PHYS 235 - Quasilinear Theory

"Mindless Mean-Field Theory"

A Summary of Pat Diamond's Notes by Steve Molesworth – Spring 2022

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1 Motivation

Linear methods of instantaneous plasma stability, while valuable, is limited by perturbation magnitudes and growth rates. Beyond this, perturbation-induced waves interact with other particles, and evolve the plasma's distribution through energy and momentum conservation. This feedback loop tends towards steady state, but a complete description is impractical. Alternatively, one can separate zeroth order (mean) quantities from their respective first order (perturbed) values and solve the evolution. This is the essence of Quasilinear Theory (QLT), which is described here in the context of weak turbulence. In weak turbulence a collective mode's self-correlation time, τ_c, is longer than the respective mode's frequency, ω, or rather $\omega\tau_c > 1$ [1]. Nonlinearities are mild in this regime as wave-wave coupling is limited [2]. A spectrum of waves relevant to statistical descriptions and independent of initial phases in long time scales is assumed.

An off-shoot from mean-field theory, QLT remains widely applicable decades after its introduction despite "dropping" details about driving and response fluctuations after its derivation. Its formulism, some uses, and its shortcomings are included below.

2 Introduction

<u>QLT concerns itself with describing and understanding the slow evolution of $\langle f \rangle$.</u>

2.1 QLT in a Vlasov Plasma

While QLT falls out from multiple approaches, it is most readily digested with a Vlasov plasma exhibiting turbulence as Vedenov, Velikov, and Sagdeev did in 1961-1962 [3,4]. To bound the problem, the gentlemen asserted that the normalized, spatially averaged, particle distribution function relaxed slowly in time relative to a collective mode's growth or damping rate, γ_k,

$$\tau_{relax}^{-1} \ll \gamma_k \;\; where \;\; \tau_{relax}^{-1} = \frac{1}{\langle f \rangle}\frac{\partial \langle f \rangle}{\partial t} \tag{2.1.1}$$

Embedded in this limit is the assumption that the distribution function is spatially homogeneous, and its fluctuation magnitudes are small enough for separation into velocity-space mean and a spectrum of fluctuating components, or

$$f = f_{\circ}(V,t) + \tilde{f}(V,t) \tag{2.2.2}$$

First $\langle f \rangle$ must be obtained via the Vlasov equation,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial v} \;=\; 0 \tag{2.1.3}$$

Averaging (2.1.3) over the ensemble, recognizing the second term drops out, and reducing the third time using $\frac{\partial f_{\circ}}{\partial v} \approx 0$ leaves a Vlasov hierarchy closure equation

$$\frac{\partial\langle f\rangle}{\partial t}+\left\langle\frac{q}{m}\frac{\partial\tilde{E}\tilde{f}}{\partial v}\right\rangle=0 \tag{2.1.4}$$

This is the generic mean field equation for a conserved order parameter, which is <f> in this case. The second term represents a velocity space flux, so (2.1.4) can be rewritten as a phase space continuity equation.

$$\frac{\partial\langle f\rangle}{\partial t}+\frac{\partial J_v}{\partial v}=0 \tag{2.1.5a}$$

$$where\ J_v=\left\langle\frac{q}{m}\tilde{E}\tilde{f}\right\rangle \tag{2.1.5b}$$

<u>Neglecting multimodal coupling and noise since fluctuations are eigenmodes, or ω = ω(k), one can express $\tilde{f}$ as linear and solve its coherent response, f_k^c, to $\tilde{E}$ in k-space,</u>

$$\tilde{f}\ \approx f_k^c\ =\ -i\frac{q}{m}\frac{E_k\partial\langle \mathrm{f}\rangle\,/\partial v}{\omega\ -\ kv} \tag{2.1.6}$$

The distribution's response to fluctuations clearly has a resonance. Plugging (2.1.6) into (2.1.5b) shows this as a flux

$$J_v=-i\frac{q^2}{m^2}\sum_{k,\omega}\frac{\left|\widetilde{E_{k,\omega}}\right|^2\partial\langle \mathrm{f}\rangle/\partial v}{\omega\ -\ kv} \tag{2.1.7}$$

Restating (2.1.5a) with (2.1.7) yields the QL equation and a diffusion coefficient,

$$\frac{\partial\langle f\rangle}{\partial t}+\frac{\partial}{\partial v}\mathrm{D(v)}\frac{\partial\ \langle f\rangle}{\partial v}=\ 0 \tag{2.1.8a}$$

$$\mathrm{D(v)}=\mathrm{Re}\left(i\frac{q^2}{m^2}\sum_k\frac{\left|\widetilde{E_k}\right|^2}{\omega-kv+i|\gamma_k|}\right) \tag{2.1.8b}$$

The absolute value of γ_k is necessary in (2.1.8b) for non-negative diffusion, which damped waves ($\gamma_k < 0$) can cause. With the marginally stable, linear dielectric function (2.1.8c) and electric field energy's temporal evolution (2.1.8d), the quasilinear evolution of $\langle f\rangle$ to its marginally stable state may be calculated by an iterative process.

$$\epsilon(\mathrm{k},\omega)=\ 0 \tag{2.1.8c}$$

$$\frac{\partial}{\partial t}|E_k|^2\ =\ 2\gamma_k|E_k|^2 \tag{2.1.8d}$$

Notice the diffusion coefficient (2.1.8b) has resonant and non-resonant components,

$$D_{res}(\mathrm{v}) \approx \frac{q^2\pi}{m^2}\sum_k |E_k|^2 \delta\left(\frac{\omega}{k} - v\right)\frac{\partial \langle f \rangle}{\partial v} \tag{2.1.9a}$$

$$D_{non-res}(\mathrm{v}) \approx \frac{q^2}{m^2}\sum_k |E_k|^2 \frac{|\gamma_k|}{\omega^2} \tag{2.1.9b}$$

Interestingly, non-resonant diffusion is related to ponderomotive energy, $|V_k|^2$,

$$D_{nr} = \frac{1}{2}\frac{\partial}{\partial t}\sum_k |V_k|^2 \ where\ |V_k|^2 = \frac{q^2}{m^2}\frac{|E_k|^2}{\omega_k^2} \tag{2.1.10}$$

The ponderomotive energy stems from “quiver” motions of particles as they “slosh” through waves. Because this motion is reversible, it cannot be derived from Fokker-Planck theory, and may be deemed “fake diffusion”. Further, this portion of diffusion evidently vanishes in a stationary state, while resonant diffusion does not necessarily. Further detail on the resonant portion will be provided in later sections.

