Notes 6 → Percolation Basics, I

→ Why Percolation?

- The problem of large Ku ...

recall: $Ku \sim \frac{l_{ac}}{\Delta_\perp} \frac{\delta B}{B}$

$\sim \nabla \tau_{ac} / \Delta_\perp$ etc.

⇒

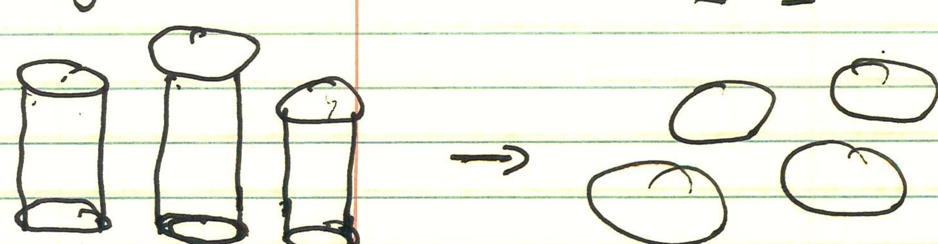
Limiting case, well defined paradigms:
(Toy models!?)

$Ku \ll 1$ → diffusion, extensively studied
$l_{ac} \to 0$

$Ku \gg 1 \to Ku \to \infty$. ⇒ $l_{ac} \to \infty$

i.e. no variation of $\delta B_\perp$ in z (or time).

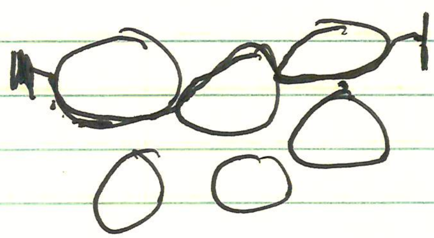
→ random (by assumption) array of magnetic cells ($\underline{\nabla} \cdot \underline{B} = 0$!)



static, disordered 2D medium.
random

→ Akin array of randomly placed static vortices, as in Taylor–McNamara problem (see notes.)

→ ignoring D_0, Θ_0 etc. need macroscopic connection — i.e. a percolation — to transport (hills and lakes, as before)



n.b. need not have connection if $D_0 \neq 0$, though expect

$$D_{eff} \sim D_0^{\alpha} D_{cell}^{\beta} \qquad (\alpha+\beta=1)$$

↑ explicit dependence on collisional diffusivity

→ Problem is percolation of static random function of <u>2 variables</u>

akin to problem of effective conductivity of medium with random mixture of conducting and insulating elements, ~ identical.

Percolation — Magnetic Example — Intro. to Theory

⇒ Lots online

✱

Some properties of system/problem, sticking to magnetic example:

→ spectrum of A, $B = \nabla A \times \hat{z}$

- random phases
- $\langle A \rangle = 0$
- $\langle A^2 \rangle = 1$

and small auto-correlation at large distances,

$$\ell_{\perp ac} \ll \underset{\text{system size}}{L}$$

characteristic scale k_0^{-1}

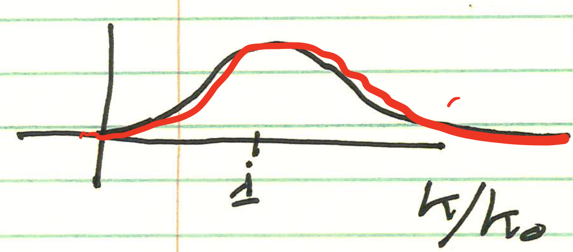
$$\langle A^2 \rangle_k \cong e^{-k^2/k_0^2}, \quad k > k_0$$

$k_0 L \gg 1$

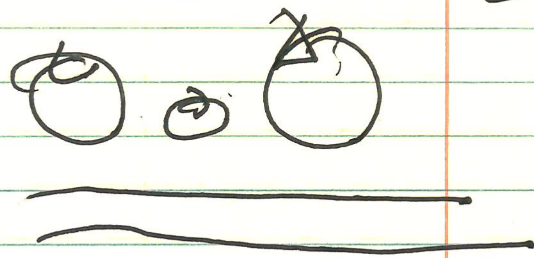
a

$$\langle A^2 \rangle_k \cong \left(\frac{k}{k_0}\right)^{2n} \qquad n > 0, \quad k < k_0$$

so $\langle A^2 \rangle_k$



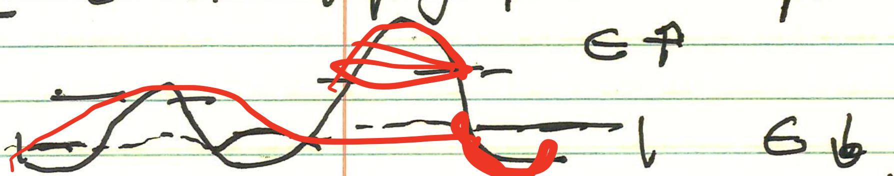
$A > \epsilon \rightarrow \begin{cases} \text{island} \\ \text{Magnetic vortex} \end{cases}$



