PHYS 235 Report on Intermittency

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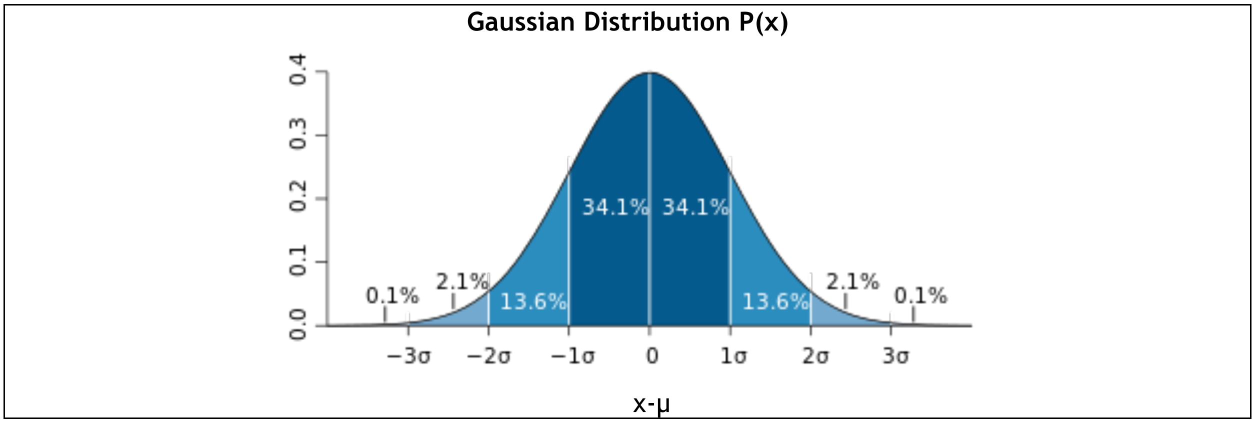
Before Intermittency – Central Limit Theorem

- To better understand the concept of intermittency, one needs to first understand the concept of probability distributions. Take a random variable x, where each value of x is independent of all previous values and also independent of any other variables. The probability distribution P(x) describes the likelihood of obtaining any particular value of x. If x is a continuous variable (rather than a discrete integer), then P(x) is also a continuous probability density function (PDF).
- Any random event can be described with a probability distribution. For example, the flip of a coin has a ½ chance of returning heads-up and a ½ chance of returning tails-up, as expressed in the following probability distribution:

$$\xi = \begin{cases} 1 \\ 0 \end{cases} (50\%\ chance)$$

- One way to analyze probability distributions is to find their moments. This involves raising every random variable in the distribution by a power p, then summing or integrating over the entire distribution. Moments of the distribution are denoted as $\langle x^p \rangle$.
    - The first moment is the mean $\mu = \langle x \rangle$
    - The second moment is the variance $\sigma^2 = \langle x^2 \rangle$
    - Higher moments encode additional information about the PDF
- Most probability distributions that one finds in nature are Gaussian, i.e., they have a mean μ and standard deviation σ, the probability distribution is bell-shaped, and the probability of any outlier event occurring decreases exponentially (~exp(-x^2)) the further one gets from the mean.

