

Notes 11 - L-stable Distributions, Levy Walks etc.

Recall many new elements:

- H-parameter $0 < H < 1/2$ - subdiffusion (sticky)
 ("random" stationary series) $1/2 < H < 1$ - superdiffusion (flights, ballistic)

$H = 1/2$ - diffusion (Gaussian Distrib.)

also: $\begin{cases} 1/f \text{ noise} \leftrightarrow H \sim 1 \\ \text{Zipf's Law } P \sim 1/\Delta x \end{cases}$

- Range of H ~~and~~ should broaden our view of random processes

Fundamental element of conventional wisdom on random processes is Central Limit Theorem.

But range of $H \rightarrow$ more than Gaussians...

Recall Central Limit Theorem:

(c.f. Chandrasekhar
Kubo, et al. "Statistics
Physics II")

- Consider a sum
of N independent
random variables

$$\Delta X_1, \Delta X_2, \dots, \Delta X_N \quad (N \gg 1)$$

$$\text{Let } X_N = \Delta X_1 + \Delta X_2 + \dots + \Delta X_N$$

- For ΔX_i ; $\langle \Delta X_i \rangle = 0$

$$\langle \Delta X_i^2 \rangle = \sigma_i^2$$

n.b. Note only variance (i.e. second moment) assumed to exist.

$$\text{Let } \sigma_N^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_N^2$$

Then $\Rightarrow$

- CLT states that Pdf of sum:

$$\text{Pdf}(y) \xrightarrow{N \rightarrow \infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$y_N = X_N / \sigma_N$$

$\rightarrow$ Gaussian,

i.e.
$$P.d.f(X_n) \stackrel{d}{\approx} \frac{1}{\sqrt{2\pi}\sigma_n} \exp(-X_n^2 / 2\sigma_n^2)$$

n >>> 7

— key points:

i.) ΔX_i all alike, ^{sum} not dominated by few.

ii.) finite variance of step prob.

Buried Bodies:

— what of higher moments? i.e.

$$\langle \Delta X_i^2 \rangle < \infty \quad \text{and} \quad \langle \Delta X_i^4 \rangle < \infty$$

$\Rightarrow$ kurtosis can induce heavy tails.

— how much? effect

Now, observe that CLT effectively states that, modulo conditions,

$X_i \rightarrow$ statistically distributed consistent with CLT

then if X_i, y_i follow CLT,

$$\text{then } (a''X_i + b'') + (a'Y_i + b') = a''Z_i + b''$$

Follows CLT.

$a, b > 0$ Gaussian
 $a', b' > 0$ non-stoch
const.

{ Adding two Gaussian distributed series $\Rightarrow$ sum is Gaussian distributed }

$\Rightarrow$ CLT $\Rightarrow$ Gaussian, modulo conditions, is attractor in function space.

The point: - Gaussian is a very special case!

- more generally, a class of such distributions exist, which are said to be L-stable (L for Levy).

Gaussian is one particular case, indeed the only which has finite variance.

class
Distrib. $\left[\begin{array}{l} X_i, Y_i \rightarrow \text{Levy Dist} \\ Z \rightarrow \text{Levy Data} \end{array} \right.$

Li

we need consider $\left. \begin{array}{l} X_i \\ Y_i \end{array} \right\}$ L-stable

$$(a X_i + b) + (a' Y_i + b') = a'' Z_i + b''$$

Z_i L-stable, $\rightarrow$ i.e. class of attractors in function space.

$\rightarrow$ Big expansion in family of allowed distributions!

$\rightarrow$ Encompasses Super-Diffusion, etc.
for $H > 1/2$

$\rightarrow$ Levy Flight!

How understand L-stable processes?

First, re-visit Central Limit Theorem in depth.

$\rightarrow$ How is Central Limit Theorem proved?

Central ideas are:

- Markov property $\rightarrow$ Chapman-Kolmogorov Equation
- convolutions

$\Rightarrow$

L-stable Distributions

Prove CLT

— "Generating" or "Characteristic" function.

Point: Fourier transform of step probability is more significant than probability.

ii) C-K Egn:

$$P_N(x) = \int dy \cdot P_{N-1}(y) P_N(x|x-y)$$

(convolution)

i.e. $\begin{cases} x-y \rightarrow x \\ \text{don't expand} \end{cases}$

Then, if F.T., and noting that F.T. of ~~convolution~~ convolution = Product of functions convolved

$$P_N(k) = \int e^{-ikx} P_N(x)$$

$$\begin{aligned} P_N(k) &= \hat{P}_1(k) \hat{P}_2(k) \dots \hat{P}_N(k) \\ &= \prod_{n=1}^N \hat{P}_n(k) \end{aligned}$$

$$P_N(x) = \int dk e^{ikx} \prod_{n=1}^N \hat{P}_n(k)$$

applies for identical steps.

