Notes 7 — Percolation Basics II

→ Recall:

Scale Invariant struct. Fcn
→ $\xi_{cor} \to \infty$
→ $n_s(p)$

⇒ 2 questions of interest:

① "why" — "Any relevance to fluid flow"
(cf J-T) ($\omega \to k\omega \to \infty$)

Origins → Broadbent and Hammersley 50's (posted)

— hydrology (like H. E. Hurst)

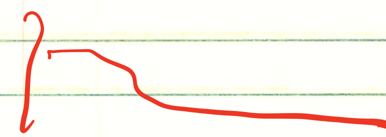
— interested in transport/flow thru porus media — water seepage in rocks.

— microscopic underpinnings of Darcy's Law and Kozeny equation:

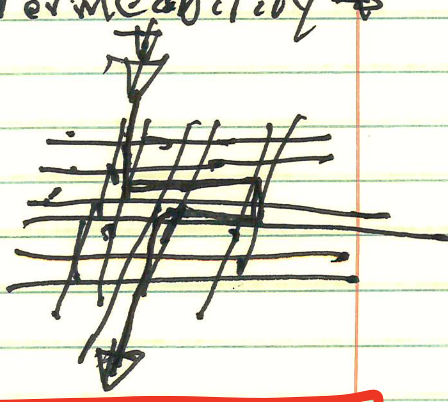
→ permeability

$$\underline{I} = -\frac{K}{\mu} \underline{\nabla P}$$

↑ flux



Permeability → net flow thru random network



Percolation as connection thru random maze

(cf pinball)

$\boxed{\text{also — avalanches}}$

P

– Also, percolation cluster distribution $\boxed{n_s(p)}$, $s n_s(p)$ etc. is measure of $\boxed{\text{emergent order}}$, and its statistical characterization

i.e. distribution of cells.

⇒ simpler problem than avalanche distribution (time)

⇒ SOC originally 'defined' in terms of 'percolation cluster' of single toppling (BTW '87)

⇒ prototype of many body, short range interaction system with universality, scaling etc.

② "How do I know it when I see it? Specifically, how identify it in a simulation?" (c.f. H. C.)

→ Percolation is intrinsically 'static' concept (i.e. snap shot)

→ suggest analyzing clustering distribution in an image

see Boffetta et. al, posted → Fig 1.

[Beautifully shows vorticity clustering in 2D turbulence. Appeals to intuition from percolation.

2D – inverse cascade

→ what of time?

- sequence of cluster images?
  With transport, should manifest avalanches i.e. large clusters discharge across the sytem
- to be continued.

Now, describe percolation by:

$\boxed{n_s(p)}$ ≡ avg #/site of s-clusters

population density

and moments:

$\sum_s n_s(p)$ → population

$\sum_s s\, n_s(p)$ → ~~[illegible]~~

probability of a cluster size s

i.e. $\boxed{w_s = \frac{n_s s}{\sum_s n_s s}}$ → probability that cluster to which an arbitrary site belongs, contains s sites

$w_s < 1$

see Stauffer

and $\boxed{\bar{s}} = \sum_s s w_s$ — $L \to$ outer, $\Delta \to$ inner,
avg size

etc.

and $\boxed{\Delta/L \to 0}$ — $N \to \infty$

Universality $\to$ scaling $\to$ power laws
"linear criticality"

Special focus on $\boxed{p \sim p_c}$

i.e. near percolation threshold, anticipate scalings $\sim |p-p_c|^\alpha$, etc.

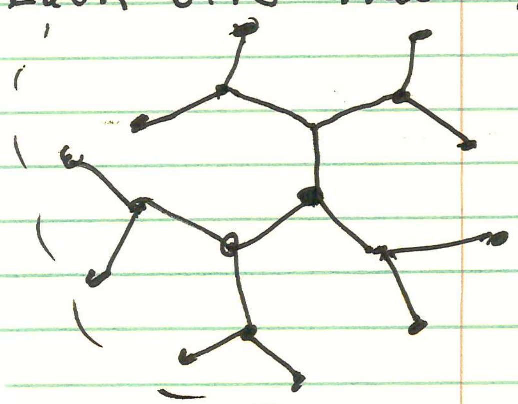
$\to$ structure of scalings $\to$ exponents.

