisfies the following

$$\frac{\partial S}{\partial J} \approx \theta_0,$$

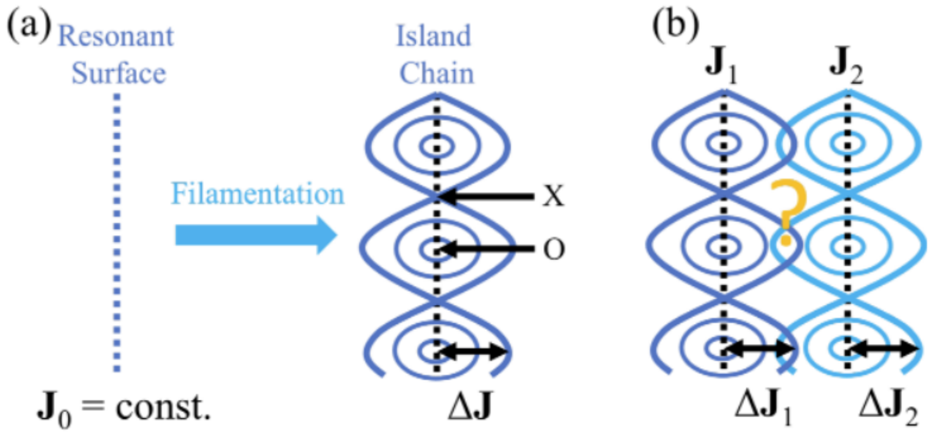

Figure 5: (a) Destruction of a single resonant surface and (b) overlapping multiple resonances.

therefore,

$$\frac{\partial\omega_0}{\partial J} = \frac{\partial}{\partial J}\frac{d}{dt}\frac{\partial S}{\partial J} \approx \frac{\partial^2}{\partial J^2}(\epsilon H_1) \approx \frac{\epsilon H_1}{\Delta J^2}$$
$$\implies \Delta J \approx \sqrt{\frac{\epsilon H_1}{\partial\omega_0/\partial J}}.$$

2.6 Multiple Resonant 2 Tori

Thus far, we've seen the fate of a single resonant surface. Now the next question is what happens when there are two resonant surfaces forming an overlapping island chains? As shown in figure 5 (b). From simple observation, one can conclude that a trajectory in the overlapping region will wander between different radius. It no longer belongs to a certain surface, but fills some volume in phase space. Therefore, this is a chaotic picture.

Chaos can be understood as the trajectory separation exhibits linear instability, which exponentially grows

$$\Delta\mathbf{J} = \mathbf{J} - \mathbf{J}_0 = \Delta\mathbf{J}_0 e^{\gamma t};\ \gamma > 0. \tag{36}$$

This is referred to as the "Lyapunov instability" where the exponent is the "Lyapunov exponent". To understand this, let's consider $\Delta\mathbf{J}_0 \to 0$. For a chaotic system, $\gamma > 0$ and implies that even an infinitesimal difference in initial conditions will diverge into considerable difference. The behavior of the system is highly sensitive to initial conditions, which is the definition of chaos. If there exists multiple resonances, chaos means that there's at least one positive Lyapunov exponent $\gamma_i > 0$.

For chaotic motion, we no longer have a deterministic solution of the Hamiltonian equations. Instead, we require a statistical approach for prediction/characterization of the system. A deterministic trajectory no longer exists, but a "probability density function" f can be defined as an alternative quantity to describe the motion. Thus, we can use the **Fokker-Planck equation** to find the evolution of f. One convenient way to simplify the Fokker-Planck problem is to use **quasilinear theory**. This assumes that the unperturbed trajectory is a good approximation to calculate the diffusion coefficient.

The quasilinear theory is concerned with describing and understanding the slow evolution of $\langle f \rangle$. The first and most applied approach to dermine spatial diffusion coefficients and other transport parameters is the so-called quasilinear theory (QLT). The quasilinear approximation is comparable to a first-order perturbation theory. Another way to put it, QLT is "mindless mean field theory". QLT is known to well describe stochastic trajectory divergence in standard map/magnetic field lines, even for static fields/fixed phases, Rechester et al., 1980. Generally, QLT encounters trouble for non-dispersive and weakly dispersive waves. Here is a question to keep in mind: can QLT equation be derived from Fokker-Planck theory? From the phase space continuity equation we can derive the QLT equation with diffusion coefficient,

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v} = 0. \tag{37}$$

The quasilinear diffusion coefficient is defined as,

$$\begin{aligned} D &= \sum_k \frac{q^2}{m^2} |E_k|^2 \left(\frac{|\gamma_{\parallel}|}{(\omega - kv)^2 + |\gamma_{\parallel}|^2} \right) \qquad (38) \\ &\cong \sum_k \frac{q^2}{m^2} |E_k|^2 \left(\pi \delta(\omega - kv) + \frac{|\gamma_{\parallel}|}{\omega_k^2} \right), \qquad (39) \end{aligned}$$

where the first term represents resonant diffusion and the second term represents non-resonant diffusion. The resonant diffusion is irreversible due to resonance overlap and it is rooted in particle stochasticity. Thus, resonant diffusion can be obtained from Fokker-Planck. On the other hand, non-resonant diffusion corresponds to "sloshing" motion energy of particles in wave. Thus, it is reversible. Due to reversibility, QLT cannot be derived from Fokker-Planck theory and the non-resonant diffusion may be viewed as a "fake diffusion". In general, the most appealing picture of plasma involves resonant particles and quasi-particles (i.e., waves). For further discussion on QLT, please see Diamond's notes on quasilinear theory and applications.

The quasilinear approach is only applicable in a limited regime. The first criterion is the **Chirikov overlap**, which states that the island chains of different resonances should "overlap". **Stochastic** field lines only appear when perturbation is large enough, which in terms of the action variable is given by

$$S_C \equiv \frac{\Delta J_1 + \Delta J_2}{J_1 - J_2} > 1. \tag{40}$$

Here 1, 2 denote the two neighbouring resonant surfaces and S_C is the Chirikov number. We can also express this in terms of the island width

$$S_C \equiv \frac{\Delta w_1 + \Delta w_2}{|r_2 - r_1|} > 1, \tag{41}$$

where r is the radius of circle on the intersection and w is the island width.

The second criterion that needs to be satisfied for the quasilinear regime is small Kubo number ($\mathrm{K_u} < 1$). The criterion here says that the random kicks are so often that the phase-space structure changes before a point has chance to bounce in the structure for once. This criterion states that using unperturbed trajectories to calculate the diffusion of trajectories is a good approximation, which is important when it comes to using quasilinear equations. In the next chapter we will discuss the Kubo number in full detail.

It should be mentioned that KAM theorem is concerned with ruggedness of **irrational** surfaces, the onset of chaos is concerned with **rational** surfaces.