2.2 Underpinnings

Before proceeding with the physical basis for QLT, its relevant scales must be defined. A finite system of size L forces quantization of wavenumbers and phase velocities,

$$k = \frac{n\pi}{L}\ and\ v_{ph,m} = \frac{\omega_n}{k_n} \tag{2.2.1}$$

Each resonance produces an island in phase space defined by separatrix width, $\Delta v \approx \sqrt{q\phi_n/m}$, as depicted in Figure 1 (left) [1]. For isolated islands, particles inside the separatrix will remain *trapped* in that region of phase space, while those beyond the separatrix will *circulate.* When separatrices overlap, they'll destroy themselves, and particles can migrate stochastically between them (Figure 1, right [1]). “Resonance hopping” provides the irreversibility necessary for diffusion on the order $D_v \approx \frac{\Delta v^2}{\tau_{ac}}$. The Chirikov number, S_c (2.2.2), relates velocity variation within two separatrices to the velocity “distance” between those resonances, and a value larger than unity suggests resonance overlapping.

$$S_c = \frac{(\Delta v_i + \Delta v_{i\pm 1})/2}{|v_{ph,i} - v_{ph,i+1}|} \tag{2.2.2}$$

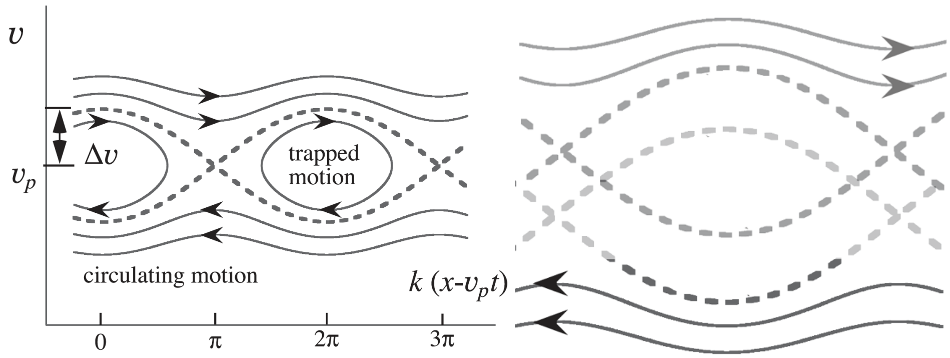

Figure 1 - Phase space island structure from wave-particle resonance (left) [1] and magnetic islands from overlapping separatrices (dashed lines) allow stochastic particle migration (right) [1]

This all relies on a linear, unperturbed orbit though! If the electric field fluctuation's wave packet duration, τ_L, is much shorter the time a particle requires to experience a 'bounce', τ_B, in its trajectory due to the field, then the quasilinear approximation holds. Otherwise, the particle trajectory exhibits some degree of trapping by the field pattern. These extremes are depicted in Figure 2 [1].

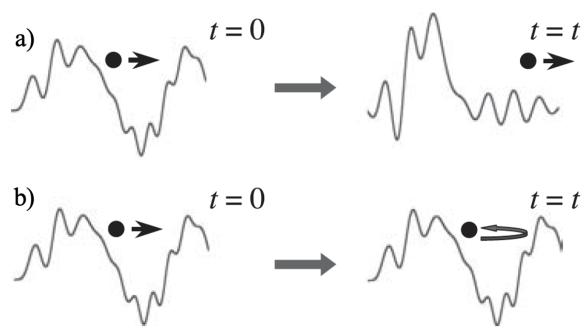

Figure 2 - Effects of (a) "long" and (b) "short" field fluctuation lifetime on particle trajectory [1]

The time to disperse one wavelength with respect to the range of dispersions between all waves, $\Delta(\omega_k, k)$, is

$$\begin{aligned} \tau_L^{-1} &= k|\Delta(\omega/k)| = k\left|\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right| \\ &= \left|\left(v_g(k) - v_{ph}(k)\right)\Delta k\right| \end{aligned} \tag{2.2.3}$$

which states that the interaction time between a resonant particle and a wave packet is limited by the difference in group and phase velocities. As wave dispersion decreases to zero, τ_L must approach infinity, and the pattern coherence time depends on shocks. To ensure $\tau_L << \tau_B$, Δk must be sufficiently large. Assuming a continuous spectrum of finite width – true for overlapping islands – in $|E_k|^2$,

$$|E_k|^2 = \frac{E_\circ^2}{\Delta k}\left[\left(\frac{k - k_\circ}{\Delta k}\right)^2 + 1\right]^{-1} \tag{2.2.4}$$

shows a Lorentzian probability density function. Δk effectively controls the function's width, and broadens the curve quadratically, so Δk >> 1 provides assurance the spectrum is "broad enough".

The Kubo number, Ku (2.2.5), which measures memory of a flow, provides another criterion for unperturbed orbits.

$$Ku = \frac{(qE/m)\tau_c}{(\Delta v)_c} \approx k(\Delta v_c)\tau_c \approx \omega_b \tau_c \tag{2.2.5}$$

Here, τ_c is the field (scatterer) correlation time, ω_b is the bounce frequency, and Δv_c is the velocity correlation length. A small Ku suggests the field pattern changes before a particle bounces, so a linear trajectory approximation is valid. Flow is ordered with a persistent memory (scatterer pattern) when Ku > 1 and QLT likely breaks down due to orbit trapping and phase space distortions.

Next, consider the electric field's autocorrelation time, τ_{AC}. The correlation function for stationary, homogeneous turbulence between two points of the field in space and time is

$$\langle E(x_1,t_1)E(x_2,t_2)\rangle = \sum_k |E_k|^2 e^{i[kx_- - \omega_k t_-]} = \int \frac{dk}{\Delta k} \frac{\mathrm{E}_\circ^2 e^{\mathrm{ikx}_{\circ -}} e^{it_-[(kv-\omega_k)]}}{\left[\left(\frac{\mathrm{k}-\mathrm{k}_\circ}{\Delta \mathrm{k}}\right)^2 + 1\right]} \tag{2.2.6}$$
$$\approx \mathrm{E}_\circ^2 e^{\mathrm{ikx}_{\circ -}} e^{-|\Delta k x_{o-}|} e^{\mathrm{i}\tau(k_\circ v - \omega_{k,\circ})} e^{-|\Delta(kv-\omega_k)|\tau}$$