(ii)

then, also have, for moments;

$$M_1 = \langle x \rangle$$

$$M_2 = \langle x^2 \rangle$$

$$\vdots$$

$$M_n = \langle x^n \rangle$$

So

$$\hat{P}(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} M_n$$

(Prob.)

FT

$$M_n = i^n \left. \frac{\partial^n \rho}{\partial k^n} \right|_{k=0}, \text{ From F.T.}$$

Series

$$\hat{P}(k) = 1 - iM_1 k - \frac{1}{2} k^2 M_2 + \dots$$

→ easily generalized to higher dimensions

(iii) Cumulants

- nonlinear combinations of moments.

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$$P(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \hat{P}(k)$$

$$= \int_{-\infty}^{\infty} e^{[ikx + \psi(k)]} \frac{dk}{2\pi}$$

$$\psi(k) = \ln \hat{P}(k)$$

and,

$$\psi(k) = -i c_1 k - \frac{1}{2} c_2 k^2 + \dots$$

expand on ik

$$\left\{ \begin{aligned} c_1 &= m_1 \end{aligned} \right.$$

$$\left\{ \begin{aligned} c_2 &= m_2 - m_1^2 = \sigma^2 \quad \text{etc.} \end{aligned} \right.$$

generally,

(assumes exist!)

$$i^n c_n = \frac{\partial^n \psi}{\partial k_1 \partial k_2 \dots \partial k_n}$$

Now, cumulants additive - assuming
identically, independently distributed (IID)
steps:

$$P_N(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} P_N(k)$$

$$= \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} (P(k))^N$$

$$= \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} (e^{\psi(k)})^N$$

$$= \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} e^{N\psi(k)}$$

$$\boxed{\psi_N(k) = N\psi(k)}$$

Proof

So, for CLT;

consider as $N \rightarrow \infty$

$$P_N(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{1}{\pi} \hat{P}_n(k) \frac{dk}{2\pi}$$

$$= \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} (\hat{P}(k))^N$$

$$= \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} e^{N\psi(k)}$$

Now, additivity $\Rightarrow$

$$\psi_N = N \psi(k)$$

$$\psi(k) = -i C_1 k - \frac{1}{2} k^2 C_2 + \dots$$

↑

Thus

$$P_N(x) = \int \frac{dk}{2\pi} e^{ikx} e^{N\psi(k)}$$

For large N , only region near $k=0$ contributes, so! (Laplace method)

$$P_N(x) = \int_{|k| \ll 1} e^{ikx} e^{N\psi(k)} \frac{dk}{2\pi}$$

$$= \int_{|k| \ll 1} \frac{dk}{2\pi} e^{ikx} \exp \left[-iNC_1 k - \frac{N}{2} k^2 C_2 + \dots \right]$$

and can re-expand ^{integration} domain to:

l.o. cumulant
+
large N

$$C_1 = 0, \text{ etc}$$

11.

$$P_N(x) = \int \frac{dk}{2\pi} e^{ikx} e^{-iN\epsilon_k} e^{-\frac{N\epsilon_k^2}{2}}$$

F.T. of Gaussian is Gaussian, $\Rightarrow$

$$P_N(x) = \frac{1}{\sqrt{2\pi(N\epsilon_2)}} e^{-x^2/N\epsilon_2}$$

C. L. T.

asymptotic

A few points:

- no questions asked about higher moments. Indeed these need not be well behaved, and induce Fat Tails.

- i.e.
$$p(x) = \frac{1}{1+x^2}$$

has diverging second moment, but

$$p(x) = \frac{2}{\pi} (1+x^4)$$

meets C. L. T., but Kurtosis diverges $\Rightarrow$ Fat Tail.

Nominally, low probability events have large influence.

N.B. Interesting to examine consequences for resonance broadening theory

→ Can show:

— Gaussian eroded (fat tail)

→ large X (how?)

$$P_N(x) \sim NA/x^4$$

Fat tail (power, not Gaussian)

Issue of CLT with fat tails is interesting and merits more analysis. Good paper topic!