$\rightarrow$ $\underset{\text{reg conf.}}{A < \epsilon}$, $\epsilon > 0 \Rightarrow$ percolating

$\epsilon = 0$, critical phase

i.e. recall topographic map



percolation/connection for $A < \epsilon$.

$\ell(\epsilon) \equiv$ length island or isoline surrounding island.

$\ell \uparrow$ as $\epsilon \downarrow$; $\ell \rightarrow L \rightarrow \infty$
percolation

→ in such 2D topography, with $D_0 \to 0$ (i.e. nothing kicking particle off field line)

$D_{eff} = 0$

i.e. turbulent diffusion approximation not applicable (i.e. all particles trapped in/near closed cells)

→ either no transport or 'burst' along macroscopic connection.

*

→ Interesting to consider: Rec $Ku \gg 1$

2D system $\longleftrightarrow$ $Ku \gg 1$.

$\langle B \rangle \neq 0$ ~ low k spectrum $\frac{\delta B}{B_0} > 1$

$\tilde{B} \gg \langle B \rangle$ $\boxed{B_{rms} \gg \langle B \rangle}$

akin: solar tachocline

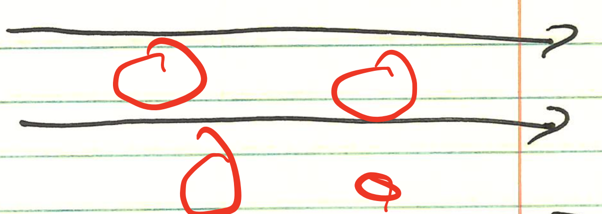
Aside: References

- "The Almighty Chance"
  - Zeldovich, Ruzmaikin, Sokolov
  - serious OV of statistical dynamics.
  - dated, highly recommended
  - percolation chapter posted.

see also:

- Ya. B. Zeldovich, 1983 (posted)
  (magnetic problem)
- Stauffer 1973 (review) (posted)
  (review of percolation theory).
- Stauffer and Aharony, monograph.

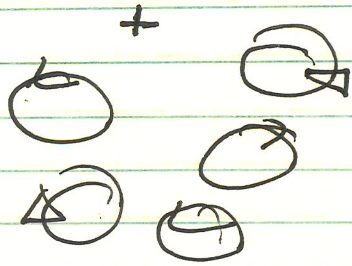
Now, view as

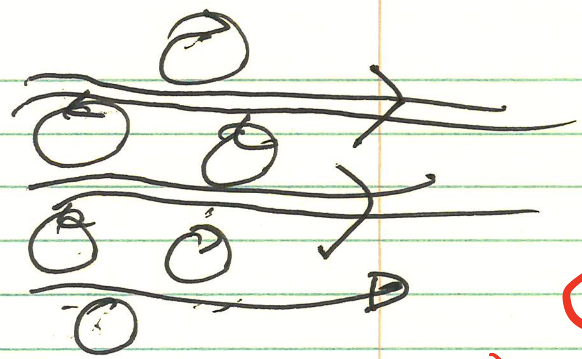


$\langle B \rangle \to B_0$

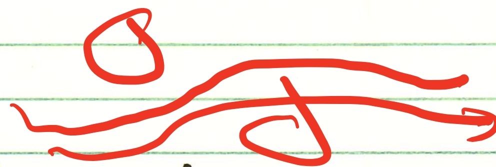
unperturbed large scale field

+

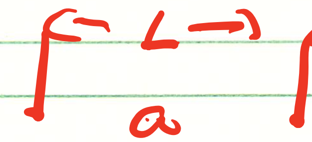


= 

B_L confined to narrow channels.



i.e. real field pattern is ensemble of cells + sinews of deformed B_L



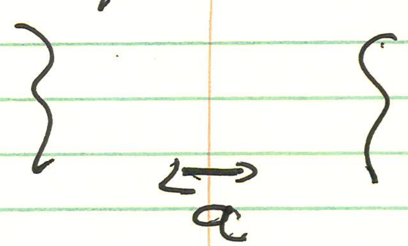
Question, does B_L percolate →
i.e. extend to a, system size
as $a \to \infty$.

