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Random walks with self-similar clusters

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ABSTRACT We construct a random walk on a lattice having a hierarchy of self-similar clusters built into the distribution function of allowed jumps. The random walk is a discrete analog of a Lévy flight and coincides with the Lévy flight in the continuum limit. The Fourier transform of the jump distribution function is the continuous nondifferentiable function of Weierstrass. We show that, for cluster formation, it is necessary that the mean-squared displacement per jump be infinite and that the random walk be transient. We interpret our random walk as having an effective dimension higher than the spatial dimension available to the walker. The difference in dimensions is related to the fractal (Hausdorff–Besicovitch) dimension of the self-similar clusters.

For a one-dimensional random walk on an infinite perfect lattice of spacing Δ, let the probability of occupancy of the lth site after n steps be denoted by $P_n(l)$ and let $p(l)$ be the probability of a step having a displacement of l sites. Then,

$$P_{n+1}(l) = \sum_{l'=-\infty}^{\infty} p(l-l')P_n(l'). \qquad [1]$$

This equation can be solved for $P_n(l)$ given any $P_0(l)$ by a combination of discrete Fourier transform and generating function techniques (1). For a walk commencing at the origin—i.e., $P_0(l) = \delta_{l,0}$—it is found that

$$\tilde{P}_n(k) = [\lambda(k)]^n, \qquad [2]$$

where

$$\tilde{P}_n(k) = \sum_{l=-\infty}^{\infty} e^{ikl}P_n(l) \qquad [3]$$

and

$$\lambda(k) = \sum_{l=-\infty}^{\infty} e^{ikl}\, p(l). \qquad [4]$$

For a symmetric random walk $[p(l) = p(-l)]$ having finite mean-square displacement per jump,

$$\langle l^2 \rangle = \sum_{l=-\infty}^{\infty} l^2 p(l), \qquad [5]$$

the "structure function" $\lambda(k)$ has the asymptotic form

$$\lambda(k) = 1 - \tfrac{1}{2}\langle l^2\rangle k^2 + o(k^2). \qquad [6]$$

Then, for large n and $k = O(n^{-1/2})$,

$$\tilde{P}_n(k) \approx \left(1 - \frac{1}{n}\cdot\frac{1}{2}\langle l^2\rangle n k^2\right)^n \approx \exp -\tfrac{1}{2}n\langle l^2\rangle k^2, \qquad [7]$$

so that

$$P_n(l) \approx \{2\pi\langle l^2\rangle n^{-1/2} \exp\{-l^2/(2\langle l^2\rangle n\}. \qquad [8]$$

This Gaussian form corresponds to a diffusing packet of probability and ensures that, as $n \to \infty$, the walk spreads out to occupy all sites with a slowly varying and essentially unstructured distribution. Walks on a multidimensional space lattice are characterized by similar equations with l being interpreted as a lattice point vector and k being interpreted as an appropriate vector.

Suppose now that the walk is symmetric but that $\langle l^2\rangle = \infty$. Then, passage to the continuum limit in space and time in the standard fashion will give one of the symmetric "stable" distributions of Lévy (2) for the probability density function $P(x,t)$ a time t after the walk begins. These distributions are most easily characterized in Fourier space by the equation

$$\tilde{P}(q,t) = \int_{-\infty}^{\infty} e^{iqx}P(x,t)dx = \exp\{-A|q|^{\mu}t\}, \qquad [9]$$

where A and μ are real positive constants and $0 < \mu < 2$. One of the many possible distributions of individual step lengths leading to Eq. **9** is the Lévy distribution of order μ itself, for which

$$p(x) \approx \text{constant} \cdot |x|^{-1-\mu} \qquad (0 < \mu < 2) \qquad [10]$$

as $|x| \to \infty$, so that the mean-square displacement per step

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 p(x)dx \qquad [11]$$

is infinite. Mandelbrot's computer simulations of such "Lévy flights" in two dimensions (3) yield much more interesting trajectories in the case $\mu < 2$ than in the "diffusive" case ($\mu = 2$ or any flight for which $\langle x^2\rangle < \infty$). Points visited appear in clusters, well separated in space. Under magnification, each cluster is found to consist of a set of clusters, each of which in turn is a set of clusters and so on, giving a nested hierarchy of "self-similar" clusters. The clustering is found to be more pronounced (i.e., tighter clusters, further apart in space) as μ decreases. In a Lévy flight, the trajectory spreads out over space as $t \to \infty$ but in a qualitatively different manner from the diffusive case.