Negative subscripts denote wave phase. Note that $|E_o|^2$ is the spectral density, Δk is the spectral width (as before), and k_o is the spectral distribution's centroid. The leading exponential in the second righthand term goes to 0 for resonance, while the final exponential introduces correlation decay due to dispersion and its interaction with resonance. The Doppler-shifted frequency, $\Delta|kv - \omega_k|$, therefore determines the spectral auto-correlation time, τ_{AC}. From this we see that for <u>resonant</u> particles,

$$\tau_{ac}^{-1} = |\Delta(\mathrm{kv} - \omega_\mathrm{k})| = \left|\left(\mathrm{v_{ph}} - \mathrm{v_{gr}}\right)\Delta k\right| \tag{2.2.7}$$

Reformulating the diffusion coefficient in Ku parameters, and relating with $\tau_c \sim \tau_{ac,packet}$ at resonance confirms τ_{ac} is relevant,

$$D \sim \frac{q^2}{m^2}\langle E^2\rangle \tau_c \tag{2.2.8}$$

The criteria for QLT applicability are thus

- $\tau_{ac} < \tau_B$, $Ku \le 1$ (unperturbed orbit approximation)
- γ_k^{-1}, $\tau_{AC} < \tau_{relax}$ (closure of ⟨f⟩)
- $\tau_{AC} < \gamma_k^{-1} < \tau_{relax}$ (QLT validity)
- $S_c > 1$ (resonance overlap)

3 Energy Conservation

Two conservation relations exist for energy and momentum conservation – resonant particles vs waves and particle vs fields!

3.1 Resonant Particle and Wave Energy Densities

First, how do resonant particles balance energy with waves? We begin by taking the energy moment of the QLT Vlasov equation,

$$\frac{\partial}{\partial t}\int dv \frac{mv^2}{2}\langle f\rangle = -\int dv \frac{mv^2}{2}\frac{q}{m}\frac{\partial\langle \tilde{E}\tilde{f}\rangle}{\partial v} \tag{3.1.1}$$

Next, applying the linear response of f-tilde and the Plemelj theorem, we find

$$\frac{\partial}{\partial t}\mathrm{E_{kin}} = -i\int dv \frac{v}{2}\frac{q^2}{m}\frac{\partial\langle \mathrm{f}\rangle}{\partial v}\sum_k |E_k|^2\left(\frac{P}{\omega - kv} - i\pi\delta(\omega - kv)\right) \tag{3.1.2}$$

where only the resonant component survives, so

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{E_{kin}^{res}} &= -\int dv \frac{\pi q^2}{m}\frac{\partial\langle \mathrm{f}\rangle}{\partial v}\sum_k |E_k|^2\pi\delta(\omega/\mathrm{k} - v) \\ &= -\frac{\pi q^2}{m}\textstyle\sum_k |E_k|^2 \frac{\omega}{k|k|}\frac{\partial\langle \mathrm{f}\rangle}{\partial v}|_{\omega/k}\end{aligned} \tag{3.1.3}$$

Now that an expression is available for resonant particle kinetic energy density (RPKED) evolution, we must relate it to the wave energy density. The marginal stability dielectric function previously found is

$$\epsilon = 1 + \frac{\omega_p^2}{k}\int dv \frac{1}{\omega - kv}\frac{\partial\langle \mathrm{f}\rangle}{\partial v} \quad and \quad \epsilon_{\mathrm{r}}(\omega_{\mathrm{k}} + \mathrm{i}\gamma_{\mathrm{k}}) + \mathrm{i}\epsilon_{\mathrm{im}} = 0 \tag{3.1.4}$$

Thus, the growth rate is

$$\gamma_k = \frac{\epsilon_{im}}{(\partial\epsilon_r/\partial\omega)|_{\omega_k}} \quad where \quad \epsilon_{im}(k,\omega_k) = -\frac{\pi\omega_p^2}{|k|k}\frac{\partial\langle \mathrm{f}\rangle}{\partial v}|_{\omega_k} \tag{3.1.5}$$

W, for an electric wave packet,

$$W = \sum_k \omega_k \frac{\partial\epsilon_r}{\partial\omega}|_{\omega_k}\frac{|E_k|^2}{8\pi} \tag{3.1.6}$$

The time evolution of W is thus

$$
\begin{aligned}
\frac{\partial W}{\partial t} &= \sum_k 2\gamma_k \omega_k \frac{\partial \epsilon_r}{\partial \omega}|_{\omega_k} \frac{|E_k|^2}{8\pi} = \sum_k 2 \frac{\epsilon_{im}}{(\partial \epsilon_r / \partial \omega)|_{\omega_k}} \omega_k \frac{\partial \epsilon_r}{\partial \omega}|_{\omega_k} \frac{|E_k|^2}{8\pi} \\
&= \sum_k -\epsilon_{im} \omega_k \frac{|E_k|^2}{4\pi}
\end{aligned}
\tag{3.1.7}
$$

and substituting (3.1.5) into (3.1.7) yields

$$
\frac{\partial W}{\partial t} = \sum_k \pi \frac{q^2}{m} \frac{\omega}{k|k|} \frac{\partial \langle \mathrm{f} \rangle}{\partial \mathrm{k}} |_{\omega/k} |E_k|^2 \tag{3.1.8}
$$

Any density dependence of $\langle \mathrm{f} \rangle$ has been eliminated for clarity in the result. With RPKED and W time variation quantified, the QLT resonant particle-wave conservation of energy is simply

$$
\frac{\partial}{\partial t}(E_{kin}^{res} + W) = 0 \tag{3.1.9}
$$

In a sense, this is a two "fluid" equation where the divide is made between resonance and non-resonance with the wave. Also, a Poynting theorem for plasma waves is buried in (3.1.9),

$$
\frac{\partial}{\partial t} W + \vec{\nabla} \cdot \vec{S} + Q = 0 \tag{3.1.10}
$$

Wave Energy + Wave Energy Density Source/Sink + Resonant Particle Heating = 0

Since $\vec{\nabla} \cdot \vec{S} = 0$ in a homogeneous system, the second term falls out. Q, the energy dissipation, can be replaced by $\langle \vec{\mathrm{E}} \cdot \vec{J} \rangle$ because the resonant particles feel a "constant" electric field.