3 Field Lines in a Torus

The discussion so far has been in regard to general Hamiltonian systems. Now I'll be discussing the trajectory of magnetic field lines in plasmas. Let $\mathbf{B}(\mathbf{x})$ denote the magnetic field vector. The field line trajectory equation gives a parametric function

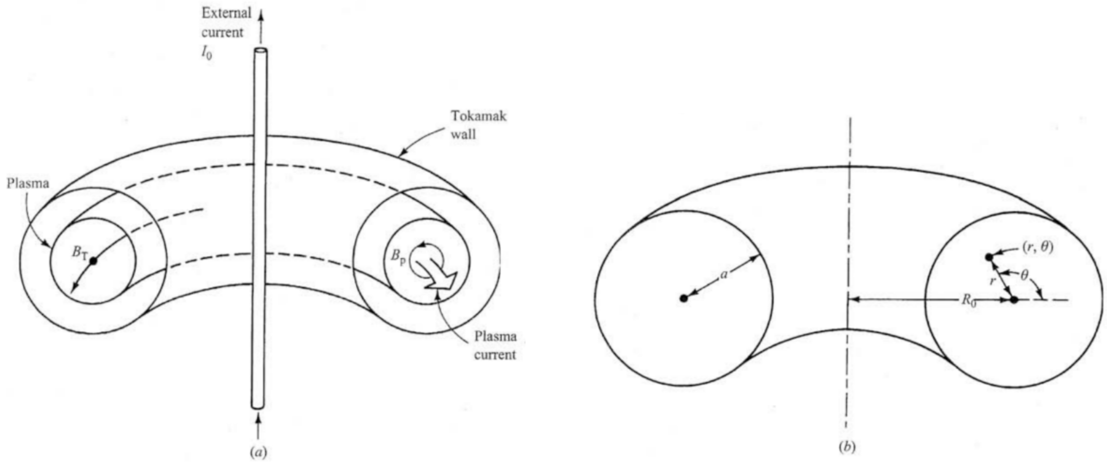

Figure 6: (a) Schematic illustration of a tokamak. (b) Toroidal coordinates.

$\mathbf{x}(s)$ for the curve on which a magnetic field line lies, where s is the parameter which we can think of as a measure of distance along the field line. The equation for $\mathbf{x}(s)$ is

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{B}(\mathbf{x}). \tag{42}$$

Since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, the equation above represents a conservative flow. Thus, the magnetic field lines in physical space are mathematically analogous to the trajectory of a dynamical system in its phase space.

Due to the nature of the Hamiltonian, this means that under many circumstances we can expect that some magnetic field line trajectories to fill up toroidal surfaces, while other field lines wander chaotically over a volume which may be bounded by tori. This is of great importance in controlled nuclear fusion where the fundamental problem is to confine a hot plasma for long enough time that sufficient energy-releasing nuclear fusion reactions take place. If the magnetic field is strong, the motion of the charged particles are constrained to follow the magnetic field lines.

The main point is that the magnetic field lines do not connect the plasma interior to the walls of the device. The example of such a configuration is provided by the tokamak device. For a schematic of the device see figure 6. Assuming that the configuration is perfectly symmetric with respect to rotations around the axis of the system, the superposition of the toroidal and poloidal magnetic fields leads to field lines that circle on a toroidal surface, simultaneously in both the toroidal and poloidal directions, filling the surface ergodically. The field lines are restricted to lie on a nested set of tori and never intersect bounding walls of the device. This is analogous to the case of an integrable Hamiltonian system.

Note that this is the situation if there is perfect toroidal symmetry, which is not the case in real life. Symmetry-breaking in magnetic field perturbations play a role analogous to nonintegrable perturbations of an integrable Hamiltonian system. Therefore, they can destroy some of the nested set of toroidal magnetic surfaces that exists in the symmetric case. If the perturbation is too strong, the chaotic field lines can wander from the interior of the plasma to the wall. This leads to rapid heat and particle loss of the plasma.

3.1 A Connection to Integrable System

Let's return to the unperturbed system and for simplicity let's consider cylindrical geometry (r, θ, z). In this system an external coil generates a constant toroidal magnetic field B_z and another coil with time-varying current induces a plasma current in the toroidal direction, thus, generating a poloidal magnetic field $B_\theta(r)$. In this case there is no radial magnetic field. The unperturbed magnetic field lines are curves winding on toroidal surfaces, which are also called magnetic surfaces. The winding rate of a field line can be found by the ratio between the poloidal and the toroidal field

$$\frac{rd\theta}{dz} = \frac{B_\theta(r)}{B_z}. \tag{43}$$

We can define the "safety factor" which is useful when dealing with MHD instabilities,

$$q(r) = \frac{d\phi}{d\theta} = \frac{dz}{Rd\theta} = \frac{rB_z}{RB_\theta}. \tag{44}$$

Here I will provide a review of the definition of an integrable Hamiltonian system with $N = 1$ degree-of-freedom (a circle) provided first in Feng-Jen Chang lecture notes. Let the action variable be x and the action angle variable be y. They should be canonically conjugate, thus, satisfying Hamilton's equations,

$$\frac{dx}{dt} = -\frac{\partial H}{\partial y}, \tag{45}$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial x}. \tag{46}$$

The Hamiltonian is not explicitly dependent on time so in this case it is a constant of motion,

$$H(x,y,t) = H(x(t),y(t)) \implies \frac{dH}{dt} = 0.$$

Now Liouville theorem states that for a Hamiltonian system the phase space volume is conserved. We can define the probability density function $f(x,y,t)$ of line density in phase space, thus, Liouville theorem can be expressed by the Liouville equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial t} - \frac{\partial H}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial f}{\partial y} = 0. \tag{47}$$

Now let's consider a perturbation by specifying $H(x,y) = H_0(x) + \tilde{H}(x,y)$. The Liouville equation becomes

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} - \frac{\partial H_0}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \tilde{H}}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial \tilde{H}}{\partial x}\frac{\partial f}{\partial y} \\ &= \frac{\partial f}{\partial t} - v_y(x)\frac{\partial f}{\partial y} - (\boldsymbol{\nabla}\tilde{H} \times \hat{z}) \cdot \boldsymbol{\nabla} f = 0.\end{aligned}$$

Now, let's check the equations for magnetic field lines in a tokamak. Recall, the incompressibility of the magnetic field ($\boldsymbol{\nabla} \cdot \mathbf{B} = 0$). The magnetic flux density ψ in any closed loop is conserved when the loop is convected in the magnetic field

$$(\mathbf{B} \cdot \boldsymbol{\nabla})\psi = 0. \tag{48}$$

For a tokamak the unperturbed field is of the form $\mathbf{B}_0 = B_0\hat{z} + B_\theta(r)\hat{\theta}$. Let's consider a perturbation term perpendicular to z

$$\mathbf{B} = B_0\hat{z} + B_\theta(r)\hat{\theta} + \tilde{\mathbf{B}}_\perp. \tag{49}$$

From here we can apply this to the conservation of magnetic flux

$$\begin{aligned}(\mathbf{B} \cdot \boldsymbol{\nabla})\psi = B_0\frac{\partial \psi}{\partial z} + \frac{B_\theta(r)}{r}\frac{\partial \psi}{\partial \theta} + \tilde{\mathbf{B}}_\perp \cdot \boldsymbol{\nabla}_\perp \psi = 0 \\ \implies \frac{\partial \psi}{\partial z} + \frac{B_\theta(r)}{rB_0}\frac{\partial \psi}{\partial \theta} + \frac{\tilde{\mathbf{B}}_\perp}{B_0} \cdot \boldsymbol{\nabla}_\perp \psi = 0 \\ \implies \frac{\partial \psi}{\partial z} + r\frac{\partial \theta}{\partial z}\frac{\partial \psi}{\partial \theta} + \frac{\tilde{B}_r}{B_0}\frac{\partial \psi}{\partial r} = 0.\end{aligned}$$

We can make a connection between the equation and that for the 1-D Hamiltonian by noting the following:

$$\begin{gathered}\psi \leftrightarrow f \\ z \leftrightarrow t;\ r \leftrightarrow x;\ rd\theta \leftrightarrow dy \\ \frac{\tilde{B}_r}{B_0} \leftrightarrow -(\boldsymbol{\nabla}\tilde{H} \times \hat{z}) \cdot \boldsymbol{\nabla}\end{gathered}$$

Note that here z plays the role of time. The winding rate corresponds to the angular velocity in an integrable Hamiltonian system:

$$r\frac{d\theta}{dz} = \frac{1}{Rq(r)} \leftrightarrow v_y(x) \leftrightarrow \omega(J)$$

Thus, the system of magnetic field lines in a tokamak is analogous to an integrable Hamiltonian system of $N = 1$.