Now,

(Integrable)

→ Levy Stability and Distributions
(Bodhuhas, Random Walks and Random Environments)

Now,

$X_i \rightarrow$ random variables

$$X_N = \sum_{n=1}^N X_n$$

generalized
CLT

Zaslavsky

Levy Distribution and Process

$P(x) \rightarrow$ normalized distribution of random variable

$$\int_{-\infty}^{+\infty} dx P(x) = 1$$

with characteristic function:

$$P(z) = \int_{-\infty}^{+\infty} dx e^{i z x} P(x)$$

(n.b. why? $\Rightarrow$ C-K for multi-step, N i.i.d steps)

L -stability: X_1, X_2, X_3 random variables
s.t.

$$C_3 X_3 = C_1 X_1 + C_2 X_2$$

all distributed according to same P .

c.e. clearly applies to CLT/Gaussian.

c.e. "distributions additive" //

General form of additive, limit distribution
 $\Rightarrow$ Levy

additivity $\Rightarrow$

$$P(x_3) dx_3 = P(x_1) P(x_2) \delta(x_3 - \frac{c_1}{c} x_1 - \frac{c_2}{c} x_2) dx_1 dx_2$$

$$P(z) = \int_{-\infty}^{\infty} dx_3 e^{izx_3} P(x_3)$$

$$= \int dx_1 \int dx_2 e^{izx_3} P(x_1) P(x_2) \delta(x_3 - \frac{c_1}{c} x_1 - \frac{c_2}{c} x_2)$$

$$= \iint dx_1 dx_2 e^{iz(\frac{c_1}{c} x_1 + \frac{c_2}{c} x_2)} P(x_1) P(x_2)$$

$$= P(\frac{c_1}{c} z) P(\frac{c_2}{c} z)$$

so

$$P(cz) = P(c_1 z) P(c_2 z)$$

$$\ln P(z) = \ln P(z) + \ln P(z)$$

105 → Abner Lewis

Solution:

$$\ln P_\alpha(z) = (Cz)^\alpha$$

if α

$$C_1^\alpha + C_2^\alpha = C^\alpha$$

See also
rescaling

regular behavior
near $z \rightarrow 0$.

$$P_\alpha(z) = \exp[-C|z|^\alpha]$$

Levy
Distribution
with
Levy Index α

magnitude $\Rightarrow$ normalization

Levy Index: $0 < \alpha \leq 2$

$\Rightarrow \rho(x)$
positive

$\alpha = 2 \rightarrow$ Gaussian

$\alpha = 1 \rightarrow$ Cauchy / Lorentzian

i.e. $P(C, Z) P(C, Z) = P(C, Z)$
condition.

13.

Now, $\hat{P}_n(k) = [P(k)]^n \rightarrow$ characteristic
fctn

$$\begin{cases} Z_n = X_n / a_n & \rightarrow \text{rescaling} \\ P_n(Z_n) = F_n(X) / a_n & \leftarrow \text{self sim.} \\ X = X_n / a_n. \end{cases}$$

and:

$F_n(a_n k) = \hat{P}_n(k)$ (some fixed fctn.,
(attractor))

Now, went $F_n(k) \xrightarrow{n \rightarrow \infty} \bar{F}(k)$

let $\lim_{n \rightarrow \infty} \frac{a_n}{a_m} = C_n$ (not any)

Condition for function as limiting case.

$$\bar{F}(k C_n) = [\bar{F}(k)]^n \quad \leftarrow \begin{cases} \text{self-} \\ \text{sim.} \end{cases}$$

↑
scale.

so, need solve:

$$\bar{F}(k u(\alpha)) = [\bar{F}(k)]^\alpha \quad \leftarrow \begin{cases} \text{fixed} \\ \text{fctn.} \\ \text{condition} \end{cases}$$

$$\psi = \ln \hat{F}(k)$$

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$$\psi(ku(\lambda)) = \lambda \psi(k)$$

$$k \left(\frac{d\lambda}{d\lambda} \right) \psi(ku(\lambda)) = \psi(k)$$

$$u(\lambda=1) = 1 \quad (\text{must.})$$

$$k u' \psi = \psi(k)$$

$$\boxed{\frac{d\psi}{dk} = \psi / u'(1) k}$$

self-sim.
power
law

$$k \frac{d\psi}{dk} = 1/k$$

$$k \frac{d \ln \psi}{dk} = 1/k$$

self-sim.

$$\psi(k) = \begin{cases} v_1 |k|^\alpha \\ v_2 |k|^{-\alpha} \end{cases}$$

$$k > 0$$

$$k < 0$$

$$\hat{F}(k) = \begin{cases} \exp(v_1 |k|^\alpha) & k > 0 \\ \exp(v_2 |k|^{-\alpha}) & k < 0 \end{cases}$$

on, in more detail,

Levy Gen Funct is general self-similar
Gen Funct for random process / G.H.