Physics :- Will Alfvénic excitation propagate? $\boxed{}$ *

- More generally, what is response of such a system to external excitation?

Now, seek $\langle B \rangle$ → mean field. what is it?

- define system as strip of width a



$k_{min} \simeq 2\pi/a$

$\to 1/a$.

- take $A_k \sim k^{m}$ (see before)

so $B_k \sim k^{m+1}$

- $$\langle B \rangle = \left(\langle b^2 \rangle_{k < 1/a} \right)^{1/2}$$

$\sim b_{rms}$

$d\underline{k} = k\,dk$

long wavelength components define mean field

so

$$\langle B \rangle = \left(\int_0^{1/a} dk\, k \; k^{2m} \, k^{2} \right)^{1/2}$$

2π

$$\cong (1/a)^{-m-2}$$

then, for $\langle B \rangle \neq 0$ as $a \to \infty$

(i.e. not decrease with increasing distance, a)

for $a \to \infty$,

$\langle B \rangle \neq 0$

$\langle B \rangle \to$ const.

$$\boxed{m = -2}$$

well behaved $a \to \infty$

$m \sim -2$

$\boxed{\text{currents white noise!}}$

$j_{z k} \approx k^2 A_k \sim k^0 \sim$ const.

- random currents

i.e. Random currents will result in percolating mean field

$\leftrightarrow$ large field structure set by currents.

N.B. More generally:

- need $\langle j(\underline{x}) j(\underline{x}+\underline{r}) \rangle \geq 0$ (correlated currents) for percolating $\langle B \rangle$.

- no percolation for anti-correlated currents.

→ Some General Features of Percolation Problems

- explore formation of $\ell \sim$ L linked paths.

$L \sim \cancel{(N)^{1/D}} \Delta x \; N.$ $\quad N \to \infty$ thermodynamic limit

$\# \sim N^D$

— Basic game is some D-lattice:

x x x x

x x x x

x x x x etc.

Dimension-ality

Geometry

With probability p of occupation of a site,

— connections ↑ with p.

— bond x—x percolation

site x x

— Universality is key concept

SOC

for $\Delta \ll 1$, $L \to \infty$,

L

details of specific site element irrelevant,

i.e. classic is details of, say, conductor or insulator shape element.

- Caveat: → Fractal structure elements can complicate universality

$\langle B\rangle \Big/ \left[\langle \tilde{B}^2\rangle_{k\to 0}\right]^{1/2}$ → $\boxed{\text{example: pancakes in 3D}}$ →

$\langle \tilde{B}^2\rangle \gg \langle B\rangle^2$

Universality →

- define as simple problem
- concept as for $\boxed{\text{phase transition}}$

$p \to$

Ginzburg → $\ell_c \to \infty$ as $T \to T_c$. $p \to p_{crit}$

$\langle \tilde{n}^2\rangle \gg \langle n\rangle^2$

$\dfrac{\langle \tilde{n}^2\rangle}{\langle n\rangle^2} \to O(1)$

Criterion?

→ $\langle \tilde{T}^2\rangle > (T-T_c)^2$ in regime interest

→ $\boxed{\text{scaling!}}$ → power laws   Mean Field Theory fails

c.f. N. Goldenfeld "Intro to Phase Transitions"

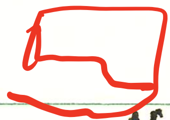
→ Landau & Lifshitz, "Stat. Mech". Vol. 5

- contrast with diffusion:
  (i.e. on lattice)

→ diffusion: randomness arises from particle / trajectory dynamics

i.e. → island overlap

→ percolation: randomness encoded in site probabilities.

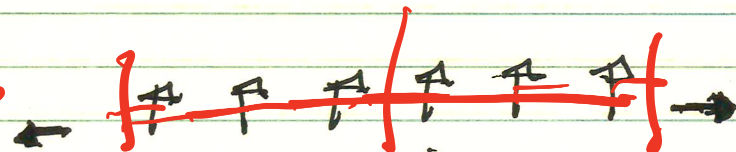


→ D matters

P → control. —

if $p = p_c$ → critical occupation dens.
threshold for percolation

1D; $p_c = 1$



all sites occupied.

2D; $p_c = .59$ triangular lattice

$p_c = .50$ square lattice

etc.

How describe percolation?