Note that if we don't treat $z = R\phi$ as time but use $\mathbf{q} = (\theta, \phi)$ coordinate system, one can show that the magnetic field lines in a tokamak is a $N = 2$ integrable Hamiltonian system (2 torus). In this system, the condition of resonance is easier to understand

$$\begin{aligned} \mathbf{m} \cdot \boldsymbol{\omega} \equiv m\frac{d\theta}{dt} - n\frac{d\phi}{dt} &= 0 \\ \implies q(r) = \frac{d\phi}{d\theta} &= \frac{m}{n}, \end{aligned}$$

for some integers m and n. Here we have the following condition:

$$\text{Resonance} \Leftrightarrow q \in \text{rational number}.$$

3.2 Line Wandering

Let's now consider the field line density f, which is analogous to ψ the magnetic flux density. Here f is analogous to the probability density function in Hamiltonian systems, which is the number of trajectories penetrating through a unit area in phase space. Again, we use the incompressibility of the magnetic field ($\boldsymbol{\nabla} \cdot \mathbf{B} = 0$). This is analogous to Liouville theorem for Hamiltonian systems,

$$\begin{aligned} \frac{\partial f}{\partial z} + \frac{B_\theta(r) + \tilde{B}_\theta}{rB_0}\frac{\partial f}{\partial \theta} + \frac{\tilde{B}_r}{B_0}\frac{\partial f}{\partial r} &= 0 \\ \implies \frac{\partial f}{\partial z} + \frac{B_\theta(r)}{rB_0}\frac{\partial f}{\partial \theta} + \frac{\partial}{\partial \theta}\left(\frac{\tilde{B}_\theta}{rB_0}f\right) + \frac{\partial}{\partial r}\left(\frac{\tilde{B}_r}{B_0}f\right) - f\left(\frac{1}{r}\frac{\partial \tilde{B}_\theta}{\partial \theta} + \frac{\partial \tilde{B}_r}{\partial r}\right) &= 0, \end{aligned}$$

The last term on the LHS vanishes and we get

$$\frac{\partial f}{\partial z} + \frac{\partial}{\partial \theta}\left(\frac{B_\theta(r) + \tilde{B}_\theta}{rB_0}f\right) + \frac{\partial}{\partial r}\left(\frac{\tilde{B}_r}{B_0}f\right) = 0. \tag{50}$$

Now let us assume $f = \langle f \rangle + \tilde{f}$, where $\langle \cdots \rangle$ denote the average along the θ direction. By averaging the equation above in θ we can drop the second term since the system has

Figure 7: Small Kubo number means field lines experience many "random kicks" when traveling an entire coherence length and large Kubo number means strong scattering and long correlation.

periodicity in θ. However, although the fast perturbations terms $(\widetilde{\ldots})$ are periodic in the z direction, the average density $\langle f\rangle$ might not be. The reason for this is because z now plays the role of time here. As one travels through the field lines, some irreversible changes might take place in the system. Note that the fast perturbation term $\tilde{B}_r$ vanishes, thus,

$$\frac{\partial\langle f\rangle}{\partial z} + \frac{\partial}{\partial r}\left\langle \frac{\tilde{B}_r}{B_0}\tilde{f}\right\rangle = 0. \tag{51}$$

Now let's recall the expression for Fick's law:

$$\Gamma_{r,B} = \left\langle \frac{\tilde{B}_r}{B_0}\tilde{f}\right\rangle = -D\boldsymbol{\nabla} f, \tag{52}$$

where $\Gamma_{r,B}$ represents the flux line density. We can then apply it to eq. 51 to get the following equation,

$$\frac{\partial\langle f\rangle}{\partial z} + \frac{\partial \Gamma_{r,B}}{\partial r} = 0. \tag{53}$$

Due to the radial perturbation $\tilde{B}_r$, will have field lines that can move (i.e., wander) in the r direction, thus, leaving their original flux surface. We can estimate the displacement δr by using equations of lines

$$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_z}, \tag{54}$$

hence,

$$\delta r \approx \int_0^l \frac{\tilde{B}_r}{B_0} dz. \tag{55}$$

Now, line trajectory de-coheres from perturbation for $l > l_{ac}$ where l_{ac} is the auto-correlation length. l_{ac} can be estimated by the z-direction spectrum $k_\parallel$ of the fluctuation $\tilde{B}_r$,

$$l_{ac} \approx \frac{1}{|\Delta k_\parallel|} \tag{56}$$

this is also the inverse spatial bandwidth. We can express now δr by l_{ac}:

$$\delta r \approx l_{ac} \frac{\tilde{B}_r}{B_0}, \tag{57}$$

this is the excursion of field lines in one l_{ac}.

Recall that $\tilde{B}_r$ also has a radial dependence. This implies that even if the structure of B_r didn't change with time, the motion of field lines in the radial direction might change due the change in $\tilde{B}_r$. Such "radial correlation length" Δr is the radial correlation length of the "scatterer". The ratio between δr and Δr is important, since the wandering of magnetic field lines is governed by different mechanisms. This ratio is the **Kubo number** ($\mathrm{K_u}$),

$$\mathrm{K_u} \equiv \frac{\delta r}{\Delta r} = \frac{\tilde{B}_r}{B_0} \frac{l_{ac}}{\Delta r}. \tag{58}$$

For $\mathrm{K_u} < 1$, we have $\delta r < \Delta r$, thus, when moving in one entire coherence length of $\tilde{B}_r$, $\tilde{B}_r$ will have changed for many times. Since $\tilde{B}_r$ is a random fluctuation, this means that the field lines have experienced many "random kicks" when traveling in an entire coherence length in radial direction. This leads to a **diffusion** process of radial wandering. On the other hand, if $\mathrm{K_u} > 1$, then $\delta r > \Delta r$, the field lines are influenced by the same $\tilde{B}_r$ structure without being disturbed by random kicks. Hence, the field lines will experience strong scattering. In this scenario we have for example, a percolation picture, which will not be discussed here. In figure 7, it shows the dynamics of both small and large Kubo number.

To conclude, for $\mathrm{K_u} \sim 1$ this is referred to as the "natural state of EM turbulence" in Kadomtsev and Pogutse, 1977. In the regime of $\mathrm{K_u} = 1$ we have the critical balance. Critical balance indicates that the linear term ($\partial/\partial t$ or $B_0\ \partial/\partial z$) and the non-linear term ($\tilde{\mathbf{v}} \cdot \boldsymbol{\nabla}$ or $\tilde{\mathbf{B}} \cdot \boldsymbol{\nabla}$) has the same strength.

For the rest of this paper we will discuss the diffusive regime of $\mathrm{K_u} < 1$ and calculate the diffusion coefficient of magnetic field lines.