To illustrate how the self-similar clusters can arise, we present a discrete analog of the Lévy flight that has the clustering property built into the transition probability distribution $p(l)$. Furthermore, we show how the effective dimensionality of the random walk is increased by the clustered nature of the random walk paths.

A RANDOM WALK WITH CLUSTERS

A random walk on the one-dimensional continuum that exhibits clustering may be constructed by taking a discrete, but unevenly spaced, distribution of allowed step lengths. Specifically, we consider a symmetric walk with step lengths $\{\Delta b^n\}$ and probabilities corresponding to these step lengths proportional to a^{-n}, where a, b, and Δ are positive constants and both a and b exceed unity. The probability density function for a step x is then

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$$p(x) = \frac{a-1}{2a} \sum_{n=0}^{\infty} a^{-n}\{\delta(x - \Delta b^n) + \delta(x + \Delta b^n)\}. \quad [12]$$

This density function has the property that a step of length Δb^n is a times more likely than the next longest step. It follows (roughly speaking) that the walker will make about a steps of a given order of magnitude (and many shorter steps), forming a cluster, before moving an order of magnitude further away and beginning a new cluster. While a determines the number of points in a cluster, b determines the spatial separation of the clusters.

The mean-square displacement per step, given by Eq. **11**, is

$$\langle x^2 \rangle = \frac{a-1}{a} \Delta^2 \sum_{n=0}^{\infty} \{b^2/a\}^n \quad [13]$$

and is thus infinite if $b^2 \geq a$. Only the case $b^2 > a$ will be discussed here. (The case $b^2 < a$ leads ultimately to diffusive behavior and therefore nothing new, and the case $b^2 = a$ requires but a few modest extensions of the present analysis.)

If b is an integer, then the walk takes place on a lattice of spacing Δ, and the probability of a displacement of l sites at a given step is

$$p(l) = \frac{a-1}{2a} \sum_{n=0}^{\infty} a^{-n} \{\delta_{l,-b^n} + \delta_{l,b^n}\}. \quad [14]$$

The structure function [4] for this lattice walk is

$$\lambda(k) = \frac{a-1}{a} \sum_{n=0}^{\infty} a^{-n}\cos(b^n k). \quad [15]$$

Its small-k behavior will determine the differential equation for $P(x,t)$ in the continuum limit and the persistence or transience of the walk (Eq. **33**). The nonanalytic behavior of $\lambda(k)$ at $k = 0$ is most easily exhibited if the cosine is replaced by its inverse Mellin transform with respect to $|k|$ (4)

$$\cos(b^n k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (b^n k)^{-s}\Gamma(s)\cos(\tfrac{1}{2}\pi s)\, ds,$$

$$0 < c = \mathrm{Re}(s) < 1. \quad [16]$$

(We have taken $k > 0$ to avoid the need to write $|k|$ everywhere.) Interchanging orders of integration and summation gives a contour integral for $\lambda(k)$, namely

$$\lambda(k) = \frac{(a-1)/a}{2\pi i} \int_{c+i\infty}^{c+i\infty} \frac{k^{-s}\Gamma(s)\cos(\tfrac{1}{2}\pi s)}{1 - a^{-1}b^{-s}}\, ds,$$

$$0 < c = \mathrm{Re}(s) < 1 \quad [17]$$

(the infinite series being identified as a geometric progression). The integrand is a meromorphic function of s, with simple poles at $s = 0, -2, -4, \ldots$ [arising from the factor $\Gamma(s)\cos(\pi s/2)$] and at $s = -\mu + 2n\pi i/\ln b$, $n = 0, \pm 1, \pm 2, \ldots$ (arising from zeros of the denominator), where