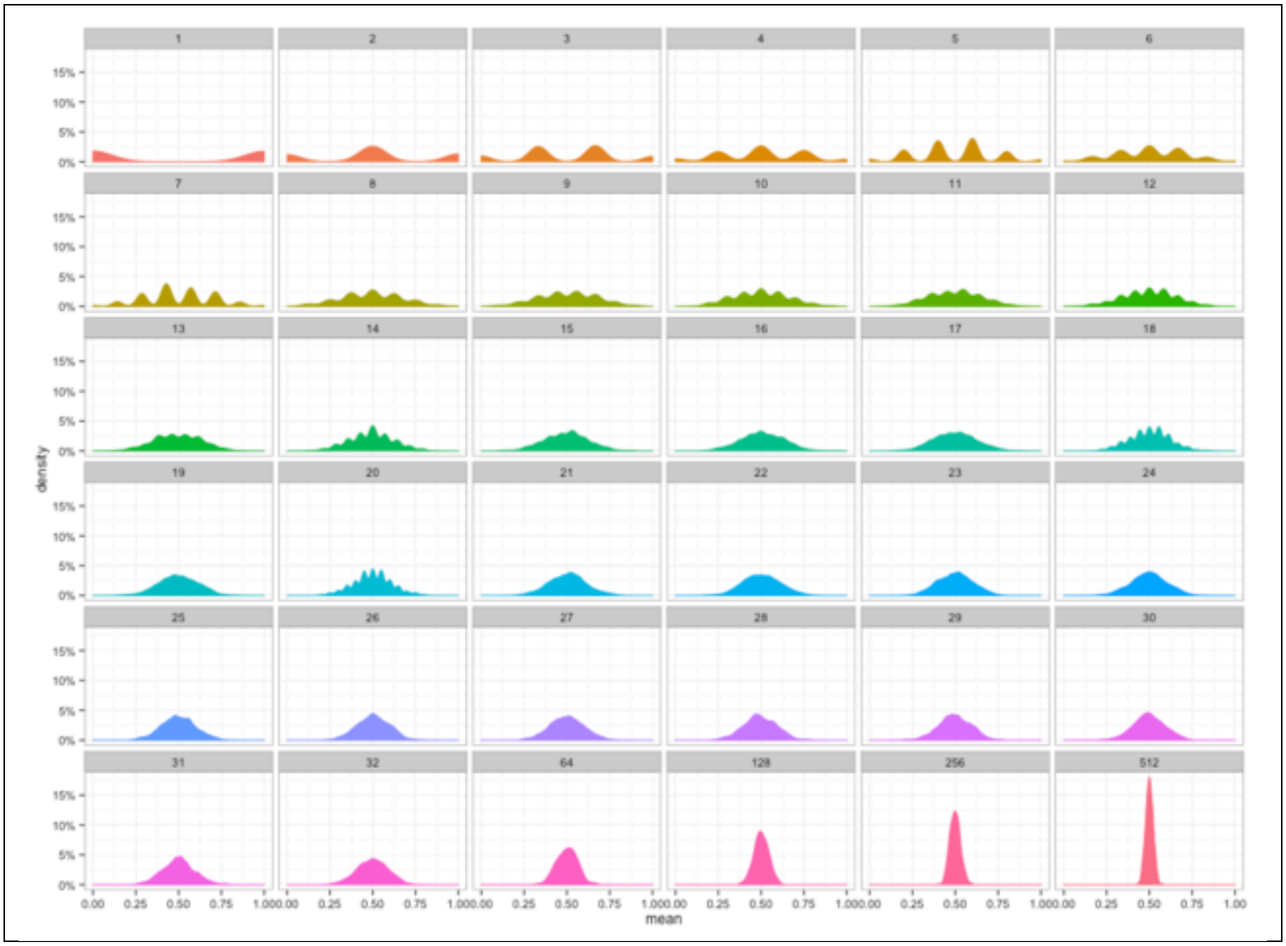
$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$



- There are numerous other naturally occurring probability distributions, such as the Binomial distribution, but the Gaussian distribution is seemingly the most common out of them all (hence why it is also known as the Normal distribution). This phenomenon is due in large part to the Central Limit Theorem.
- The Central Limit Theorem states that, given a sample of N random variables x_1, x_2, … x_N picked from a probability distribution that isn't necessarily Gaussian, the sample mean $\bar{x}$ tends towards a Gaussian distribution anyways as N goes to ∞.

- Returning to the example of the coin toss, except now repeating the coin toss N times. Despite the individual coin flips having an equal chance of landing on either 1 or 0, the sample mean tends towards a Gaussian distribution about the mean $\bar{x} = 0.5N$ that gets narrower with increasing N.