3.2 Total Particle and Field Energy Densities

The multi-component nature of the QLT diffusion coefficient and wave energy density means total particle kinetic energy density (PKED) conserves with field energy density (FED) too,

$$
\begin{gathered}
\frac{\partial}{\partial t} FED + \frac{\partial}{\partial t}(RPKED + NRPKED) = 0 \\
\frac{\partial}{\partial t} FED + \frac{\partial}{\partial t} PKED = 0
\end{gathered}
\tag{3.1.11}
$$

Considering waves as quasi-particles provides further reasoning for our second energy theorem. The PKED term may be rewritten without Plemelj decomposition, unlike (3.1.2), as

$$
\frac{\partial}{\partial t} \mathrm{E}_{\mathrm{kin}} = -\sum_k \int dv\, kv \frac{\omega_p^2}{k} \frac{|E_k|^2}{4\pi} \frac{1}{\omega - \mathrm{kv}} \frac{\partial \langle \mathrm{f} \rangle}{\partial v} \tag{3.1.12}
$$

Recalling (3.1.5), we obtain

$$\frac{\partial}{\partial t}\mathrm{E_{kin}} = -\,i\sum_{k}\frac{|E_k|^2}{4\pi}\int dv\,\frac{\omega_p^2}{k}\frac{(kv-\omega+\omega)}{\omega-\mathrm{kv}}\frac{\partial\langle \mathrm{f}\rangle}{\partial v} \tag{3.1.13}$$

$(kv-\omega)$ in the numerator of (3.1.16) cancels with the denominator since its residual is odd and energy is real. Then, by setting $\epsilon(k,\omega_k)=0$ by ω_k's definition,

$$\frac{\partial}{\partial t}\mathrm{E_{kin}} = -i\sum_{k}\frac{|E_k|^2}{4\pi}\omega_k = -\sum_{k}\frac{|E_k|^2}{4\pi}\gamma_k \;=\frac{\partial}{\partial t}FED \tag{3.1.14}$$

This directly proves (3.1.12) without separation of resonant and non-resonant particles. Momentum conservations follow similarly with the caveat that an electromagnetic field is necessary for field momentum contributions.

4 Applications

<u>Within the detailed limitations, QLT describes macro plasma responses to common instabilities, while providing information about the nonlinearities involved.</u>

4.1 Bump-on-Tail Instability

The well-known Bump-on-Tail (BOT) instability provides great proving grounds for QLT. Energy is available in the system when the electron distribution function transitions from purely Maxwellian to having a slight beam of fast particles, which presents as a bump on its positive tail as in Figure 3 [1]. Phase velocities in the range where $\partial\langle f\rangle/\,\partial v>0$ are unstable, so the system seeks to convert this potential energy to kinetic energy amongst the electron distribution.

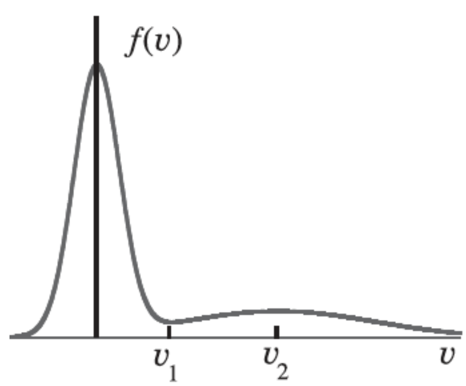

Figure 3 - Distribution Function Required for Bump-on-Tail Instability [1]

It is evident that resonant and non-resonant mechanisms are in play due to the phase velocity dependence of the BOT instability. We can surmise that waves will gain energy at the expense of resonant electrons between v_1 and v_2 – a velocity space diffusive process – and the bulk of electrons must conserve momentum through reversible plasma oscillations, or non-resonant diffusion. Together, one should expect a flattening of the tail with a positive shift of the bulk, as in Figure 4, when the distribution reaches marginal stability [1].

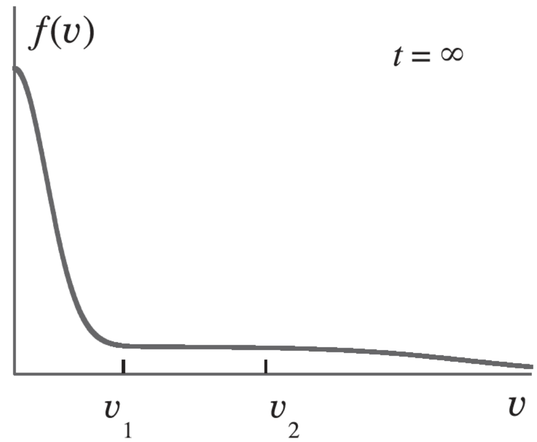

Figure 4 - Hypothesized distribution function after BOT instability [1]

Starting with the resonant range, and again limiting the distribution fluctuations to coherent linear responses, the Zeldovich's 1957 theorem can be generalized to our primary QLT equation. For conserved phase space, assume ⟨f⟩ goes to zero at the boundaries, so $\partial^2\langle f\rangle/\,\partial v^2 < 0$ around $\langle f\rangle_{max}$ and $\partial\langle f\rangle\,/\partial t\ is\ necessarily\ negative\ definite. Squaring\ \langle f\rangle$ in (2.1.8a) and integrating by parts over velocity space proves $\partial\langle f\rangle\,/\partial t \rightarrow 0$,

$$\frac{\partial}{\partial t}\int_{\mathrm{res}}\frac{\langle \mathrm{f}\rangle^2}{2} = -\int_{res} dv\, D_{res}\left(\frac{\partial\langle \mathrm{f}\rangle}{\partial v}\right)^2 \tag{4.1.1}$$

At instability saturation a stationary state is achieved, so the RHS of (4.1.1) must disappear either by D_R=0 or $\partial\langle f\rangle/\,\partial v = 0$. Physically, the former suggests fluctuations decay and damp, while the latter means a plateau forms and growth ceases. Assuming resonant diffusion vanishes, $D_R(t) \rightarrow 0$ as t →∞, suggests $\langle f(v,t)\rangle\ \approx \langle f(v,t)\rangle$, which contradicts damped wave influence ($\partial\langle f\rangle/\,\partial v\ < 0$). Thus, a plateau in the distribution function must form as the instability decays.

The plateaued condition must be found by applying resonant and non-resonant diffusion mechanisms. Using the first energy theorem, equate the change in RPKED to the change in W,

$$\begin{gathered}\frac{\partial}{\partial t}WED + \frac{\partial}{\partial t}RPKED = 0\\ \Delta\left(\int_{\mathrm{v}_1}^{\mathrm{v}_2} dv\ \frac{\mathrm{mv}^2}{2}\langle \mathrm{f}\rangle\right) = -\Delta\left(\int_{\mathrm{k}}^{k_2}\omega_{\mathrm{k}}\, dk\right)\ where\ \omega_k = 2\epsilon(k)\end{gathered} \tag{4.1.2}$$

The field grows from nearly zero to its saturation level, so the above simplifies to

$$\Delta\left(\int_{v_1}^{v_2} dv\, \frac{mv^2}{2}\langle f\rangle\right) = -\Delta\left(\int_{k}^{k_2} E^f(k)\, dk\right) \tag{4.1.3}$$

An analytical approach can be used to find the RPKED change. The beam flattens to lower velocities until the resonant particles balance each other's energies, so a partitioned rectangle over the resonant region, as in Figure 5, shows the total change and settled value near the average of f(v) over the resonant region [1]. Beyond v_2 is an area that flattens in response to the reduced v_2 peak, corresponding in magnitude to the bulk adjustment below v_1 required for conservation and function continuity. Of course, this process is slightly indirect: diminishing over- and undershoots can be expected until the distribution settles.