$$\mu = \ln a/\ln b. \quad [18]$$

Translating the integration contour to $\mathrm{Re}(s) = -\infty$ and taking account of the residues at the poles crossed, we find that, for $k > 0$,

$$\lambda(k) = \frac{a-1}{a} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{k^{2m}}{\{l - b^{2m}/a\}} + k^{\mu}Q(k), \quad [19]$$

$$Q(k) = \frac{a-1}{a\ln b} \sum_{n=-\infty}^{\infty} \Gamma(s_n)\cos(\tfrac{1}{2}\pi s_n)\exp(-2n\pi i \ln k/\ln b),$$

$$s_n = -\mu + 2n\pi i/\ln b. \quad [20]$$

By bounding certain contour integrals, the analysis leading to Eqs. **19** and **20** can be justified rigorously when $\tfrac{1}{2} < \mu < 2$. For $0 < \mu \leq \tfrac{1}{2}$, a convergence factor is needed and the same result is obtained if the summation in Eq. **20** is interpreted appropriately. An alternative approach, via Poisson's summation formula, is given in the *Appendix*. It follows from Eq. **15** that $\lambda(k)$ satisfies the functional equation

$$\lambda(k) = a^{-1}\lambda(bk) + \frac{a-1}{a}\cos k. \quad [21]$$

The singular part of $\lambda(k)$ at $k = 0$ has the form of a power (k^{μ}) modulated by an oscillatory function $Q(k)$, which is periodic in $\ln k$ with period $\ln b$, making $\lambda(k)$ a much more complicated structure function than is usually encountered in lattice walks.

One can generalize the random walk with clusters defined above to multidimensional space lattices. We quote here the results for a two-dimensional square lattice (the extension to d dimensions being obvious). When Eq. **14** is replaced by

$$p(\mathbf{l}) = \frac{a-1}{4a} \sum_{n=0}^{\infty} a^{-n}\{(\delta_{l_1,-b^n} + \delta_{l_1,b^n})\delta_{l_2,0}$$

$$+ (\delta_{l_2,-b^n} + \delta_{l_2,b^n})\delta_{l_1,0}\}, \quad [14a]$$

the structure function becomes

$$\lambda(\mathbf{k}) = \frac{a-1}{2a} \sum_{n=0}^{\infty} a^{-n}\{\cos(k_1 b^n) + \cos(k_2 b^n)\} \quad [15a]$$

$$= \frac{a-1}{2a} \sum_{m=0}^{\infty} \frac{(-1)^m({k_1}^{2m} + {k_2}^{2m})}{(2m)!\,\{1 - b^{2m}/a\}}$$

$$+ \tfrac{1}{2}\{|k_1|^{\mu}Q(|k_1|) + |k_2|^{\mu}Q(|k_2|)\}, \quad [19a]$$

and we have the functional equation

$$\lambda(\mathbf{k}) = a^{-1}\lambda(b\mathbf{k}) + \frac{a-1}{2a}\{\cos k_1 + \cos k_2\}. \quad [21a]$$

To show the connection between the clustered random walk and Lévy distributions, we consider the general random walk defined by Eq. **12**, so that b is no longer restricted to integer values. The probability density function for the position after n steps satisfies the recursion relation

$$P_{n+1}(x) = \int_{-\infty}^{\infty} p(x - x')P_n(x')dx', \quad [22]$$

so that, if the steps occur at equal time intervals τ,

$$\frac{1}{\tau}\{P_{n+1}(x) - P_n(x)\}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\tau}\{p(x - x') - \delta(x - x')\}P_n(x')dx'. \quad [23]$$

In the limit $\tau \to 0$, we obtain an equation for the probability density function at time t:

$$\frac{\partial}{\partial t} P(x,t) = \int_{-\infty}^{\infty} \lim_{\tau \to 0} \frac{1}{\tau}\{p(x - x') - \delta(x - x')\}P(x',t)dx'. \quad [24]$$