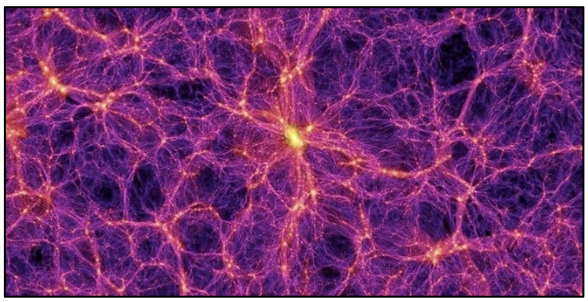
$$x = \sum_{i=1}^{N} \xi_i = \xi_1 + \xi_2 + \cdots + \xi_N, \qquad \xi_i = \begin{cases} 1 \\ 0 \end{cases} (50\%\ chance)$$



- Crucially, a probability distribution is gaussian if the second moment of the step probability $\langle x^2 \rangle = \sigma^2$ exists and is constant.
- Note that the Gaussian distribution is an inherently additive distribution. Adding together two Gaussian-distributed variables results in another Gaussian-distributed variable. This is true even in the case of the central limit theorem, where the process of calculating the sample mean involves adding together N randomly distributed variables x.
- As we shall see, not every probability distribution in nature is additive, and those outliers tend to defy the central limit theorem in spectacular ways.

Intermittency

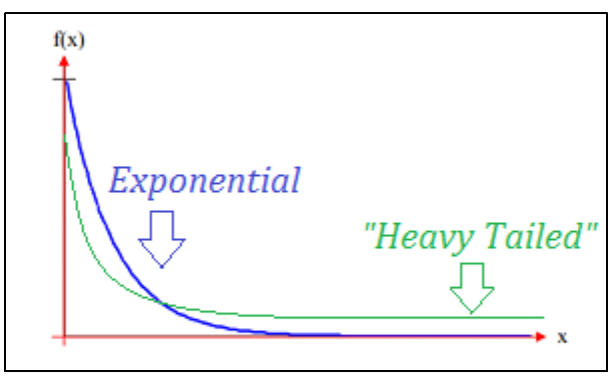
- Informally, an intermittent process is one which occurs in bursts (in time) or patches (in space). Inside of each peak, the probability density is much higher than the average probability density. These peaks occur in random places and times, surrounded by large valleys where little happens.
- Systems that are intermittent have their physical properties dominated by the actions of these rare but influential peaks, and thus, they cannot be ignored.
- The probability density functions that give rise to intermittency are not smooth or Gaussian, and don't abide by the central limit theorem or the law of large numbers. These axioms underpin thermodynamics, which is part of why thermodynamics is applicable to an extremely wide range of problems but is ill-equipped to describe intermittency. Phenomena such as the spontaneous organization of water molecules into ice crystals, the orderly arrangement of convection cells in a liquid, the organized layers within jasper and agate crystals, and indeed life itself, all seemingly violate the laws of thermodynamics by locally going against the maximization of entropy.
- Another example of intermittency appears in the grand structure of the universe, where the action of gravity pulls the universe away from being uniformly dense everywhere and instead intermittently coalescing into grand filaments of matter and galaxies, rather than the entropy-maximizing heat death of the universe predicted by thermodynamics.



- Hence, intermittency is closely tied to the phenomenon of spontaneous self-organization and symmetry-breaking. Here, structure arises *because* of randomness, not despite it.
- One of the characteristics of intermittency is the relationship between successive statistical moments of their PDFs $\langle x\rangle, \langle x^2\rangle, \ldots, \langle x^p\rangle$, namely that they grow exponentially: $\langle x^p\rangle \sim e^p$. The PDF is intermittent if one can define an exponential moment growth rate γ_p, such that

$$\gamma_p = \frac{\log_2\langle x^p\rangle}{N} > 0$$

    - Here, N is a surrogate for quantities like time, the number of steps, samples, etc.
- Intermittent processes can arise from many distinctly non-Gaussian probability distributions. In contrast to the normal distribution, the higher moments of an intermittent distribution are not only non-flat, but also diverge to infinity. This results in said probability distributions having "heavy tails".