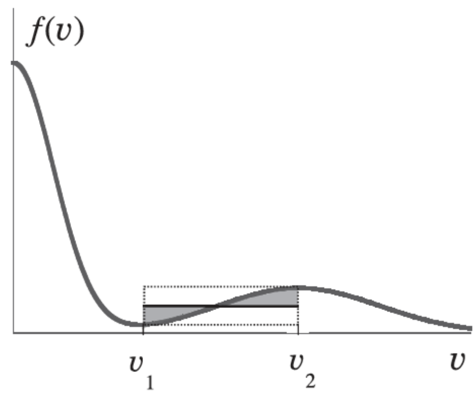

Figure 5 - Bump flattening causes a balanced change in resonant particle energy by increasing or decreasing energy depending on particle initial velocity(gray) [1]

Conserving total momentum requires the bulk distribution to shift like mentioned earlier in this section. For that portion, consider QLT equation for non-resonant diffusion from $0 < v < v_1$.

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial t} D_{NR}\frac{\partial\langle f\rangle}{\partial v} \approx \frac{8\pi q^2}{m^2}\int dk\, E^f(k)\frac{\gamma_k}{\omega_{pe}^2}\frac{\partial^2\langle f\rangle}{\partial v^2} \tag{4.1.4}$$

Inserting γ_k's modifies (4.1.8) to

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{1}{nm}\frac{\partial}{\partial t}\int dk\, E^f(k)\frac{\partial^2\langle f\rangle}{\partial v^2} \tag{4.1.5}$$

and define $\tau(t) = \frac{2}{n_e}\int dk\, \epsilon(k,t)$ for

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{1}{2m}\frac{\partial^2\langle f\rangle}{\partial v^2} \tag{4.1.6a}$$

$$\langle f\rangle = \left[\frac{m}{2\pi\big(T + \tau(t) - \tau(0)\big)}\right]^{1/2} e^{-\frac{mv^2/2}{T+\tau t-\tau(0)}} \tag{4.1.6b}$$

where a Maxwellian distribution is assumed for the bulk at t=0. Notice non-resonant particles at saturation gain temperature and the bulk evolution ceases when $\partial\langle f\rangle/\partial v > 0$,

$$T \to T + \frac{2}{n}\int dk\left[E^f(k,\infty) - E^f(k,0\}\right] \tag{4.1.7a}$$

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{1}{2m}\frac{\partial}{\partial v}\frac{\partial\langle f\rangle}{\partial v} = 0 \tag{4.1.7b}$$

This effect appears to come from an overall increase in field energy albeit reversible as with non-resonant diffusion mentioned previously.

4.2 Current-Driven Ion Acoustic Instability

QLT also demonstrates anomalous resistivity via analysis of the current-driven ion acoustic (CDIA) instability. Consider first the classic Sweet-Parker magnetic reconnection model where two opposing magnetic fields approach each other like in the image below.

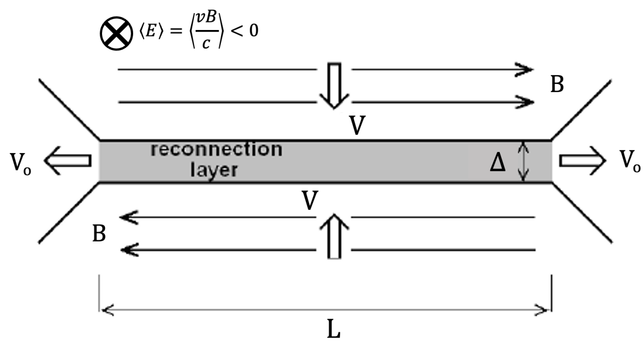

Figure 6 - Sweet-Parker magnetic reconnection problem

As the field lines close in on each other, the magnetic topology changes to include a central stagnation point, and plasma expulsion occurs due to conversion of magnetic field energy to particle kinetic energy. The relevant equations, (4.2.1a-c), are provided for discussion, but derivation is beyond the scope of this summary.

$$vL = v_A\Delta \tag{4.2.1a}$$

$$2vB^2L = \eta J^2 L\Delta \tag{4.2.1b}$$

$$\Delta^2 = \frac{L\eta}{v_A} \to \frac{\Delta}{L} \approx 1/\sqrt{R_M} \tag{4.2.1c}$$

Solving for mean electron drift velocity with the above parameters and $J = nq\overline{v_e}$,

$$\begin{gathered}\frac{cB}{4\pi\Delta} = nq\overline{v_e}\\ \overline{v_e} = cB/4\pi nq\Delta = d_{skin,e}^2\Omega_e/\Delta\end{gathered} \tag{4.2.2}$$

$\overline{v_e}$ is the electron drift speed, $d_{skin,e}$ is the electron skin depth and Ω_e is the electron gyrofrequency. It is then evident that $\overline{v_e} \approx B/\Delta n$, so $\overline{v_e}$ goes up as layer thickness decreases, density decreases (fewer charge carriers), and magnetic field increases.

These shifts in the electron distribution higher in velocity space, depicted in Figure 7 [1], cause a gap between electron and ion distribution centroids, and destabilize CDIA modes.