It is in fact also necessary to let $\Delta \to 0$ in Eq. **24** to ensure a finite result. The analysis is most easily performed in Fourier space:

$$\frac{\partial}{\partial t}\tilde{P}(q,t) = \lim_{\Delta,\tau\to 0}\left[\frac{1}{\tau}\{\tilde{p}(q) - 1\}\right]\tilde{P}(q,t). \quad [25]$$

For a random walk on a lattice having finite mean-square displacement per jump,

$$\tilde{p}(q) = 1 - \tfrac{1}{2}\Delta^2\langle l^2\rangle q^2 + o(\Delta^2), \quad [26]$$

so that we obtain

$$\frac{\partial}{\partial t}\tilde{P}(q,t) = -Dq^2\tilde{P}(q,t), \quad [27]$$

provided that Δ,τ tend to zero in such a manner that

$$D = \lim_{\Delta,\tau\to 0} \tfrac{1}{2}\Delta^2\langle l^2\rangle/\tau \quad [28]$$

is finite. Eq. **27** is the Fourier transform of the diffusion equation. For the clustered walk, however, we have

$$\tilde{p}(q) = \lambda(\Delta q). \quad [29]$$

It follows that, if $a \to 1+$ and $b \to 1+$ as Δ,τ tend to zero, such that

$$\left\{\begin{matrix} a = 1 + \alpha\Delta + o(\Delta),\ b = 1 + \beta\Delta + o(\Delta) \\ \Delta^\mu/\tau = \text{constant}\end{matrix}\right\},\ 0 < \alpha < 2\beta, \quad [30]$$

then

$$\lim_{\Delta,\tau\to 0}\left[\frac{1}{\tau}\{\tilde{p}(q) - 1\}\right] = \frac{-(\pi/2)|q|^{\alpha/\beta}}{\Gamma(\alpha/\beta)\sin(\tfrac{1}{2}\pi\alpha/\beta)}\cdot\lim_{\Delta,\tau\to 0}\Delta^{\alpha/\beta}/\tau. \quad [31]$$

Hence (where D_1 is a constant),

$$\frac{\partial}{\partial t}\tilde{P}(q,t) = -D_1\,|q|^{\alpha/\beta}\,\tilde{P}(q,t), \quad [32]$$

which implies, from Eq. **9**, that P has a Lévy distribution of order α/β.

THE EFFECTIVE DIMENSIONALITY OF A CLUSTERED WALK

A lattice walk is called *persistent* if the walker is certain to return to his starting point and transient otherwise. A persistent random walk will reach every lattice point with certainty. Thus, walks that have clustering must be transient. It can be shown (1) that, for a symmetric random walk on an N-dimensional lattice, the probability that the walker escapes (never returns to his starting point) is 1/u, where

$$\mathrm{u} = \frac{1}{(2\pi)^N}\int_{-\pi}^{\pi}\cdots\int \{1 - \lambda(\boldsymbol{k})\}^{-1}d^N\boldsymbol{k}. \quad [33]$$

The walk is persistent if and only if this integral diverges. The convergence or divergence of this integral is the same as the convergence or divergence of the integral of $\{1 - \lambda(\boldsymbol{k})\}^{-1}$ over an N-dimensional sphere of radius $r < \pi$, centered on $\boldsymbol{k} = \boldsymbol{0}$, namely

$$\int\cdots\int_{|\boldsymbol{k}|\le r} \{1 - \lambda(\boldsymbol{k})\}^{-1}d^N\boldsymbol{k} = S_N\int_0^r \{1 - \lambda(\boldsymbol{k})\}^{-1}|\boldsymbol{k}|^{N-1}d|\boldsymbol{k}|, \quad [34]$$

where S_N is the surface area of a unit sphere in N dimensions. (We have transformed to spherical polar coordinates, assuming that λ is a function only of $|\boldsymbol{k}|$.) For a walk with $\langle l^2\rangle < \infty$, the right-hand side of Eq. **34** has the same behavior as

$$\int_0^r |\boldsymbol{k}|^{N-3}d|\boldsymbol{k}| = \left\{\begin{matrix}\infty \text{ if } N \le 2 \\ \text{finite value if } N > 2\end{matrix}\right\}, \quad [35]$$