15.

$$\hat{F}(k) = \exp \left[-a |k|^\alpha \left(1 - i\beta \tan \frac{\alpha\pi}{2} \operatorname{sgn}(k) \right) \right]$$

↑
skewness

Now, take $\beta = 0 \rightarrow$ Levy
Distribution

$$L_\alpha(a, k) = \hat{F}(k) = \exp(-a|k|^\alpha)$$

Gen
Funct.

$\alpha = 2$ $\hat{F}(k) = \exp[-a k^2]$ $0 < \alpha \leq 2$
 $\rightarrow$ Gaussian

$\alpha = 2$ is case of CLT ✓
 $P(x) = \exp[-x^2/a]$

$\alpha = 1$ $\hat{F}(k) = e^{-a|k|}$ ✓
 $\rightarrow$ Cauchy Lorentzian

$$P(x) = \frac{1}{\sqrt{a^2 + x^2}}$$

Note $\alpha = 2$ is max, and
 only Levy stable distribution
 with 2nd moment finite.

Weierstrass Random Walks

→ General structure for Fractional Kinetics

→ 1D, spacing 1
 p_j prob. to make step a_j

$$P(l) = \frac{1}{2} \sum_{j=1}^{\infty} p_j [\delta(l-a_j) + \delta(l+a_j)]$$

$\int$
 prob to
 make step l

$$\int_{-\infty}^{\infty} P(l) dl = 1$$

→ Require scaling:

$$a_j = a^j$$

$$p_j = C p^j$$

$$C = (1-p) \quad \text{norm.}$$

Q

$$P(l) = \frac{1}{2} (1-p) \sum_{j=0}^{\infty} p^j [\delta(l-a^j) + \delta(l+a^j)]$$

Characteristic function then:

$$P(k) = \int_{-\infty}^{+\infty} dl e^{ikl} P(l) = (1-p) \sum_{j=0}^{\infty} p^j \cos(k a^j)$$

Weierstrass Function

$$P(ka) = (1-p) \sum_{j=0}^{\infty} p^j \cos(ka a^j)$$

scaling ↓

Q

$$P(k) = p P(ka) + (1-p) \cos k$$

{ - pick up
factor
- read.

Now, solution of $P(k)$ equation can be written as:

$$P(k) = \underbrace{P_S(k)}_{\text{singular}} + \underbrace{P_R(k)}_{\text{regular}}$$

Now, singular part satisfies:

$$P_S(k) = \frac{1}{k} B(ak)$$

only $k \rightarrow 0$.

resembles:

RG eqn.

$$F(g) = l^{-d} F(g') + \epsilon(\epsilon)$$

$\rightarrow$ regular

$F \equiv$ free energy

$g =$ coupling const

$d =$ dims

Near $k \rightarrow 0$,

$$P_S(k) \sim 1/k^4 \odot(k)$$

②

$$|k|^u Q(k) = p(k)^u Q(k) \quad (Q \text{ out})$$

$$\Rightarrow u \ln |k| = \ln p + u \ln |k| + \ln q$$

$$u = - \frac{\ln p}{\ln q}$$

$$u = \frac{\ln p}{\ln q}$$

$$(\ln p < 0)$$

$\hookrightarrow$ Prob

Connects u to $\frac{|\ln p|}{\ln q}$

$\downarrow$
Heavy Index

③ Demonstrates importance of self-similarity in scaling.