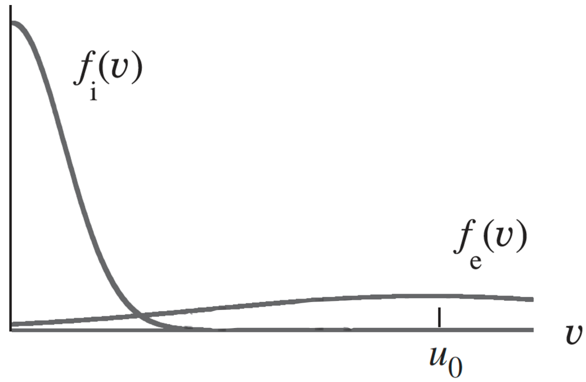

Figure 7 - Shifted electron distribution function relative to the ion's creates an unstable region (gray). Note $u_\circ = \overline{v_e}$. [1]

Like the BOT instability, there's an unstable range in velocity space where $df_e/dv > 0$, so one expects momentum exchange between waves and electrons to return the system to a marginally stable state. The new feature here is the addition of anomalous resistivity, $\eta_a(v)$, in the thickness definition,

$$\Delta^2 = L\big(\eta + \eta_a(\bar{v})\big)/v_A \tag{4.2.3}$$

which will help with resolving the system evolution. More details on anomalous resistivity will come later. The first method of calculating η_a in a 1D configuration begins with converting the electron kinetic equation to averaged momentum form in the vertical direction,

$$m_e v\left[\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial v} = C(f)\right] \tag{4.2.4a}$$

$$\frac{\partial\langle p_e\rangle}{\partial t} - q\left\langle E\int dv\, f\right\rangle = -\nu_{e,i} n_\circ m_e \overline{v_e} \tag{4.2.4b}$$

$$\frac{\partial\langle p_e\rangle}{\partial t} - q\big(n_\circ\langle E\rangle + \langle\tilde{E}\tilde{n}\rangle\big) = -\nu_{e,i} n_\circ m_e \overline{v_e} \tag{4.2.4c}$$

The righthand side of (4.2.4c) represents collisional losses to ions from which flux freezing breaks down. An approximately stationary state may be assumed, which, with some rearrangement, reduces (4.2.4c) to

$$\langle E\rangle + \langle\tilde{E}\tilde{n}/n_\circ\rangle = \eta\langle J\rangle \tag{4.2.5}$$

In order, these terms represent the driving field ($\sim\langle vB\rangle$), electron acceleration by turbulence ("anomalous resistivity"), and collisional resistivity. Solving for the anomalous resistivity term in k-space and making use of the electron density perturbation, $\widetilde{\rho_{k,e}}$,

$$\langle\tilde{E}\tilde{n}/n_\circ\rangle = \sum_k ik\,\widehat{\phi_{-k}}\,\frac{\widetilde{n_{k,e}}}{n_\circ} = \int dv\sum_k ik\,\widehat{\phi_{-k}}\widetilde{\rho_{k,e}} \tag{4.2.6}$$

The reader is here left to step back through the previous equations with this last result to apply QLT. Resonant transport must occur because of the stationarity assumption, so an energy theorem may also be applied for an alternative solution.

Interest in resonant particle interactions makes use of

$$\frac{\partial}{\partial \mathrm{t}}\left(\mathrm{E}_{\mathrm{rp}}+\mathrm{E}_{\mathrm{w}}\right)=0 \tag{4.2.7a}$$

$$\frac{\partial}{\partial \mathrm{t}}\left(p_{\mathrm{rp}}+p_{\mathrm{w}}\right)=0 \tag{4.2.7b}$$

and recalling the wave energy and momentum equations, respectively

$$\mathrm{E}_{\mathrm{k,w}}=\omega_k \frac{\partial \epsilon_r}{\partial \omega}|_k \frac{|E_k|^2}{8\pi}=\omega_k N_k \quad where\ N_k \in \mathrm{Z} \tag{4.2.8a}$$

$$p_{\mathrm{k,w}}=\frac{kE_{k,w}}{\omega}=kN_k \tag{4.2.8b}$$

CDIA waves are electrostatic, so the wave momentum only has a wave component, p_w, to counter resonant particle momentum, $p_{rp,e}$ (RPM),

$$\frac{\partial}{\partial \mathrm{t}} p_{\mathrm{rp,e}}=-\frac{\partial}{\partial \mathrm{t}} p_w=-\sum_k 2\,\gamma_{k,e} \frac{kE_{k,w}}{\omega_k} \tag{4.2.9}$$

Notice that while $\gamma_{k,e}$, the resonant electron growth rate, is positive, RPM decreases. Going further, represent p_w by a macroscale collision-dependent expression,

$$\frac{\partial}{\partial \mathrm{t}} p_{\mathrm{rp,e}}=-n m_e \nu_{eff} \bar{v} \tag{4.2.10}$$

Now, solve for $\bar{v}$ using (4.2.9),

$$\bar{v}\ =\sum_k 2\,\gamma_{k,e} \frac{kE_{k,w}}{\omega_k n m_e \nu_{eff}} \tag{4.2.11}$$

and relate the $\eta_a(\bar{v})$ term in Δ with $\bar{v}$,

$$\Delta^2=\mathrm{L}\left(\eta+\mathrm{c}^2 \nu_{\mathrm{eff}}/\omega_{\mathrm{pe}}^2\right)/\mathrm{v}_{\mathrm{A}} \tag{4.2.12}$$

Finally, proceed per usual with iterations of QLT: apply linear theory to a CDIA instability, solve $\epsilon_{k,w}$ at nonlinear saturation, retrieve $\langle f_e \rangle$, and repeat until marginally stable.

Linear analysis of CDIA modes shows,

$$\nabla^2 \tilde{\phi}=-4\pi|\mathrm{e}|(\tilde{\mathrm{n}}_\mathrm{i}-\tilde{\mathrm{n}}_\mathrm{e}) \tag{4.2.13a}$$

$$\tilde{n_i}/n_\circ = \frac{k^2 C_s^2}{\omega^2}\frac{|e|\phi}{T} \tag{4.2.13b}$$

$$\widetilde{n_e}/n_\circ = \frac{|e|\phi}{T}\big(1 - ir(k)\big) \tag{4.2.13c}$$

where $C_s^2 \approx T_e/m_i$. Then, to find r(k) we apply the Vlasov equation on distribution fluctuations,

$$\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x} + \frac{|q|E}{m_e}\frac{\partial \tilde{f}}{\partial v} = 0 \tag{4.2.14a}$$

$$\tilde{f} = \frac{|e|\phi}{T}\langle f\rangle + g \tag{4.2.14b}$$

and solve for the constant of integration responsible for the distribution shift, g,

$$\frac{\partial \tilde{g}}{\partial t} + v\frac{\partial \tilde{g}}{\partial x} = -v\frac{\partial}{\partial x}\frac{|e|}{T}\tilde{\phi}\langle f\rangle - \frac{\partial}{\partial t}\frac{|e|}{T}\langle f\rangle\tilde{\phi} + \frac{|q|}{m_e}\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial\langle f\rangle}{\partial v} \tag{4.2.15a}$$

$$= v\frac{\partial\tilde{\phi}}{\partial x}\frac{|e|}{T}\langle f\rangle + \frac{|q|}{m_e}\frac{\partial\tilde{\phi}}{\partial x}\frac{\bar{v} - v}{T/m_e}\langle f\rangle - \frac{\partial}{\partial t}\tilde{\phi}\frac{|e|}{T}\langle f\rangle \tag{4.2.15b}$$

$$= \bar{v}\frac{\partial\tilde{\phi}}{\partial x}\frac{|e|}{T}\langle f\rangle - \frac{\partial}{\partial t}\tilde{\phi}\frac{|e|}{T}\langle f\rangle \tag{4.2.15c}$$