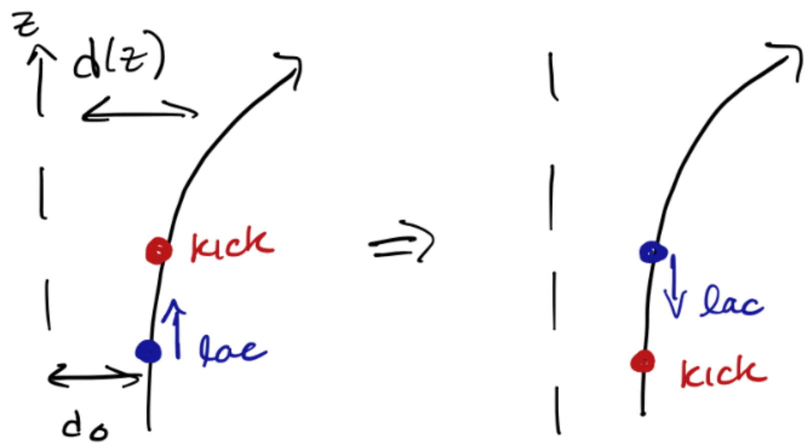

Figure 8: Physical picture of auto-correlation length, l_{ac}. Here l_{ac} represents the coherence or the memory line length for scattering fields.

3.3 Diffusion of Magnetic Field Lines

To derive the diffusion of magnetic field lines we proceed via quasilinear theory. For this to be valid we will work in the regime of $\mathrm{K_u} < 1$ and $S_C > 1$. In this regime, field line distribution is stochastic and has no memory about its history. Here will take the Fourier series expansion of both equation 51 and 52:

$$-i\left(k_z - k_\theta \frac{B_\theta}{B_0}\right)\tilde{f}_{\mathbf{k}} + \tilde{B}_{r,\mathbf{k}}\frac{\partial\langle f\rangle}{\partial r} = 0. \tag{59}$$

We then solve for f and get the following,

$$\tilde{f}_{\mathbf{k}} = \frac{-i}{k_z - k_\theta \frac{\langle B_\theta\rangle}{B_0}} \frac{\tilde{B}_{r,\mathbf{k}}}{B_0}\frac{\partial\langle f\rangle}{\partial r}. \tag{60}$$

Now, we can expand the density flux equation,

$$\Gamma_M = \left\langle \sum_{\mathbf{k}',\mathbf{k}} \frac{B_{r,\mathbf{k}'}}{B_0}\tilde{f}_{\mathbf{k}} \right\rangle \tag{61}$$

$$= -i\left\langle \sum_{\mathbf{k}',\mathbf{k}} \left(\frac{\tilde{B}_{r,\mathbf{k}'}\tilde{B}_{r,\mathbf{k}}}{B_0^2}\right) \frac{1}{k_z - k_\theta \frac{B_\theta}{B_0}} \right\rangle \frac{\partial\langle f\rangle}{\partial r} \tag{62}$$

$$\equiv -D_M \frac{\partial\langle f\rangle}{\partial r}. \tag{63}$$

Now, working with the magnetic diffusivity we take the average along θ and z direction and normalize via corresponding periods

$$D_M = \sum_{\mathbf{k}} \left| \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \right|^2 \pi\delta\left(k_z - k_\theta \frac{B_\theta}{B_0}\right). \tag{64}$$

Recall the winding rate of field lines

$$\frac{B_\theta}{B_0} = \frac{rd\theta}{dz} = \frac{k_z}{k_\theta}, \tag{65}$$

thus, implying

$$k_z - k_\theta \frac{B_\theta}{B_0} = 0. \tag{66}$$

Hence, we can rewrite the magnetic diffusion as

$$D_M = \sum_{\mathbf{k}} \left| \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \right|^2 \pi\delta(k_\parallel). \tag{67}$$

Let $\delta(k_\parallel) = 1/\Delta k_\parallel$, where $\Delta k_\parallel$ corresponds to the smallest difference in $k_\parallel$. We can say $1/\Delta k_\parallel \propto l_{ac}$ is a reasonable approximation. Thus, we have the following expression for magnetic diffusivity,

$$D_M \approx \left\langle \left(\frac{\delta B_r}{B_z}\right)^2 \right\rangle l_{ac}. \tag{68}$$

For the reader who is interested in details of the derivation of D_M, please see Rosenbluth et al., 1966 for a rigorous derivation of the result.

The question that remains is what is l_{ac}? To estimate what l_{ac} we have a spatial scale of spectral width (Δr) sets $|k_\parallel| \sim \left|\frac{k_\theta \Delta r}{L_S}\right|$, thus we have

$$l_{ac} \approx \frac{L_S}{|k_\theta|\Delta r}. \tag{69}$$

This is the parallel correlation length of the fluctuation field. Above l_{ac} the fluctuation structure changes, so that the field lines experience “kicks” or “scattering”. Therefore, l_{ac} can also be viewed as the “memory length” of field lines.

We can also solve for a “de-correlation length” l_c in the z direction, over which the field lines is scattered from its unperturbed trajectory. We can estimate a scaling by first noting

$$r\frac{d\theta}{dz} = \frac{dy}{dz} = \frac{B_\theta(r_0)}{B_0}, \tag{70}$$

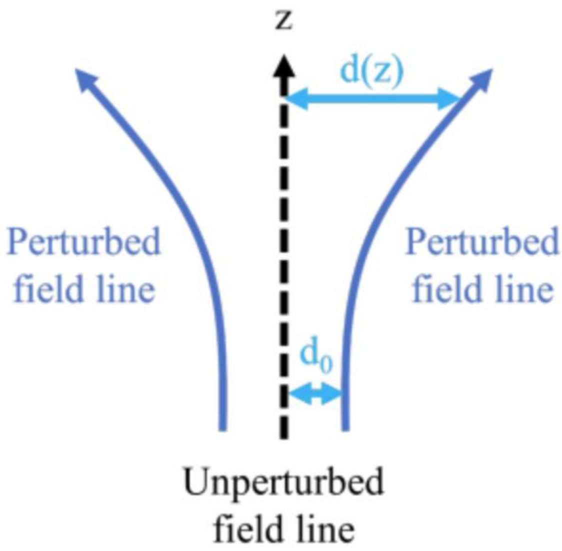

Figure 9: Physical picture of de-correlation showing the separation of field lines by "stretching" them in the perpendicular direction. The effect of field line stretching is $\propto e^{z/l_c}$ where l_c is the characteristic length in the z direction. This effect is also called stochastic instability, which is the Lyapunov instability which gives rise to chaos in the system.

here B_θ has perturbation. We denote the effect of radial wandering by δ so that:

$$\frac{dy}{dz} = \frac{\mathbf{B}_\theta(r_0)}{B_0} + \frac{1}{B_0}\left[\frac{\partial \tilde{B}_\theta(r_0)}{\partial r}\delta r\right]$$

$$\implies \frac{d(\delta y)}{dz} \approx \frac{1}{B_0}\left[\frac{\partial \tilde{B}_\theta(r_0)}{\partial r}\delta r\right],$$

we can then integrate and average to get

$$\langle \delta y^2 \rangle = \frac{1}{B_0^2}\tilde{B}_\theta'^2 Z^2 \langle \delta r^2 \rangle.$$

Then from quasilinear diffusion equation

$$\langle \delta r^2 \rangle \approx D_M Z$$

$$\implies \langle \delta y^2 \rangle \approx \frac{\tilde{B}_\theta'^2}{B_0^2} D_M Z^3.$$