→ Large x expansion: $0 < \alpha < 2$

$$L_\alpha(a, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-a|k|^\alpha} dk$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos(kx) e^{-a|k|^\alpha} dk$$

imp

$$= \frac{ax}{\pi} \int_0^{\infty} dk \frac{\sin k|x|}{|k|} \exp[-a|k|^\alpha] k^{\alpha-1} dk$$

$$= \frac{ax}{\pi |x|^{1+\alpha}} \int_0^{\infty} \Sigma^{\alpha-1} \sin \Sigma \exp\left[-\frac{a \Sigma^\alpha}{|x|^\alpha}\right] d\Sigma$$

Now, $\Sigma = k|x|$

$$d\Sigma = dk|x|$$

$$L_\alpha(a, x) = \frac{ax}{\pi |x|^{1+\alpha}} \int_0^{\infty} \Sigma^{\alpha-1} \sin \Sigma d\Sigma$$

so

$$L_\alpha(a, x) \sim \frac{1}{|x|^{1+\alpha}}$$

$$C \sim \frac{a \alpha \Gamma(\alpha) \sin \pi \alpha}{\pi}$$

Point: Levy Distributions $L(a, \alpha)$
 have power law α
 tails (Fixed pt. fn $\leftrightarrow$
 self-similarity)

$$L_\alpha(a, x) \sim \frac{1}{|x|^{1+\alpha}}$$

$$0 < \alpha \leq 2$$

obviously need $\alpha = 2$ for
 convergent 2nd moment.

- width, for N steps, $\sim N^{1/\alpha}$
- $\alpha > 2 \rightarrow$ super-diffusive.

*

$\rightarrow$ Now, obvious analogy question
 arises

~~Diffusion~~ : Diffusion :: Levy Distribution : $L_\alpha(a, x)$

Gaussian

→ Levy Process / Flight.

see Zaslowsky, Chapt. 15

- analogue / generalization of diffn.
→ Levy Flight

- time dependent

- Levy dist. at infinitesimal time

Now, C-K Egn:

$$P(x_0, t_0 | x_N, t_N) = \int dx_1 \int dx_2 \dots \int dx_{N-1} \quad (+ \text{ understood})$$

$$\times [P(x_0, t_0; x_1, t_1) P(x_1, t_1; x_2, t_2) \dots P(x_{N-1}, t_{N-1}; x_N, t_N)]$$

now $t_{j+1} - t_j = \Delta t$

$$t_N - t_0 = N \Delta t$$

$$N \gg 1$$

and assume process uniform in space and stationary in time, so:

so

$$P(x_j, t_j; x_{j+1}, t_{j+1}) = P(x_{j+1} - x_j, \Delta t)$$

 $\Rightarrow$

$$P(x_N - x_0; N\Delta t) = \int dy_1 \dots \int dy_N P(y_1, \Delta t) \dots P(y_N, \Delta t)$$

Now,

$$P(z) = \int dy_j e^{izy_j} P(y_j, \Delta t)$$

$$\left\{ \begin{aligned} P_N(z) &= \int dy^N \dots e^{izy^N} P(y^N, N\Delta t) \\ y^N &= \sum_{i=1}^N y_i = y_N - y_0 \end{aligned} \right.$$

so

$$P_N(z) = [P(z)]^N$$

(identical steps)

, as before.

but

Now, take generating fn. to

be

Levy Distribution.

Now,

$$P(z) = P(q|x, c)$$

here:

$$P(z) = P_x(z, \Delta C)$$

$$= L_x(z, \Delta C)$$

and need

$$P_N(z) = P_x(z, C_N)$$

above consistent of:

$$\begin{aligned} C_N &= N \Delta C = \underbrace{N \Delta t}_{\text{const}} \frac{\Delta C}{\Delta t} \\ &= \cancel{N \Delta t} (N \Delta t) C \\ &= tC = ct. \end{aligned}$$

$$\begin{aligned} P_N &\Rightarrow P_x(z, ct) = \exp[-c \underbrace{N \Delta t}_{\text{const}} |z|^x] \\ &= \exp[-ct |z|^x] \end{aligned}$$

characteristic fctn of Levy Process

Characteristic fctn:

$$P_\alpha(\xi, t) = \exp(-ct|\xi|^\alpha)$$

$$P_\alpha(x, t) = \int \frac{d\xi}{2\pi} e^{i\xi x} e^{-ct|\xi|^\alpha}$$

$\alpha = 2 \rightarrow$ diffusion ✓

$\langle x^2 \rangle \rightarrow \infty$, for $\alpha < 2$, at any t .

for $x \rightarrow \infty$

$$P_\alpha(x, t) \sim t / |x|^{\alpha+1}$$

— 'accelerating tail' distribution
'expanding tail' *

— i.e. P equal at $t > t'$
 $\Rightarrow (\Delta x) > (\Delta x)'$

Next:

- Physics: Weeks, Swinney experiment
- how extend Fokker-Planck Theory?
 - $\sim$ CTRW
 - $\sim$ Fractional Kinetics.