By converting g to k-space,

$$g_k = -\frac{(\omega - k\bar{v})}{(\omega - kv)}\tilde{\phi}\frac{|e|}{T}\langle f\rangle \tag{4.2.16}$$

the distribution functions can be integrated and inserted into (ne/no) for the i*r(k) result,

$$-ir(k) = \int dv\frac{(\omega - k\bar{v})}{(\omega - kv)}\langle f\rangle = (k\bar{v} - \omega)\frac{i\pi}{|k|v_{th}}\bar{f}|_{\omega/kv_{th}}$$
$$where\ \ \bar{f}|_{\omega/kv_{th}} = \frac{1}{\sqrt{\pi}}e^{\frac{-(\omega/k-\bar{v})^2}{v_{th}^2}} \tag{4.2.17}$$

Next, using the relation $\omega_k^2 = (kC_s)^2/(1 + k^2\lambda_d^2)$ with (4.2.1.a-c) simplifies (4.2.17) to

$$(1 + k^2\lambda_d^2) = \frac{(kC_s)^2}{\omega^2} + \frac{(\omega - k\bar{v})i\pi}{|k|v_{th}}\bar{f}|_{\omega/kv_{th}} \tag{4.2.18}$$

Taking $\omega \to \omega + \delta\omega$ to represent a growing wave and setting the lefthand side of (4.2.18) leaves

$$0 = -\frac{2\delta\omega}{\omega} - \frac{(\omega - k\bar{v})i\pi}{|k|v_{th}}\bar{f}|_{\omega/kv_{th}} \to \frac{\delta\omega}{\omega} = -\frac{(\omega - k\bar{v})i\pi}{2|k|v_{th}}\bar{f}|_{\omega/kv_{th}} \tag{4.2.19}$$

Since $\delta\omega = i\gamma_k$, the critical velocity must be $\bar{v} = C_s$ by

$$\gamma_k = -\frac{(\omega - k\bar{v})\pi}{2|k|v_{th}} \bar{f}|_{\omega/kv_{th}} \; where \; \gamma k > 0 \text{ for } \bar{v} > C_s \tag{4.2.20}$$

Now, for the mean distribution's evolution we again return to the primary quasilinear equation and substitute 'g' from above,

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial v}\sum_k \widetilde{E_k}\frac{e}{m_e}\widetilde{g_k} \tag{4.2.21a}$$

$$= \frac{\partial}{\partial v}\sum_k \widetilde{E_k}\frac{e}{m_e}\frac{-(\omega - k\bar{v})}{(\omega - kv)}\widetilde{\phi_k}\frac{|e|}{T}\langle f\rangle \tag{4.2.21b}$$

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial v}\sum_k -v_{th}^2\,\pi\left|\frac{|e|\widetilde{\phi_k}}{T}\right|^2\langle f\rangle k(\omega - k\bar{v})\delta(\omega - k\mathrm{v}) \tag{4.2.21c}$$

Notice a minimal assumption regarding $\langle f\rangle$'s structure because,

$$\langle f\rangle = \left\langle f\,\frac{(v-\bar{v})^2}{2v_{th}^2}\right\rangle \rightarrow \frac{\partial\langle f\rangle}{\partial v} = \frac{v-\bar{v}}{v_{th}^2}\langle f\rangle' \;\; and \;\; \langle f\rangle' = -\langle f\rangle \tag{4.2.22}$$

A temporal evolution equation for $\bar{v}$ also becomes available,

$$\frac{\partial\bar{v}}{\partial \mathrm{t}} = \int dv\sum_k v_{th}^2\,\pi\left|\frac{|e|\widetilde{\phi_k}}{T}\right|^2\langle f\rangle k^2(\omega/\mathrm{k} - \bar{v})\,\delta(\omega - k\mathrm{v}) \tag{4.2.23}$$

For $\omega/\mathrm{k} < \bar{v}$, $\partial\bar{v}/\partial\mathrm{t} < 0$ and vice versa.

The remaining piece missing is a fluctuation intensity level of the field. A general form of this is

$$\frac{\partial \mathrm{E_{w,k}}}{\partial \mathrm{t}} = \gamma_k E_{w,k} - \left(\sum_{k'}\omega_k c_1(k,k')E_{w,k'}/nT\right)E_{w,k} - \left(\sum_{k',k''}\omega_k c_2(k,k',k'')E_{w,k'}E_{w,k''}/(nT)^2\right)E_{w,k} \tag{4.2.24}$$

In order from left to right, the righthand terms are

- Linear growth
- Quadratic Nonlinearities
  - Resonant mode coupling where $\omega_k + \omega_{k'} = \omega_{new} \; and \; k_k + k_{k'} = k_{new}$
  - Ion-wave interaction-driven quadratic nonlinearity
- Wave coupling-driven cubic nonlinearity

Resonant mode coupling and strong ion-wave interactions effects are minimal because all waves involved must be system eigenmodes and ion energies are predominately too low to overcome Landau damping, respectively. This leaves linear and cubic growth in (4.2.24), or

$$\frac{\partial \mathrm{E_{w,k}}}{\partial \mathrm{t}} = \left[\gamma_k - \omega_k B(\omega, k)\left(\frac{E_{w,k}}{nT}\right)^2\right] E_{w,k} \tag{4.2.25}$$

Applying stationarity modifies the field fluctuation intensity and growth rate to

$$\mathrm{E_{w,k}} = nt\sqrt{\gamma_k/\omega_k \mathrm{B}} \tag{4.2.26a}$$
$$\gamma_k = \frac{(\bar{\mathrm{v}} - \mathrm{C_s})\pi}{2|k|v_{th}} k\omega_k \bar{f}|_{\omega/kv_{th}} \tag{4.2.26b}$$