From 1-D vlasov diffusion

$$\langle \delta x^2 \rangle = D_v \frac{T^3}{3}. \tag{71}$$

hence,

$$\langle \delta y^2 \rangle = \frac{\tilde{B}_\theta'^2}{3B_0} D_M Z^3. \tag{72}$$

Now for orbit de-correlation length:

$$k_\theta^2 \langle \delta y^2 \rangle \sim \frac{k_\theta^2 B_\theta'^2}{3B_0^2} D_M Z^3$$

which results in

$$l_c \sim \left(k_\theta^2 \frac{B_\theta'^2}{3B_0^2} D_M \right)^{-1/3} \sim \left(\frac{k_\theta^2}{L_S^2} \frac{D_M}{3} \right)^{-1/3}. \tag{73}$$

where we have the definition of magnetic shear length:

$$\frac{1}{L_S} = \left| \frac{1}{B_0} \frac{\partial \tilde{B}_\theta}{\partial r} \right|. \tag{74}$$

For the quasilinear theory to be valid, we need the unperturbed trajectory to be a good approximation. This means that the field lines will not have a chance to deviate from its unperturbed trajectory before being "kicked" by de-coherence in the z direction. This criterion is stated

$$\mathrm{K_u} \equiv \frac{\delta r}{\Delta r} = \frac{\tilde{B}_r}{B_0} \frac{l_{ac}}{\Delta r} < 1.$$

When the field lines deviate from its unperturbed trajectory we have

$$\langle \delta r^2 \rangle \approx D_M l_c. \tag{75}$$

The radial displacement can be viewed as the result of field lines wandering in the radial direction due to the radial fluctuation $\tilde{B}_r$

$$\sqrt{\langle \delta r^2 \rangle} \approx \frac{\tilde{B}_r}{B_0} l_c$$
$$\Longrightarrow \frac{\tilde{B}_r}{B_0} \approx \sqrt{\frac{D_M}{l_c}},$$

thus,

$$\mathrm{K_u} \approx \sqrt{\frac{D_M k_\theta^2}{L_S^2 l_c} l_{ac}^2} < 1.$$

From here we can apply l_c to get the following criterion

$$\mathrm{K_u} \approx \left(\frac{l_{ac}}{l_c}\right) < 1, \tag{76}$$

which implies that

$$l_{ac} < l_c, \tag{77}$$

is the criterion for the quasilinear theory to be valid.

The most common form of transport is collisional transport. The characteristic length of collision is the mean-free path l_{mpf}. In a tokamak, charged particles move helically around the field line, the deviation of field lines from their unperturbed trajectory also causes transport. The relation between the de-correlation length l_c and the mean-free path l_{mfp} gives us different regimes:

$$l_{ac} < l_c < l_{mfp} \rightarrow \textbf{collisionless} \text{ regime.}$$
$$l_{ac} < l_{mfp} < l_c \rightarrow \textbf{collisional} \text{ regime.}$$

4 Transport in Stochastic Fields ($\mathbf{K_u} < 1$)

So far we've discussed the diffusion of field lines, but in reality no one cares about "line" diffusion. So one might ask why then do this? We do this because people such as experimentalist care about the transport of particles, momentum, and heat.

In this chapter, we will discuss perpendicular heat transport of electrons for both the collisionless and collisional regimes. The key points that we hope the reader gets from this chapter is (1) how irreversibility is generated and (2) the interacting processes. A must read for this section is the article by Rechester and Rosenbluth, 1978.

4.1 Collisionless Transport

In this regime we can imagine that the perpendicular electron heat transport is mainly contributed by the wandering of field lines rather than the perpendicular diffusion of electrons. The thermal diffusivity χ is given to us by

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T, \tag{78}$$

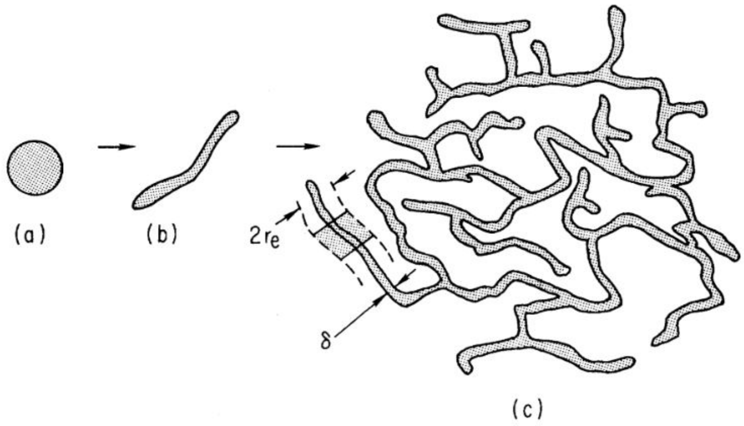

Figure 10: The evolution of area mapping.

where T is the temperature. We can imagine that the collisionless $\chi_\perp$ will be directly proportional to the quasilinear diffusion coefficient of field lines D_M and the thermal velocity of electrons v_{th},

$$\chi_\perp \approx v_{th} D_M, \tag{79}$$

but is it this simple? Let's consider a thought experiment. Let's assume parallel collisions (only) happen (i.e., particle stays on line). Therefore, the motion along the line is diffusive

$$\delta z^2 \approx D_\parallel t \approx \chi_\parallel t, \tag{80}$$

where $D_\parallel$ is the parallel particle diffusion coefficient and $\chi_\parallel$ is the parallel thermal diffusion. For the heat transport in perpendicular direction we have

$$\langle \delta r^2 \rangle \approx D_M \delta z \approx D_M (\chi_\parallel t)^{1/2}. \tag{81}$$

The perpendicular thermal diffusivity is then

$$\chi_\perp \equiv \frac{d\langle \delta r^2 \rangle}{dt} \approx D_M \sqrt{\frac{\chi_\parallel}{t}}, \tag{82}$$

which goes to zero when $t \to \infty$. This then tells us that we won't get perpendicular thermal transport. The problem here is that particles get "kicked back" by collisions along the line. These kicks cause the particle motion along the perturbed field line to be a diffusive process. Thus, electrons won't really travel along field lines to anywhere far away. Then the effect of stochastic instability is not able to take place, thus, no net

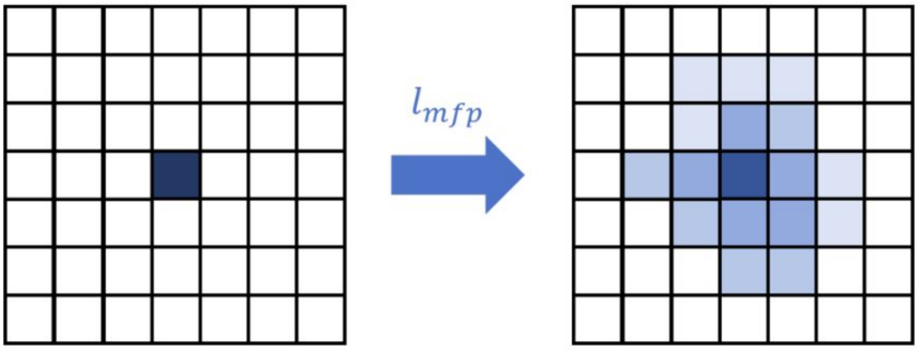

Figure 11: Effect of coarse graining on particle density function.

radial wandering of particles. This implies that we need **irreversibility** for the particle motion so that they won't get kicked back. Here **collisions control irreversibility**. The key point is that particles need to get kicked off the field lines.