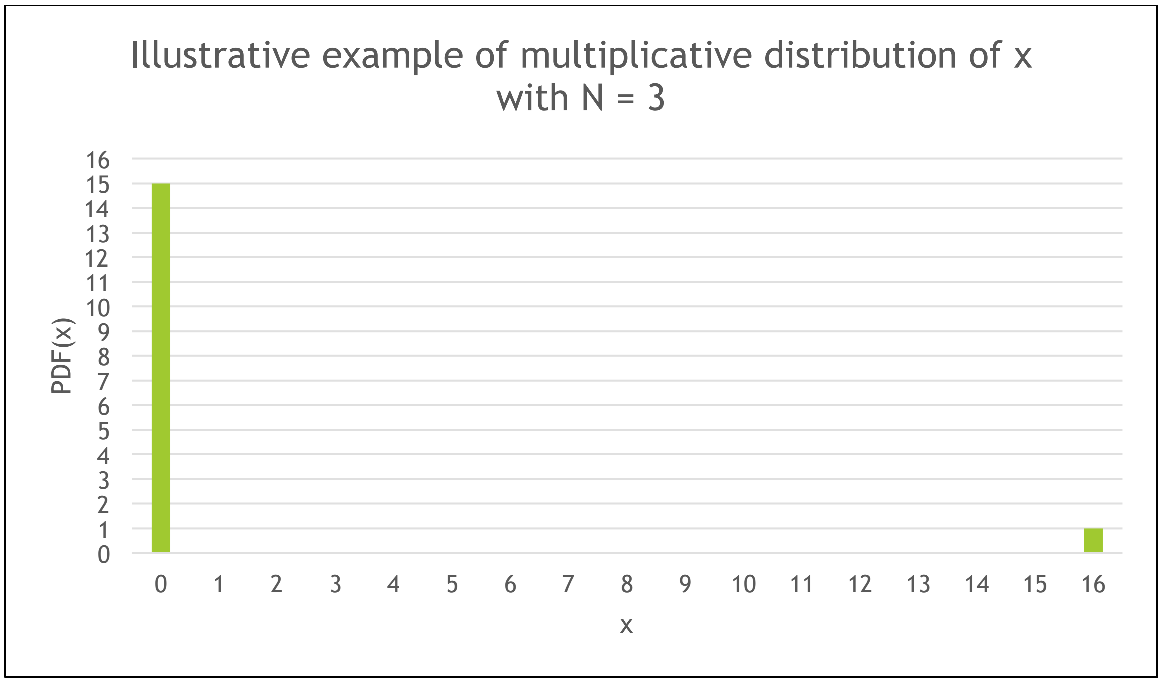


- Intermittency Example: Random Media. Charged particles with temperature T move around inside a random Gaussian-distributed electric potential field $\phi(x,\omega),\ P(\phi) \sim \exp(-\phi^2/2\sigma^2)$. However, the resulting random density of the particles varies with $\rho = \rho_0 \exp(-\phi/k_b T)$, which is nonlinear and not Gaussian.
    - To begin solving this problem, we separate variables $P\big(\rho(\phi)\big) = \rho(\phi)P(\phi) = \rho_0 \exp(-\phi/k_b T) \exp(-\phi^2/2\sigma^2)$
    - The most probable density occurs at $\phi_{max}/\sigma \sim -\sigma/T,\ P_{max} = n_0 \exp(\sigma^2 \backslash 2T)$
    - Statistical moments: $\langle n^p \rangle^{1/p} = n_0 \exp(p\sigma^2/2T^2)$, resulting in large fluctuations
    - Moment Growth Rate: $\gamma_p = \log_2\left(n_0^p \exp\left(p^2 \frac{\sigma^2}{2T^2}\right)\right) = p \log_2(n_0) + p^2 \frac{\sigma^2}{T^2}\frac{1}{2\ln 2} > 0$
    - Therefore, the density distribution is intermittent. Large fluctuations, far above the average, dominate the density distribution of the particles.

Multiplicative Processes

- Intermittency generally occurs in multiplicative processes, or any process where events arrange themselves into chains which cause the value of each successive event to grow multiplicatively, and each event has a small but nonzero chance of abruptly ending the chain.
- An Illustrative example of a multiplicative process that gives rise to intermittency: a random quantity x is the result of a series of N identically distributed random variables ξ_N multiplied together; these variables can randomly take on the values of 0 or 2 with identical probabilities of ½. This example is the multiplicative version of the coin toss from earlier.

$$x = \prod_{i=1}^{N} \xi_i = \xi_1 \xi_2 \xi_3 \ldots \xi_N, \qquad \xi_i = \begin{cases} 2 \\ 0 \end{cases} (50\%\ Chance)$$