Then, estimate the turbulent collision frequency, ν_{eff}, which partially sets the anomalous resistivity and rate of magnetic reconnection,

$$\nu_{\mathrm{eff}} = \frac{2}{\mathrm{nm}\bar{\mathrm{v}}}\sum \gamma_{\mathrm{k,e}}\mathrm{E_{w,k}}\mathrm{k}/\,\omega_{\mathrm{k}} \sim \frac{T(1 - \mathrm{C_s}/\bar{\mathrm{v}})}{\mathrm{m}\bar{\mathrm{v}}\omega_{\mathrm{k}}B|k|v_{th}}\gamma_{\mathrm{k}}\mathrm{k}^2\bar{f}|_{\omega/kv_{th}} \tag{4.2.27a}$$
$$\sim \frac{(1 - \mathrm{C_s}/\bar{\mathrm{v}})\mathrm{k}^2 v_{th}\bar{f}}{|k|}\left(\frac{(\bar{\mathrm{v}} - \mathrm{C_s})\mathrm{k}}{B|kv_{th}|}\bar{f}\right)^{1/2} \sim \left((\bar{\mathrm{v}} - \mathrm{C_s})\mathrm{k}\bar{f}\right)^{3/2}\frac{\mathrm{k}(v_{th}/\bar{\mathrm{v}})\bar{f}}{|k||kv_{th}|^{1/2}}$$
$$\nu_{eff} \approx \left((\bar{\mathrm{v}} - \mathrm{C_s})\mathrm{k}\bar{f}\right)^{3/2}\frac{\mathrm{k}v_{th}}{\bar{\mathrm{v}}|k||kv_{th}|^{1/2}} \tag{4.2.27b}$$

A coupling between collisions and macro response becomes apparent. In a collisionless scenario where $\nu_{eff} = 0$, current is finite, and $\Delta \sim \sqrt{L\nu/V_A}$, we conclude a self-consistent feedback model exists (Figure 8). The CDIA therefore hovers near marginal stability via the η/ν-Δ relationship! Decreasing η also decreases Δ, which subsequently increases Δ, and the process repeats. Outrunning this intrinsic regulation might be accomplished with a stronger magnetic field, which would drive wave-ion interaction nonlinearity and phase space turbulence (i.e. granulation) resulting in distortion of all particle species' distribution functions.

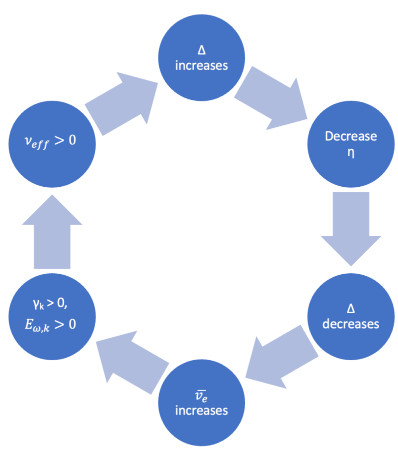

Figure 8 - Collisionless plasma CDIA feedback loop

5 Exercises

1. Derive the QLT momentum conservation equation analogous to the first energy theorem (resonant particle kinetic energy density vs wave energy density).
2. Derive the QLT equations and resonant diffusion from Fokker-Planck theory
    a. Use Hamiltonian dynamics to eliminate the dynamical friction term
3. Apply QLT to drift waves with the following configuration.
    a. What are the diffusion coefficients?
    b. Compare this 3D case to 1D Vlasov turbulence
    c. Form an expression for <f> plateaus

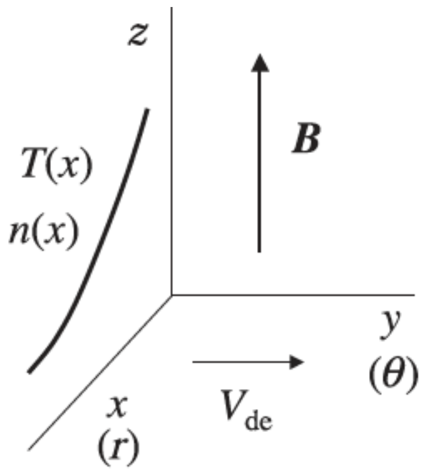

Figure 1 - 3D setup for problem 4

6 References

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7 Selected Solution

1. Derive the QLT momentum conservation equation analogous to the first energy theorem.

Take the momentum moment of the Vlasov equation in the x-direction for resonant electrons,

$$m\int v\,dv\left[\frac{\partial\langle f\rangle}{\partial t}+v\frac{\partial\langle f\rangle}{\partial x}=-\frac{\partial}{\partial v}\mathrm{D_{res}(v)}\frac{\partial\langle f\rangle}{\partial v}\right]$$

$$\frac{\partial P_{\mathrm{e}}}{\partial t}+\frac{\partial\Gamma_{\mathrm{e}}}{\partial x}=-\frac{\pi q^2}{m}\int v\,dv\sum_{\mathrm{k}}|\mathrm{E_k}|^2\delta\left(\frac{\omega}{k}-v\right)\frac{\partial\langle f\rangle}{\partial v}$$

Note P_e is the total resonant electron momentum and Γ_e is the resonant electron momentum flux.

$$\frac{\partial P_e}{\partial t}+\frac{\partial\Gamma_e}{\partial x}=-\frac{\pi q^2}{m}\int v\,dv\sum_k|E_k|^2\frac{\partial\langle f\rangle}{\partial v}|_{\omega/k}=-2\sum_k|E_k|^2\gamma_k\frac{k}{\omega}$$

Electrostatic waves like Langmuir oscillations have energy density evolutions described by

$$\frac{d|E_k|^2}{dt}=\frac{\partial|E_k|^2}{\partial t}+V_{gr}\frac{\partial|E_k|^2}{\partial x}-\frac{\partial\omega}{\partial x}\frac{\partial|E_k|^2}{\partial k}=2\sum_k|E_k|^2\gamma_k$$

Taking $V_{gr}=\frac{\partial\omega}{\partial k}$, $\omega=\omega_p$, and neglecting spatial density variations in ω_p leaves

$$\frac{\partial P_e}{\partial t}+\frac{\partial\Gamma_e}{\partial x}=-\left[\frac{\partial}{\partial t}\sum_k|E_k|^2\frac{k}{\omega}+\frac{\partial\omega}{\partial k}\frac{\partial}{\partial x}\sum_k|E_k|^2\frac{k}{\omega}\right]=-\frac{\partial P_w}{\partial t}-\frac{\partial\Gamma_w}{\partial x}$$

Like before, $P_w=\sum_k|E_k|^2\frac{k}{\omega}$, so multiplying by the group velocity yields wave momentum flux density. Thus, resonant electron momentum is balanced with wave momentum,

$$\frac{\partial}{\partial t}(P_e+P_w)+\frac{\partial}{\partial x}(\Gamma_e+\Gamma_w)=0$$

The spatial derivative can be generalized to additional directions via divergence as applicable.