Now we ask what is the mechanism of this perpendicular kicking? Recall that isotropically thermalized electrons also have perpendicular velocity, so that they move helically around the field lines. Thus, there is uncertainty in the perpendicular position due to the gyro-motion. This "smears" the electron location in a circle of electron gyro-radius ρ_e on the perpendicular plane. Here we can define the "minimum resolution scale" of electron location on the perpendicular plane. This is referred to as **coarse graining**.

To further understand we will use the diagram of figure 10. In the figure, the plane perpendicular to the field lines, the electron motion is smeared on a disk with radius ρ_e. As electrons travel in the parallel direction within one l_{mfp}, which means the longest range without parallel collision, the field line deviates from its unperturbed trajectory. This leads to the deformation of the disk. The length of it increases due to stochastic instability

$$l \sim \rho_e e^{l_{mfp}/l_c}.$$

Now recall that $\nabla \cdot \mathbf{B} = 0$ preserves the area of it so the width of the area becomes

$$w \sim \rho_e e^{-l_{mfp}/l_c},$$

Thus, the original disk is deformed into a more complicated contour as shown in figure 10.

Note that coarse graining also occurs in the new contour (see figure 11). If we divided the plane into cells of minimal resolution, after travelling in one l_{mfp}, the particle density function is redistributed onto the nearest cells. The total density function is given as

$$A_{CG}\bar{f} = Af_0. \tag{83}$$

Ludwig Boltzmann assures us that there is no memory between steps (1 l_{mfp}/collision time). So initial spot **expands**, with random walk, as

$$\langle \delta r^2 \rangle \sim D_M l_{mfp}, \tag{84}$$

in one l_{mfp}. In other words, the coarse graining interval sets $\langle \delta r^2 \rangle$ step!

Next, we estimate the collisionless thermal diffusivity in a stochastic field:

$$\chi_\perp \sim \frac{\langle \delta r^2 \rangle}{\tau_c} \sim D_M \frac{l_{mfp}}{\tau_c},$$

which results in

$$\chi_\perp \sim v_{th} D_M. \tag{85}$$

We find that the diffusivity is independent of collisionality. However, the mechanism is clearly dependent on collisions and coarse graining. The lesson here is that coarse graining is essential to irreversibility. In other words, coarse graining is essential to kick particle off field line or else collisional back-scattering reverses wandering.

Here are some suggested **exercises** by Professor Diamond:

1. Derive the magnetic diffusivity with magnetic drifts. How do these modify D_M? Explain why high energy particles (runaways) are confined longer than thermal ones. **For solution to this problem, please see chapter 5.**

2. Formulate the theory of diffusion due to stochastic fields in toroidal geometry using ballooning mode formalism for the fluctuations.

3. What happens to net cross field transport in a standing spectrum of electrons and magnetic perturbations. When might transport vanish? Why?

4.2 Collisional Transport

Let's now consider the transport in collisional regime. In this case we have the following scales, $l_{ac} < l_{mfp} < l_c$. We've shown that radial transport does not happen if there is only collision. What causes radial transport is perpendicular spread of particle trajectory due to coarse graining in stochastic field. From the quasilinear equation we have the perpendicular spread

$$\langle \delta r^2 \rangle \approx D_M l_{c,\delta}, \tag{86}$$

where $l_{c,\delta}$ is the parallel correlation length (δ signifies diffusive regime). A point to make is that $l_{mfp} < l_c$, is when the parallel collision take place such that irreversibility has already been produced by coarse graining within l_c. The parallel motion is diffusive. Here the $l_{c,\delta}$ that sets irreversibility here must be a longer length compared to l_{mfp}. Within this length scale particles experience many collisions, so that the motion is diffusive, but the time is set by

$$\begin{aligned} \frac{\chi_\parallel}{t} &\sim \frac{1}{t} \\ \implies \frac{\langle \delta r^2 \rangle}{t} &\sim \frac{\chi_\parallel}{l_{c,\delta}^2} D_M l_{c,\delta} \sim D_M \frac{\chi_\parallel}{l_{c,\delta}}. \end{aligned}$$

Thus, the perpendicular diffusivity is

$$\chi_\perp = D_M \frac{\chi_\parallel}{l_{c,\delta}}. \tag{87}$$

Now what is $l_{c,\delta}$? Notice that $l_{c,\delta}$ is set by competition between two processes: 1) width δ increases due to diffusion (coarse graining) so

$$(d\delta)^2 \sim D_\perp dt \implies d\delta \sim (D_\perp dt)^{1/2},$$

but,

$$\frac{\chi_\parallel}{(dL)^2} \sim \frac{1}{dt} \implies d\delta \sim \left(\frac{D_\perp}{\chi_\parallel} (dL)^2 \right)^{1/2},$$

therefore,

$$d\delta \sim \left(\frac{D_\perp}{\chi_\parallel} \right)^{1/2} dL. \tag{88}$$

Now 2) width shrinks due to stochastic instability and area conservation:

$$\frac{d\delta}{dL} = -\frac{\delta}{l_c}, \tag{89}$$

then balance at

$$d\delta \sim \left(\frac{D_\perp}{\chi_\parallel}\right)^{1/2} dL \sim \frac{\delta}{l_c} dL,$$

so

$$\delta \sim l_c \left(\frac{D_\perp}{\chi_\parallel}\right)^{1/2}. \tag{90}$$

It should be mentioned that this length scale can also be derived from thermal energy conservation:

$$\begin{aligned} \frac{\partial T}{\partial t} - \chi_\parallel \nabla_\parallel^2 T - D_\perp \nabla_\perp^2 T = 0 \\ \implies \frac{\chi_\parallel}{l_c^2} \sim \frac{D_\perp}{\delta^2} \\ \delta \sim l_c \left(\frac{D_\perp}{\chi_\parallel}\right)^{1/2}. \end{aligned}$$

Finally, need correlation length $l_{c,\delta}$ for chunk size δ. We can assume that this is set by k_θ ($\delta_c \approx 1/|k_\theta|$):

$$\begin{aligned} \frac{\delta_c}{\delta} = e^{l_{c,\delta}/l_c} \\ \implies l_{c,\delta} = l_c \ln\left(\frac{\delta_c}{\delta}\right) = l_c \ln\left(\frac{1}{|k_\theta| l_c}\sqrt{\frac{\chi_\parallel}{D_\perp}}\right). \end{aligned}$$

The log function is not a strong dependence function, thus, we can drop it to have

$$l_{c,\delta} \approx l_c, \tag{91}$$

which is expected because l_c is the de-correlation length in collisionless regime, at which irreveresibility is generated. We can now plug into the equation for $\chi_\perp$

$$\chi_\perp = D_M \frac{\chi_\parallel}{l_{c,\delta}} \approx D_M \frac{\chi_\parallel}{l_c} \approx v_{th} D_M \left(\frac{l_{mfp}}{l_c}\right),$$

where $l_{mfp}/l_c < 1$. Let's compare this result with the collisionless thermal diffusivity:

$$\frac{\chi_{\perp,\text{collisional}}}{\chi_{\perp,\text{collisionless}}} \approx \left(\frac{l_{mfp}}{l_c}\right)_{\text{collisional}} < 1.$$

Here are some of the lessons we learned:

1. Collisions reduce $\chi_\perp$ by (l_{mfp}/l_c) from the "collisionless" case. One should remember that even the collisionless transport requires collisions.