$\to$ relation between critical exponents.

Also – exactly solvable (albeit trivial) 1D model —

– Bethe lattice / Cayley tree has z bonds, d dimension

e.g. Each site has $z = 3$ bonds,



surface dense!

(circle ∞ iteration)

Bethe lattice

also solvable, d dimensions.

i.e. where do scalings come from?

c.f. → Stauffer, Aharony 2.4 {not disc. here}

⇒ $\boxed{\text{extract general trends of } n_s(p) \text{ scalings exploiting exact solutions.}}$

⇒ Toward a Scaling Solution for Cluster Numbers ($n_s(p)$)

– critical exponents
– Σ sum.
Relation between.

Recall: $n_s(p) = (1-p)^2 p^s$

⇒ $\boxed{s \to \infty}$ $n_s(p) \sim e^{-cs}$

For $\boxed{\text{Bethe lattice}}$ need generalize:
→ power law. – could guess from self-similarity at p_c.

$$\boxed{n_s(p) \sim s^{-\tau} e^{-cs}}$$
$s \to \infty$.

$\tau = 5/2$ Bethe lattice

Of course: $c = c(p)$ (not a strict constant)

$s \to \infty$.

For Bethe lattice: $c(p) = (p-p_c)^2$,

$C(p) = (p-p_c)^{1/\sigma}$

More generally

$$\boxed{c \sim |p - p_c|^{1/\sigma}}$$

($\sigma = 1/2$, Bethe)

assumes scale invariance

at criticality.

Point: Relation Between Critical Exponents.

Ultimately need simul. RG to get $\underline{1}$.

Note now have two exponents .....

Observe then:

$$\boxed{n_s(p) \sim \frac{1}{s^{\tau}} \exp\left[-|p-p_c|^{1/\sigma} s\right]}$$

→ defines effective cut-off on range of cluster sizes

i.e. $s' < s$[redacted] $\sim [(p-p_c)]^{-1/\sigma}$
cut off

only contribute to cluster averages,

there

$n_s(p) \sim s^{-\tau}$ — scaling | at critical pt.

$s > s_{c.o.}$ → exponentially rare.

→ s_{co} defines cross-over from

from critical clusters → contribute

to

non-critical → don't contribute.

Useful general form.

so: $$\boxed{n_s(p) \sim s^{-\tau} \exp\left[-(p-p_c)^{1/\sigma} s\right]}$$

→ working model.

Now ⇒ a limitation is validity for large clusters, only

→ improve by examining ratio

$$v_s = n_s(p) / n_s(p_c)$$

so
$$\boxed{v_s(p) \sim \exp[-cs]}$$
→ exceedingly simple.

S & A: $\boxed{n_s(p_c) \sim s^{-\tau}}$ ⇐ ! scaling at crit. point

"You might violate the Official Secrets Act if you now conclude and say loudly this is what theoretical physicists do; make calculations if they are easy irrespective of whether the assumptions are correct or wrong."

Some calculations:

①

⇒ P_∞
⇓
P

→ fraction of sites belonging to infinite network. relative.

For P

- site — empty
  — occupied — ∞ cluster
  — finite cluster

- $\sigma = \infty$, $n_s = 0$ ✓ $\qquad \frac{1}{N} \to 0$

i.e. in ∞ lattice, $\underline{\underline{1}}$ ∞ network,
$\#$ ∞ networks / lattice site $= 0$.