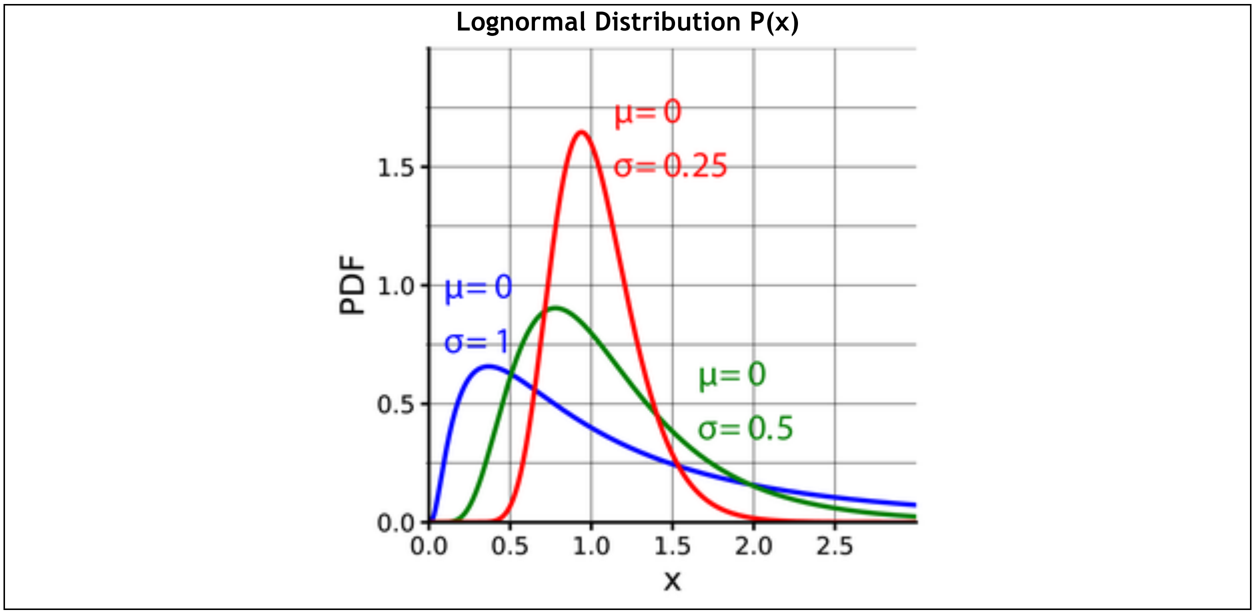


- As a result, the random quantity x is almost always 0, except in the 2^{-N} chance where all variables are 2, in which case it takes the value of 2^N.
  - The mean of this distribution, despite all the zeroes, is $\langle x \rangle = (0 + 0 + \cdots + 2^N)/\, 2^N = 1$. This means that the one rare outlier event dominates the distribution.
  - The higher moments of the distribution are even more skewed: $\langle x^P \rangle = \ 2^{NP}/\, 2^N = 2^{(P-1)N}$
  - When we compare the values of subsequent moments, we find that they grow exponentially: growth rate $\gamma_P \equiv \log_2 \langle x^P \rangle \ /N = P - 1$. Therefore, the PDF is intermittent.
- This is an idealized multiplicative process and an extreme case, meant to showcase the effect of the concept of a heavy tailed probability distribution. The Lognormal distribution provides a more realistic case for study.

Lognormal Distribution

- The lognormal distribution is the multiplicative version of the normal distribution.

$$P(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$$



- While the random variable x in this distribution is multiplicative, value of ln(x) is additive and normal distributed. This means that, despite the distribution itself being multiplicative, we can still apply statistical concepts such as the Central Limit Theorem on its logarithm.