so that we have *Polya's theorem:* An N-dimensional symmetric lattice walk with $\langle l^2\rangle < \infty$ is persistent if $N = 1$ or 2 and transient if $N \ge 3$. As $1 - \lambda(\boldsymbol{k})$ can vanish no faster than $|\boldsymbol{k}|^2$ as $|\boldsymbol{k}| \to 0$, irrespective of the value (finite or infinite) of $\langle l^2\rangle$, all walks are transient if $N \ge 3$. In fewer than three dimensions, the leading order behavior of $1 - \lambda(\boldsymbol{k})$ as $|\boldsymbol{k}| \to 0$ is all-important.

Gillis and Weiss (5) have considered a one-dimensional lattice walk, in which $p(l) = \text{constant}\cdot|l|^{-(1+\alpha)}$ $(0 < \alpha \le 2)$, for which it can be shown that $1 - \lambda(k) \approx \text{constant}\cdot|k|^\alpha$ if $0 < \alpha < 2$, and $1 - \lambda(k) \approx \text{constant}\cdot|k|^2\ln|k|$ if $\alpha = 2$. They show that, in the case where $\alpha = 1$, the number of distinct sites visited after n steps has the same asymptotic form as that for a walk with $\langle l^2\rangle < \infty$ on a two-dimensional lattice, so that $\langle l^2\rangle = \infty$ increases the effective dimensionality of the walk. From Eq. **35**, we see that the Gillis–Weiss walk is persistent if $\alpha \ge 1$ and transient otherwise.

These considerations lead us to define a lattice walk having finite variance on a space of dimension F (not necessarily integral) as a walk for which

$$\{1 - \lambda(\boldsymbol{k})\}^{-1}|\boldsymbol{k}|^{N-1} \approx \text{constant}\cdot|\boldsymbol{k}|^{F-3} \text{ as } |\boldsymbol{k}| \to 0. \quad [36]$$

If $\langle l^2\rangle < \infty$, then F is the same as the spatial dimension N. However, if $1 - \lambda(\boldsymbol{k}) \approx \text{constant}\cdot|\boldsymbol{k}|^\mu$, with $0 < \mu < 2$, we have

$$\{1 - \lambda(\boldsymbol{k})\}^{-1}|\boldsymbol{k}|^{N-1} \approx \text{constant}\cdot|\boldsymbol{k}|^{N-1-\mu}, \quad [37]$$

so that

$$F = N + 2 - \mu. \quad [38]$$

The effective dimension thus exceeds the spatial dimension if $\mu < 2$. [In the case where $\mu = 1$, we see that $F = N + 1$, which is consistent with the Gillis–Weiss result that the effective dimensionality of a one-dimensional walk is increased to two if $1 - \lambda(\boldsymbol{k}) \propto |\boldsymbol{k}|$.] A random walk is persistent if $F \le 2$ and transient otherwise.

The ideas outlined above in the context of lattice walks may be extended to walks in continuous space. To simplify the discussion here, we consider only symmetric walks in one spatial dimension. A walk in *continuous* space is called persistent if the walker is certain to return infinitely often to any neighborhood of the starting point and transient otherwise. In one dimension, it can be shown (6) that a walk is persistent if, for every $c > 0$,

$$\int_0^c \{1 - \tilde{p}(q)\}^{-1}\,dq = \infty \quad [39]$$

and transient if, for some $c > 0$, this integral is finite.

It has been shown that, for the random walk that has clusters defined by Eq. **12**, $\tilde{p}(q) - 1 \approx Q(|\Delta q|)|\Delta q|^\mu$, where Q is bounded as $q \to 0$ and $0 < \mu = (\ln a/\ln b) < 2$. This suggests that the walk is persistent for $\mu \ge 1$ and transient for $0 < \mu < 1$. The easiest way to establish this rigorously is to use two criteria given by Feller (6): (*i*) if $|1 - \tilde{p}(q)| < \text{constant}\cdot|q|$ in a neighborhood of the origin, then a symmetric walk is persistent and (*ii*) if, for some $\rho > 0$,