- Recall fraction of lattice sites in ∞ network obtained from subtracting from occupied sites those belonging to finite clusters

~ i.e. using $P + \sum_{s\ \text{finite}} n_s s = p$

$P \neq p$

$\therefore$ at $p = p_c$, $P = 0$ above $(p \to P_-)$

$$\therefore \boxed{\sum_{s\ \text{finite}} n_s s = p_c}$$

Perit ↔ moment n_s distrib

Now, $n_s \sim s^{-\tau}$ so need

$$\boxed{\tau > 2 \text{ for convergence}}$$

(Large powers for convergence.)

$n_s \sim s^{-\tau}$

conv. constraint on decay (tail)

So, re-writing: $P = P_{sct} - (p - p_{crit})$

$$f = p - \sum_{s}^{fin} n_s s$$

n_s

$$= \sum_s \left(n_s(p_c) - n_s(p)\right) s + \cancel{O(p - p_c)}$$

above

$$f \simeq \sum_s s^{1-\tau}\left[1 - \exp(-cs)\right]$$

(large s contrib dominated)

so

$$f = \int ds\, s^{1-\tau}\left[1 - \exp(-cs)\right]$$

$z = cs$ and integration by parts:

$$f \approx c \int s^{2-\tau} \exp(-cs)\, ds$$

$$= c^{\tau-2} \underbrace{\int z^{2-\tau} \exp(-z)}_{\Gamma(3-\tau)}$$

Gamma Fctn

and can integrate:

$$f \sim c^{(\tau-2)} \rightarrow (p - p_{crit})^{1/\sigma}$$

but $c \sim (p - p_c)^{1/\sigma}$ so

$$P \simeq (p-p_c)^{(\tau-2)/\sigma}$$
$$\simeq (p-p_c)^{\beta}$$

$$\boxed{\beta = \frac{\tau-2}{\sigma}} \rightarrow$$

i) first relation between scaling exponents

ii) what we seek:

β ↓ exponent for P

$\sigma \sim$ exponent for $(p-p_c)$

$\tau \sim$ exponent $n_s(s)$.

Landau + Lifshitz

How scale near criticality

② How does mean cluster size diverge?

Recall: $\bar{S} \sim \sum_s s^2 n_s / \sum_s s n_s$ moments

but $p \to p_c$, $\boxed{\sum_s s n_s = p_c}$

so, $\bar{S} \simeq \sum_s s^2 n_s / p_c$

$$\sim \int ds \; s^2 n_s$$
$$\sim \int ds \; s^{2-\tau} e^{-cs} \, ds$$

$p-p_c$

$n_s(p) \sim e^{q_s \tau}$ 11

$$\bar{S} \sim \boxed{c^{3-\tau}} \int z^{2-\tau} e^{-z} dz$$

finite

[(∞ single cluster neglected)]

$$\bar{S} \sim \boxed{c^{\tau-3}}$$

$$\sim |p-p_c|^{(\tau-3)/\sigma}$$

$$\sim |p-p_c|^{-\gamma}$$

$$|p-p_c|^{\frac{\tau-3}{\sigma}}$$

$$\boxed{\gamma = \frac{3-\tau}{\sigma}}$$

γ scale (av. of critical γ)

$$\boxed{\gamma = \frac{3-\tau}{\sigma} \qquad \beta = \frac{\tau-2}{\sigma}}$$

For $\beta > 0$, $\sigma > 0$

$$\boxed{2 < \tau < 3}$$

The cynic: [Do we have a new exponent for every quantity?]

→ Not Have $\boxed{2}$ "free", undetermined exponents σ, τ!