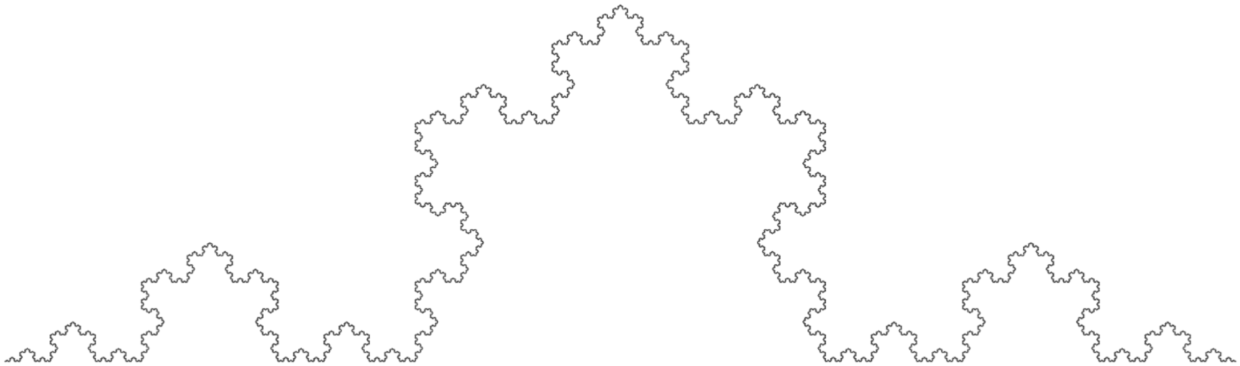
$$x = \prod_{i=1}^{N} \xi_i = \xi_1\xi_2\xi_3 \dots \xi_N \rightarrow \ln x = \sum_{i=1}^{N} \ln \xi_i = \ln \xi_1 + \ln \xi_2 + \ln \xi_3 + \cdots + \ln \xi_N$$

- Multiplying two lognormal-distributed variables together yields another lognormal-distributed variable. This is analogous to the additive property of the normal distribution.
- The statistical moments of the lognormal distribution grow exponentially, confirming that it is intermittent and has a heavy tail.

$$\langle x^p \rangle = \int_0^\infty \frac{x^p}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right) dx = \exp(p^2 N\sigma^2/2)\,, \qquad \gamma_P = \ln\langle x^P \rangle^{-N} = P^2\sigma^2/2$$

- Examples of lognormally distributed variables include:
  - The number of moves in a chess game
  - The number of posts in an online forum thread
  - The incomes of the lower 97% of the population (the richest 3% are Pareto-distributed)
- Lognormal distributions tend to appear in random variables that arise from a chain of other random variables multiplying together, each one carrying a small but nonzero chance of breaking the chain. As a result, lognormally distributed variables are highly sensitive to the actions of a few highly influential outliers.

Fractals



- Fractals are mathematical objects that are inherently self-similar, one can zoom into them and continue to find more structure ad infinitum. As a result, fractals tend to appear in situations that lack a well-defined length scale or have self-similar governing equations behind them.
- Since fractals have infinite detail and endlessly display more detail at finer length scales, measuring their dimensions (such as length, area, volume, etc.) turns up different results depending on the size of the measuring stick. For a classic example, the length of coastline of Britain (or any coastline) grows larger as one measures it with finer and finer measuring sticks. The same can be said of the perimeters of fractals in the 2D plane, such as the famous Mandelbrot set.
- This behavior is encapsulated in the concept of a *fractal dimension*: find the relationship between the perimeter of a fractal and the length of the measuring stick, then take the limit as the measuring stick becomes infinitely fine to obtain a dimensionless number unique to that fractal.
- There are may ways to measure a fractal dimension, one of which being the box-counting dimension. To start, cover your fractal in cubes/squares/lines depending on whether the fractal is embedded in 3D/2D/1D, respectively. For a given box side-length ϵ, count the number of boxes N that cover the fractal, then calculate the box-counting dimension from…

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln N}{\ln 1/\epsilon}$$

- Example of a simple fractal: the Middle-Third Cantor Set

  - Take a line (1D) and cut it into thirds, then remove the middle third, leaving behind two lines, each one-third the length of the original line. Repeat ad infinitum.
  - After n iterations, there are 2^n lines, each with a length of $(1/3)^n$, resulting in a total length of $(2/3)^n$.

$$D_0 = \lim_{n \to \infty} \frac{\ln(2^n)}{\ln\left(\frac{1}{(1/3)^n}\right)} = \frac{\ln(2)}{\ln(3)} \approx 0.63 < 1$$

  - The fractal dimension is generally less than the embedding dimension (in this case, 1D)
- The study of fractals has numerous applications in intermittency and other self-similar phenomena. One of the most practical applications of fractals is in the study of turbulence.