$$t^{-1-\rho}\int_{-t}^{t} x^2p(x)dx \to \infty \text{ as } t \to \infty, \quad [40]$$

then the walk is transient. (Let $t = b^n$ with $n \to \infty$. Then the left-hand side is divergent if $\rho < 1 - \mu$. To ensure that $\rho > 0$, it suffices to have $\mu < 1$.) By analogy with the lattice walk, we describe the clustered walk as taking place in a space of effective dimension

$$F = 3 - \mu,\ \mu = \ln a/\ln b. \quad [41]$$

The exponent μ can be considered to be the fractal dimension of the self-similar clusters. A definition of the fractal (Hausdorff–Besicovitch) dimension of a set may be phrased as follows: If a finite part Ω of the set can be divided into γ identical parts, each of which is geometrically similar to Ω, with a similarity ratio r, then $D = \ln\gamma/\ln(1/r)$. For our clustered random walk, we note that each cluster can be divided into a set of clusters one order lower in the hierarchy and there are, on the average, about a of these lower order clusters, so that $\gamma \simeq a$. Each lower order cluster, when expanded by a factor b, returns the original cluster, giving $r \simeq 1/b$. It follows that

$$D \simeq \ln a/\ln b = \mu. \quad [42]$$

This argument is not precise but gives an immediate interpretation of μ as the expected Hausdorff–Besicovitch dimension of the trajectory of the walker. When $D \geq 1$, the set of sites visited will fill the one-dimensional space in which the walk takes place. This is why $\mu \geq 1$ implies persistence. On the other hand, if $\mu < 1$, then $D < 1$, the set of sites visited is not dense, and the walk is transient.

Our discrete analog of a transient Lévy flight in one spatial dimension shows how self-similar clusters can arise in a stochastic process. Other fractal random walks, as well as walks that have non-self-similar clusters, in one or more spatial dimensions, can be explored in a similar manner. A curious analogy between the functional equation (Eq. **21**) and the transformation equation for the free energy of a system of lattice spins under the renormalization group (7) merits further consideration.

A REMARK ON THE HISTORY OF THE FUNCTION $\lambda(k)$

The structure function $\lambda(k)$, defined by Eq. **15**, can be expressed in terms of Weierstrass' function

$$W(x) = \sum_{n=0}^{\infty} A^n \cos(B^N \pi x), \quad [43]$$

which is continuous in x for $0 < A < 1$ but does not possess a finite derivative at any value of x when $AB \geq 1$. A history of $W(x)$ and similar functions considered by Riemann and Cellérier can be found in books by Singh (8) and Mandelbrot (3). The definitive treatment of $W(x)$ is that of Hardy (9), who established the result that, if $\mu = [\ln(1/A)/\ln B] < 1$, then

$$W(x+h) - W(x) = \mathcal{O}(|h|^{\mu}) \quad [44]$$

at every value of x but at no value of x is

$$W(x+h) - W(x) = o(|h|^{\mu}). \quad [45]$$

The transformation of $W(x)$ established by Eq. **19** is an illustration of Hardy's result at the point $x = 0$ and appears to be new although Berry and Lewis (10) have obtained a similar transformation, via the Poisson summation formula, for the Weierstrass–Mandelbrot function:

$$\sum_{n=-\infty}^{\infty} [1 - e^{i\gamma^n t}] e^{in\mu} \gamma^{n(D-2)} = (\ln\gamma)^{-1} \exp(\tfrac{1}{2} i\pi D - \tfrac{1}{2}\mu\pi/\ln\gamma) \sum_{m=-\infty}^{\infty} \Phi_m(t) \quad [46]$$

$(1 < D < 2,\ \gamma > 1)$, with

$$\Phi_m(t) = e^{-\pi m^2/\ln\gamma}\, \Gamma\left(D - 2 + \frac{i(\mu + 2\pi m)}{\ln\gamma}\right) t^{2-D} \exp\left\{-\frac{i(\mu + 2\pi m)\ln t}{\ln\gamma}\right\}. \quad [47]$$

There is no regular component of the function corresponding to the first series in Eq. **19** because of the absence of a smallest scale. The Weierstrass spectrum γ^n $(-\infty < n < \infty)$ is shown by Berry and Lewis to be generated by Schrodinger's equation with a weakly singular potential.