Really

recall: τ → lead power law of n_s, $n_s \sim s^{-\tau}(\ )$

$p \to p_c$

$p \to p_c$

$n_s(p) \sim \underbrace{1/s^{\tau}}$ $\qquad$ $n_s(p) \sim \underbrace{p^s}(1-p)^2$

$p \to p_{crit}$ $\qquad$ $n_s(p) \sim s^{-\tau} e^{-cs}$ $\qquad$ $s \to \infty$ $\sim e^{-cs}$

$c = c(p-p_c) \sim (p-p_c)^{-1/\sigma}$

$\left.\begin{array}{l}\tau \\ \sigma\end{array}\right\}$ num. of RG

$\sigma \Rightarrow$ scaling of c

$$c \sim (p-p_c)^{1/\sigma}$$

determines $s_{c.o.}$ cut-off.

$\boxed{\text{Can relate all else to these!}}$

✱ [of course, need σ, τ from simulations, etc.] ✱ scaling

i.e. consider general case:

$$M_k = \sum_s s^k n_s$$

$$\sim \sum_s s^{k-\tau} e^{-cs}$$

$$\sim \int ds\; s^{k-\tau} \exp(-cs)\, ds$$

$$\sim c^{\tau-1-k} \int dz\; z^{k-\tau} e^{-z}$$

so

$$M \sim c^{\tau-1-k}$$

$$\sim |p-p_0|^{(\tau-1-k)/\sigma}$$

$M_k \sim$

exponent $\sim \boxed{(\tau-1-k)/\sigma}$

internally self-contained

Caveat: – not ~~rigorous~~, find the glitches!

– yet, captures essence of scaling game

→ A Quick Look at More General Derivation

"If you have read this far thru the book, it is presumably too late for you to return it for a re-fund".

More general formulation:

→ stretched exponential version of previous

$$\boxed{n_s(p) = f(z)}$$

$$z = (p-p_c)\, s^{\sigma}$$

$$\boxed{n_s(p) \sim s^{-\tau} f\left[(p-p_c)\, s^{\sigma}\right]} \quad \begin{cases} p \sim p_c \\ s \gg 1 \end{cases}$$

$\boxed{f(z)}$ is TBD from computation

→ $n_s \sim s^{-\tau} f[(p-p_c)s^{\sigma}]$

now behaves well, all cases

check:

$$\bar{S} \sim \sum s^2 n_s$$

$$\sim |p-p_c|^{(\tau-3)/\sigma}$$

$$\sim |p-p_c|^{-(3-\tau)/\sigma} = |p-p_c|^{\gamma}$$

$\therefore$

$$\boxed{\gamma = \frac{3-\tau}{\sigma}}$$

Exponents: — β, γ universal

— independent lattice structure

$\beta \leftrightarrow P$

$\gamma \leftrightarrow \bar{S}$

Some RG evidence for γ

$\bar{S}$

→ Can extend the fn to correlation lengths, perimeters (in 2D) etc.

→ Cluster perimeter can be fractal. (usually is)

$D = \ldots$

$\xi \rightarrow \infty$

General Message:

- $\boxed{\text{Large scale } \boxed{\text{emergent}} \text{ behavior occurs in systems with local interaction}}$
- system can self-organize hierarchy of clusters, divergent at criticality.
- universality → scaling → power laws
- scaling theory is really useful $\boxed{\text{phenomenology}}$ which links $\boxed{\text{scaling (critical) exponents,}}$
- $\boxed{\text{properties described by scaling exponents i.e. the "answer"}}$ *

⇒ Emergent critical behavior,

$\ell_c \rightarrow \infty \Leftrightarrow$ bass?

[redacted] Transport Phenomena exist which are not captured by random walk models.

$\boxed{\text{Intermittency}}$ → beyond random walks

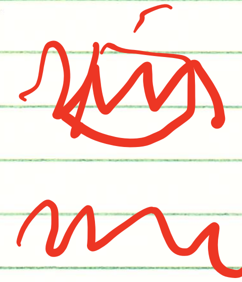
(percolation phenomena are good example)

→ Hurst exponent

→ Fractional, CTRW

$Ku \gg 1$ is good example.
relevant limit

$\boxed{\zeta(p)}$



→ Now, return to $Ku \to \infty$ magnetic problem.