Kolmogorov 1941 Model of Turbulence and the 4/5ths Law

- A key feature of turbulence is the turbulent energy cascade, where the large-scale kinetic energy of a fluid is wrapped up into vortices which repeatedly stretch and transfer the kinetic energy to smaller and smaller length scales, all the way down to the viscous length scale where the kinetic energy is dissipated into heat.
- Part of the difficulty in modelling turbulence is in accurately describing the behavior of the velocity field at different length scales. To begin the derivation, we consider the slight differences in the velocity field over small displacements l. These are the velocity increments:

$$\overrightarrow{\delta v}(\vec{r},\vec{l}) = \vec{v}(\vec{r}+\vec{l}) - \vec{v}(\vec{r})$$

- These velocity increments follow a non-gaussian probability distribution. When we consider only the component of the velocity increment that is parallel to l and integrate over the fluid domain r, we can find the moments of this probability density function, known as the Structure Functions:

$$S_p(l) \equiv \langle \overrightarrow{\delta v_\parallel}(\vec{r},\vec{l})^p \rangle$$

- In 1941, Kolmogorov published a series of papers in which he derived these structure functions from the Navier-Stokes equations for homogeneous and isotropic turbulence in the limit of infinite Reynolds Number. The paper provided a scaling law to derive structure functions of any order:

$$S_p(l) \equiv \langle \overrightarrow{\delta v_\parallel}(\vec{r},\vec{l})^p \rangle = C_p(\epsilon l)^{p/3}$$

- The K41 model assumes a finite, non-vanishing, and constant mean energy dissipation rate per unit mass: ϵ. The increment l is assumed to be small compared to the integral scale. The constants C_p cannot depend on the Reynolds number due to the assumption of infinite Reynolds number.
- From this equation, it becomes evident that only the value of p = 3 causes the structure function to become dimensionless, while any other value is dimensional and thus dependent on the geometry of the system. Indeed, one of the most important results of this work is the result for the third moment, or the third-order structure function:

$$S_3(l) = \langle \overrightarrow{\delta v_\parallel}(\vec{r},\vec{l})^3 \rangle = -\frac{4}{5}\epsilon l$$

- This "four-fifths" law is universal and is one of the most important rigorous results in fully developed turbulence. Despite this, the other moments are not universal, but they are scale-invariant. As evidence, take the constancy of the skewness of the distribution:

$$\frac{S_3(l)}{S_2(l)^{3/2}} = \frac{-\frac{4}{5}\epsilon l}{(C_2(\epsilon l)^{2/3})^{3/2}} = -\frac{4}{5}C_2^{-3/2} = const.$$

- In reality, the process of turbulent dissipation is an intermittent process: the energy cascade is "patchy" with some length scales more affected than others. This strikes at the heart of one of the basic assumptions of Kolmogorov's theory: that the self-similarity of the turbulence has a single scaling exponent h = 1/3, when it is better to think of the self-similarity as a fractal. This exposes a shortcoming of the K41 model: its scale invariance prevents it from modelling intermittency effects.
- Kolmogorov addressed this shortcoming in 1962, where he incorporated intermittency into the model via a lognormal distribution for ϵ, resulting in the K62 model.

Beta Model of Turbulence

- In an attempt to improve upon Kolmogorov's models, researchers Frisch, Sulem, and Nelkin developed the beta model in 1977, which implements a self-similar fractal scaling law.
- Take a 3D velocity flow field v that forms turbulent eddies with characteristic length scale l_0. Those turbulent eddies go on to form a turbulent energy cascade, with each step in the cascade having a length scale half as large as the previous step:

$$l_n = l_0/2^n, \qquad n = 0,1,2,\ldots$$

- The kinetic energy per unit mass at each length scales with n. This relation comes from the energy spectrum E(k), which shows how the kinetic energy of the flow is distributed at different length scales.