2. The collisional heat transport is the interplay of perpendicular and parallel diffusions.

3. Same as the collisionless case, it is critical to knock particles off field lines to produce irreversibility.

4.3 A Hydrodynamic Approach

The calculation performed in the last section requires **thought**, but it is much more convenient to crank mindlessly. There is an alternative approach using Hydrodynamics introduced by Kadomtsev and Pogutse, which is not mindless but more systematic. In this approach, let's consider heat flux along wiggling fields

$$\mathbf{q} = -\chi_\parallel \nabla_\parallel T \hat{b} - \chi_\perp \boldsymbol{\nabla}_\perp T, \tag{92}$$

where $\hat{b}$ is the unit vector along the field direction. The first term in the RHS of the equation is the parallel heat conduction and the second term is the perpendicular heat conduction. In this setup, we include the perturbation of the field

$$\mathbf{b} = \mathbf{b}_0 + \tilde{\mathbf{b}}. \tag{93}$$

We define the z axis to be along the unperturbed field direction, so we have

$$\nabla_\parallel = \frac{\partial}{\partial z} + \tilde{\mathbf{b}} \cdot \boldsymbol{\nabla}_\perp. \tag{94}$$

The wiggling of field lines contributes to the perpendicular transport. Now let's plug this into the equation for $\mathbf{q}$ and average the radial heat flux, we have the following

$$\langle q_r \rangle = -\chi_\parallel \left\langle \tilde{\mathbf{b}}_r^2 \frac{\partial \langle T \rangle}{\partial r} \right\rangle - \chi_\parallel \left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z} \right\rangle - \chi_\parallel \left\langle \tilde{b}_r \tilde{b}_r \frac{\partial \tilde{T}}{\partial r} \right\rangle - \chi_\perp \boldsymbol{\nabla}_r \langle T \rangle. \tag{95}$$

Let's denote the following terms as (1), (2), (3), and (4), respectively. Term (1) and term (2) are the usual quadratic terms of perturbation, term (4) is the perpendicular heat conduction, but now a cubic term (3) arises. Let's take the ratio of (3) and (2) to see its influence

$$\frac{(3)}{(2)} \approx \frac{\chi_\parallel \tilde{b}_r \tilde{b}_r \tilde{T}/\Delta_r}{\chi_\parallel \tilde{b}_r \tilde{T}/l_{ac}} = \tilde{b}_r \frac{l_{ac}}{\Delta_r} = \frac{\tilde{B}_r}{B_0} \frac{l_{ac}}{\Delta r} = \frac{\delta r}{\Delta r} = \mathrm{K_u}.$$

The cubic non-linearity clearly then does not dominate in the regime of small Kubo number. Therefore, we can drop term (3).

To compute $\langle q_r \rangle$ we need to retain (1), (2), and iterate $\tilde{T}$ using $\boldsymbol{\nabla} \cdot \mathbf{q} = 0$ (i.e., ala QLT)

$$\langle q_r \rangle \approx -\chi_\parallel \left[\left\langle \tilde{b}_r^2 \right\rangle \frac{\partial T}{\partial r} + \left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z} \right\rangle \right] - \chi_\perp \nabla_r \langle T \rangle. \tag{96}$$

We can apply linearization to $b_r \partial_r \langle T \rangle + \partial_z \tilde{T}$. Hence,

$$\langle q_r \rangle \approx -\chi_\parallel \left[\left\langle \tilde{b}_r \widetilde{\mathbf{b} \cdot \boldsymbol{\nabla} T} \right\rangle \right] - \chi_\perp \nabla_r \langle T \rangle, \tag{97}$$

where we need

$$\widetilde{\mathbf{b} \cdot \boldsymbol{\nabla} T} \neq 0, \tag{98}$$

to drive net heat flux $\langle q_r \rangle \neq 0$. In other words, **to drive parallel heat flux, temperature can't be constant along the field line**. Thus, $\boldsymbol{\nabla} \cdot \mathbf{q} = 0$. The result imply $\chi_\perp$ dependence to balance the heat flux, so

$$\langle q_r \rangle \approx -\chi_\parallel \left[\left\langle \tilde{b}_r^2 \right\rangle \frac{\partial T}{\partial r} + \left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z} \right\rangle \right]. \tag{99}$$

Now consider the total heat flux

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{q} &= 0 \\ \implies \nabla_\parallel \tilde{q}_\parallel + \boldsymbol{\nabla}_\perp \cdot \tilde{\mathbf{q}}_\perp &= -\chi_\parallel \frac{\partial}{\partial z} \left(\tilde{b}_r \frac{\partial \langle T \rangle}{\partial r} \right), \end{aligned}$$

in other words,

$$\mathbf{q} = -\chi_\parallel \left[\left(\frac{\partial}{\partial z} + \tilde{\mathbf{b}} \cdot \boldsymbol{\nabla} \right) (T_0 + \tilde{T})(\mathbf{b}_0 + \tilde{\mathbf{b}}) \right] - \chi_\perp \nabla_\perp T. \tag{100}$$

Next, we insert the expression for $\mathbf{q}$ into the total heat flux equation

$$\begin{aligned} \boldsymbol{\nabla} \cdot \left(-\chi_\parallel \left[\left(\frac{\partial}{\partial z} + \tilde{\mathbf{b}} \cdot \boldsymbol{\nabla} \right) (T_0 + \tilde{T})(\mathbf{b}_0 + \tilde{\mathbf{b}}) \right] - \chi_\perp \nabla_\perp T \right) &= 0 \\ \implies -\chi_\parallel \boldsymbol{\nabla} \cdot \left(\left(\partial_z \tilde{T} \right) \mathbf{b}_0 + \left(\tilde{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \right) T_0 \boldsymbol{b}_0 \right) - \chi_\perp \nabla_\perp^2 T &= 0, \end{aligned}$$

hence,

$$-\chi_\parallel \frac{\partial^2 \tilde{T}}{\partial z^2} - \chi_\perp \nabla_\perp^2 \tilde{T} = -\chi_\parallel \frac{\partial}{\partial z} \left(\tilde{b}_r \frac{\partial \langle T \rangle}{\partial r} \right). \tag{101}$$

Next, we take the spatial Fourier series expansion of the equation

$$\tilde{T}_{\mathbf{k}} = -\frac{\chi_{\parallel} i k_z \tilde{b}_{\mathbf{k}}\, \partial\langle T\rangle/\partial r}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2}. \tag{102}$$

Now we apply this to term (1) and (2)

$$\begin{aligned}
&-\chi_{\parallel}\left\langle \tilde{b}^2\right\rangle \frac{\partial\langle T\rangle}{\partial r} - \chi_{\parallel}\left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z}\right\rangle \\
&= -\chi_{\parallel}\sum_{\mathbf{k}}\left(-\frac{\chi_{\parallel} k_{\parallel}^2 \left|\tilde{b}_{\mathbf{k}}\right|^2}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2} + \left|\tilde{b}_{\mathbf{k}}\right|^2\right)\frac{\partial\langle T\rangle}{\partial r} \\
&= -\chi_{\parallel}\frac{\partial\langle T\rangle}{\partial r}\sum_{\mathbf{k}}\left(\frac{-\chi_{\parallel} k_{\parallel}^2 + \chi_{\parallel} k_{\parallel}^2 + \chi_{\perp} k_{\perp}^2}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2}\right)\left|\tilde{b}_{\mathbf{k}}\right|,
\end{aligned}$$

Thus, we have:

$$\langle q_r\rangle_{\mathrm{NL}} = -\chi_{\parallel}\frac{\partial\langle T\rangle}{\partial r}\sum_{\mathbf{k}}\frac{\chi_{\perp} k_{\perp}^2 |b_{\mathbf{k}}|^2}{\chi_{\parallel} k_{\parallel}^2 + \chi_{\perp} k_{\perp}^2}. \tag{103}$$

Note the **explicit dependence** on $\chi_{\perp}$! The expression of $\langle q_r\rangle_{\mathrm{NL}}$ tells us the importance of coarse graining in perpendicular heat transport.