APPENDIX: DERIVATION OF THE ASYMPTOTIC FORM OF $\lambda(k)$ NEAR $k = 0$

It is possible to give a brief derivation of Eqs. **19** and **20** (rigorous for $1 < \mu < 2$) by using Poisson's summation formula (11)

$$\sum_{n=0}^{\infty} f(n) = \tfrac{1}{2} f(0) + \int_0^{\infty} f(t)dt + 2\sum_{m=1}^{\infty} \int_0^{\infty} f(t)\cos(2\pi mt)\, dt, \quad [A1]$$

where $f(t) = a^{-t}\cos(b^t k) = \exp(-t\ln a)\cos(k\exp[t\ln b])$, so that

$$\sum_{n=0}^{\infty} a^{-n}\cos(b^n k) = \tfrac{1}{2}\cos k + (\ln b)^{-1} \sum_{m=-\infty}^{\infty} \int_0^{\infty} \exp\left\{\left(-\mu + \frac{2\pi mi}{\ln b}\right)t\right\} \cos(ke^t)dt \quad [A2]$$

$$= \tfrac{1}{2}\cos k + (\ln b)^{-1} \sum_{m=-\infty}^{\infty} \int_1^{\infty} v^{-\mu + \frac{2\pi mi}{\ln b} - 1} \cos(kv)dv.$$

For $0 < \mathrm{Re}(z) < 1$ and $k > 0$

$$k^{-z}\Gamma(z)\cos(\tfrac{1}{2}\pi z) = \int_0^{\infty} v^{z-1}\cos(kv)dv$$

$$= \int_1^{\infty} v^{z-1}\cos(kv)dv + \int_0^1 v^{z-1} \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n} v^{2n}}{(2n)!}\, dv$$

$$= \int_1^{\infty} v^{z-1}\cos(kv)dv + \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!(2n+z)}. \quad [A3]$$

By the principle of analytic continuation, Eq. **A3** holds for $z \neq 0, -2, -4, \ldots$, so that

$$\sum_{n=0}^{\infty} a^{-n}\cos(b^n k) = g(k) + \frac{k^{\mu}}{\ln b} \sum_{m=-\infty}^{\infty} \exp\left(\frac{-2\pi mi\ln k}{\ln b}\right) \Gamma\left(-\mu + \frac{2\pi mi}{\ln b}\right) \cos\left\{\frac{\pi}{2}\left(-\mu + \frac{2\pi mi}{\ln b}\right)\right\}, \quad [A4]$$

where

$$g(k) = \tfrac{1}{2}\cos k - (\ln b)^{-1} \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!\left(2n - \mu + \frac{2\pi mi}{\ln b}\right)}. \quad [A5]$$

Identifying the double series as

$$\lim_{M\to\infty} \sum_{m=-M}^{M} \sum_{n=0}^{\infty} = \lim_{M\to\infty} \sum_{n=0}^{\infty} \sum_{m=-M}^{M}$$

and noting (12) that

$$\sum_{m=0}^{\infty} (m^2 + z^2)^{-1} = \frac{\pi}{2z}\left\{\coth(\pi z) + \frac{1}{\pi z}\right\}, \quad \text{[A6]}$$

we find, after a little algebra, that $g(k)$ reduces to the required series. This derivation is rigorous only for $1 < \mu < 2$ because Eq. **A1** is derived on the assumption that f is of bounded variation. However, a more powerful form of Poisson's summation formula due to Borgen (13) can be used to extend the analysis to the case $0 < \mu \leq 1$. It can be shown in this manner that Eqs. **19** and **20** remain valid for $0 < \mu \leq 1$, provided that if $\mu \leq \frac{1}{2}$ the series for $Q(k)$ is summed by Abel's means (14)—i.e., a convergence factor $\exp(-\delta|n|)$ is inserted and the limit $\delta \to 0$ is taken after the evaluation of the sum.

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