$$E_n = \int_{k_n}^{k_{n+1}} E(k)\, dk\,, \qquad k_n = 2\pi/l_n$$

- From this rescaled kinetic energy, we obtain a characteristic velocity at each scale: $E_n \sim v_n^2$.
- The mean energy dissipation rate at each scale is given by $\epsilon = \beta_n v_n^2/l_n$. Here, β_n is the fraction of the flow that is actively dissipating, at the n^{th} step of the energy cascade. This term is what gives the beta model its namesake, as it is the structure function which varies with the cascade step n and has a fractal dimension D.
- To calculate ß, we need to calculate the total volume of all of the eddies at each length scale. We note that, at each length scale, the turbulent eddies have half the radius, a quarter of the surface area, and an eighth of the volume as those of the previous length scale. We also know that ß follows a power law scaling:

$$\beta_n = \beta^n = (N/2^3)^n = (N * 2^{-3})^n$$

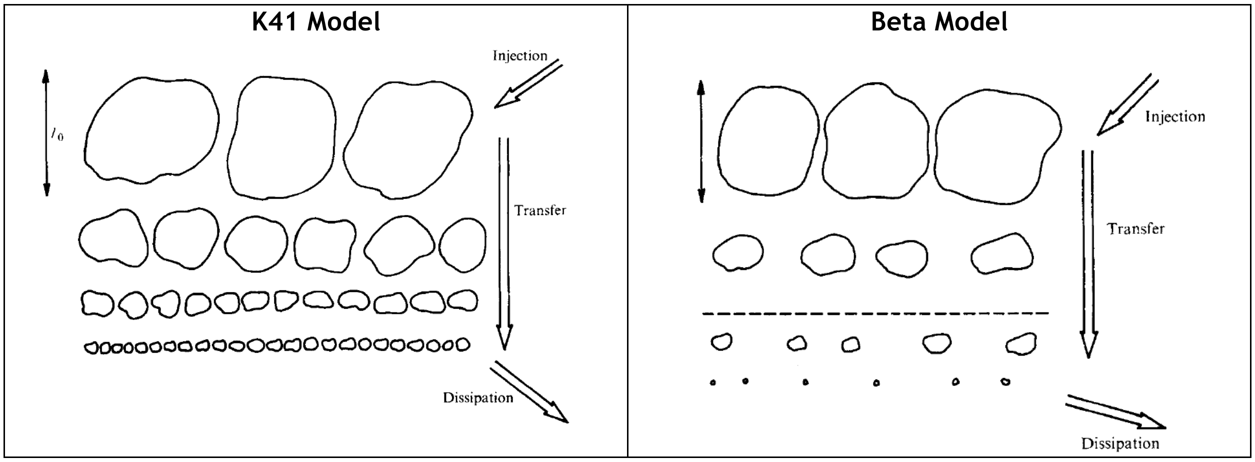
- The number of eddies N is fractal and follows a fractal dimension scaling law.

$$N \equiv 2^D, \qquad \beta_n = (2^{D-3})^n$$

- The value of the fractal dimension D must be deduced from experiments at high Reynolds numbers, which come out to around 2.7-2.8
- When we plug everything back in, we find that the original eddy length scale l_0 is still present due to the power law.

$$\epsilon = \left(\frac{l_n}{l_0}\right)^{3-D} \frac{v_n^3}{l_n} \rightarrow v_n = (\epsilon l_n)^{1/3} \left(\frac{l_n}{l_0}\right)^{(3-D)/3}$$

- The 1941 Kolmogorov model assumes that the turbulent eddies are inherently space-filling, i.e., the gaps between turbulent eddies with radius l_0 are filled with smaller eddies with radius l_1, which themselves have gaps filled in with more eddies with radius l_2, so on until the cascade hits the viscous length scale. The results from the beta model contradict this assumption; the fraction of the space filled in with these eddies changes as the step changes: small-scale structures of turbulence become less space-filling as scale size decreases.



- It is worth noting that the K41 model assumes no fractal dimension but a constant self-similar scaling exponent h = 1/3, the beta model assumes one fractal dimension D = 2.7-2.8, and the reality is multifractal with a different fractal dimension for each statistical moment p.