Now let's replace the summation by integration,

$$\begin{aligned}
\langle q_r\rangle_{\mathrm{NL}} &\approx -\chi_{\parallel}\frac{\partial\langle T\rangle}{\partial r}\int d\mathbf{k}_{\perp}\int dk_z \frac{\chi_{\perp} k_{\perp}^2 |b_{\mathbf{k}}|^2}{\chi_{\parallel}\left(k_{\parallel}^2 + \chi_{\perp} k_{\perp}^2/\chi_{\parallel}\right)} \\
&= -\frac{\partial\langle T\rangle}{\partial r}\int d\mathbf{k}_{\perp}\int dk_z \frac{\chi_{\perp} k_{\perp}^2 |b_{\mathbf{k}}|^2}{\left(\frac{k_z^2}{(\chi_{\perp}/\chi_{\parallel})k_{\perp}^2} + 1\right)\left(\frac{\chi_{\perp}}{\chi_{\parallel}} k_{\perp}^2\right)} \\
&= -\frac{\partial\langle T\rangle}{\partial r}\int d\mathbf{k}_{\perp}\frac{k_{\perp}^2(\chi_{\parallel}\chi_{\perp})^{1/2}}{\sqrt{k_{\perp}^2}}\left|\tilde{b}_{\mathbf{k}}\right|^2 l_{ac},
\end{aligned}$$

auto-correlation l_{ac} enters via normalization

$$\langle q_r\rangle_{\mathrm{NL}} \approx -\sqrt{\chi_{\parallel}\chi_{\perp}}\left\langle \tilde{b}^2\right\rangle l_{ac}\left\langle \sqrt{k_{\perp}^2}\right\rangle \frac{\partial\langle T\rangle}{\partial r}, \tag{104}$$

where $\sqrt{\chi_{\parallel}\chi_{\perp}}$ represents Bohm diffusion and $\left\langle \tilde{b}^2\right\rangle l_{ac}$ is magnetic diffusion. Note,

1. Need $\nabla_{\parallel}\tilde{T} \neq -\tilde{b}_r$ $(\widetilde{\mathbf{B}\cdot\boldsymbol{\nabla}T} \neq 0)$ for perpendicular heat flux.

2. $\langle \tilde{b}^2 \rangle l_{ac} \approx D_M$

3. $\sqrt{k_\perp^2} \approx 1/\Delta_\perp$

so,

$$\langle q_r \rangle \approx -\chi_{\perp,\mathrm{eff}} \frac{\partial \langle T \rangle}{\partial r} - \chi_\perp \frac{\partial \langle T \rangle}{\partial r}, \tag{105}$$

where

$$\chi_{\perp,\mathrm{eff}} \approx \sqrt{\chi_\parallel \chi_\perp} \frac{D_M}{\Delta_\perp}, \tag{106}$$

and

$$\chi_\parallel \chi_\perp \approx \frac{v_{the}^2}{\gamma} \rho_e^2 \gamma \approx D_B, \tag{107}$$

where D_B represents the Bohm diffusion coefficient. Plugging this into thermal diffusivity results in

$$\chi_{\perp,\mathrm{eff}} \approx \frac{D_B}{\Delta_\perp} D_M. \tag{108}$$

Therefore, we find that

1. $\chi_{\perp,\mathrm{eff}}$ scales with Bohm diffusion rather than Spitzer diffusion ($\chi_\parallel$).

2. Kicking particles off the field lines is important, again.

To compare with Rechester and Rosenbluth:

$$\chi_\perp \approx \sqrt{\chi_\parallel \chi_\perp} \frac{\langle \tilde{b}^2 \rangle}{\Delta_\perp} l_{ac}, \tag{109}$$

but what is $\Delta_\perp$? It is the thickness of a small layer that enters the spectrum

$$\frac{\chi_\parallel}{l_c^2} \approx \frac{\chi_\perp}{\Delta_\perp^2}$$
$$\Delta_\perp \approx l_c \sqrt{\chi_\parallel \chi_\perp},$$

therefore, $\Delta_\perp$ is set by diffusion. We can plug it into $\chi_\perp$,

$$\chi_\perp \approx \sqrt{\chi_\parallel \chi_\perp} \frac{\langle \tilde{b}^2 \rangle l_{ac}}{l_c (\chi_\parallel / \chi_\perp)^{1/2}}$$
$$\implies \chi_\perp \approx \frac{\chi_\parallel}{l_c} D_M = v_{th} D_M \left(\frac{l_{mfp}}{l_c} \right).$$

In this derivation we found the following,

1. Modulo $k_\perp$, $\Delta_\perp$; agrees with R+R to within a logarithmic factor.

2. $\chi_\perp \approx v_{th} D_M \frac{l_{mfp}}{l_c}$

This covers diffusion in the quasilinear regime of $\mathrm{K_u} < 1$. The lesson here is that we should take care of **coarse graining**, since it is crucial to **irreversibility**!

5 Solution to Suggested Exercise

First consider the drift-kinetic equation

$$\frac{\partial f}{\partial t} + v_\parallel \hat{n} \cdot \nabla f - \frac{c}{B} \nabla \phi \times \hat{i} \nabla f + \mathbf{v}_D \cdot \nabla f - \frac{|e|}{m_e} E_\parallel \frac{\partial f}{\partial v_\parallel} = 0, \tag{110}$$

let's consider a steady state and a constant potential, thus, we get

$$v_\parallel \hat{n} \cdot \nabla f = 0, \tag{111}$$

where $\hat{n}$ can be expressed as

$$\hat{n} = \frac{\mathbf{B}_0 + \tilde{\mathbf{B}}}{|B_0|}. \tag{112}$$

We can then expand eq.111 using eq.112 to get

$$v_\parallel n_0 \cdot \nabla \langle f \rangle + v_\parallel \frac{\partial}{\partial r} \left\langle \tilde{b}_r \tilde{f} \right\rangle = 0, \tag{113}$$

now note that we have the following expression of the form

$$n_0 \cdot \nabla \langle f \rangle + \frac{\partial}{\partial r} \Gamma = 0, \tag{114}$$

where

$$\Gamma = -D_M \frac{\partial \langle f \rangle}{\partial r} = \left\langle \tilde{b}_r \tilde{f} \right\rangle. \tag{115}$$

In addition, it should be mentioned that all drifts are cold! See Chen. Now, how do magnetic drifts modify D_M? Let's look at the following,

$$v_n \hat{n} \cdot \nabla f + \mathbf{v}_D \cdot \nabla f = 0, \tag{116}$$

and Fourier transform to get

$$\left(i k_\parallel v_\parallel + i k_\perp v_D \right) \tilde{f} = -\tilde{b} \frac{\partial \langle f \rangle}{\partial r}. \tag{117}$$

Here it can be noted that there is a shift of resonance due to the perpendicular term. Finally, runaways are better to confinement than thermal ones due to stochastic **B**.

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