→ *Scaling Theory of Percolation Clusters* — c.f. Stauffer
(also relevent to Avalanches)

percolation — lattice

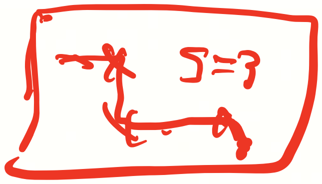
random occupation p yes

$1-p$ no.

$p > p_c$ → 1 cluster spanning lattice. | p.c.

$p < p_c$, no spanners

$p_c \to L_{syst}$



→ **s-cluster**

a cluster of s-sites, nearest neighbor coupled, ends empty.

→ seek describe scaling of cluster properties as $p \to p_c$

i.e. → characterize distribution of cluster sizes

→ how behave as $p \to p_c$

n_s = # s-clusters "$f(s)$"

n_s = avg # / site of s clusters $n_s(p)$ $n_s(p)$

$n_s = n_s(p)$ → population density for p of s

→ of interest is:

n_s, $\sum_s n_s(p)$ → cluster population

$\sum_s s\, n_s(p)$ → cluster size probability

$\boxed{\xi(p) \equiv \text{correlation length}}$

etc.

- Aim of scaling theory is $\boxed{n_s}$.

This yields more info.

i.e. P_∞ ≡ percolation probability

P_{crit}

= fraction of sites belonging a percolation network

~ "strength" of infinite network.

Now, can relate different quantities;

i.e. any lattice site can be:

→ empty → Prob = $1-p$

→ part of a percolation cluster

Prob = P_∞

→ part of a finite cluster

Prob = $p(1-P_\infty)$

$$1-p \quad +P_\infty + \sum_s s n_s = 1$$

of course, being part of a finite cluster

$$\text{Prob} = p(1-P_\infty) = \sum_s s n_s \qquad \boxed{\sum_s s n_s}$$

$$p(1-P_\infty) = \sum_s s n_s$$

$$\boxed{1-p + P_\infty + \sum_s s n_s = 1} \qquad p(1-P_\infty) = \sum_s s n_s$$

$$1-p+P_\infty+\sum_s s n_s = 1$$

so computing n_s gives P_∞!

$$p - P_\infty = \sum_s s n_s \qquad \sum_s s n_s - p + P_\infty = 0 \; +1$$

→ what we are interested in:

→ scaling exponents, near p_c.

As for phase transitions, aim will be to (understand / predict) the relationship between scaling exponents. So, determining 1 from simulation / experiment / RG calculation, (!, ?) etc. gives all.

→ Scaling Exponents: 1

$$\sum_s n_s(p) \sim |p - p_0|^{2-\alpha}$$

$$\sum_s s\, n_s(p) \sim |p - p_0|^{\beta}$$

$$\sum_s s^2 n_s(p) \sim |p - p_0|^{-\gamma}$$

$$\xi(p) \sim (|p - p_0|)^{-\nu}$$

$\nu > 0$

etc.

For 2D, $\alpha \sim -0.7$

$\beta \sim .14$

$\gamma \sim 2.4$

~~$\delta \sim 18$~~

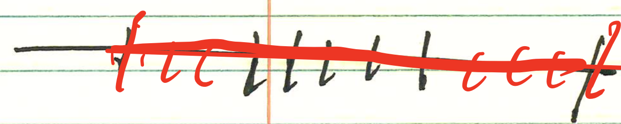
$\nu \sim 1.35$

Aim is to get relation.

→ 1D Percolation → $\boxed{\text{Trivial but Illustrative}}$

of course $p_c = 1$.

L



- prob occ ~ p
- cluster → s sites + 2 ends (empty !)

i.e. $P(s \text{ cluster}) \sim p^s (1-p)^2$ → s sites, 2 ends

$\boxed{L \gg s}$ ⇒

$$\boxed{n_s = p^s (1-p)^2}$$

n_s → # s clusters/site

{ can generate distribution

~ thermodynamic limit ($N \to \infty$)

→ no worry re: over-hang

$n_s \to 0$ for $s \to \infty$ $\boxed{\text{(exponentially)}}$

$p < 1$

$$n_s \sim \boxed{p^s} (1-p)^2$$

$s \to \infty$

Prob. of site in cluster of size s = $s n_s$

Prob any cluster = $\sum_s n_s s = p$

$\sum_s n_s s^{I}$ $\qquad = \left(\frac{d}{dp}\right)^{I}$

how?

$$\sum_s \underbrace{p^s (1-p)^2}_{n_s} s = (1-p)^2 \sum_s p \frac{d}{dp} p^s$$

$$= (1-p)^2 p \frac{d}{dp} \sum_s p^s$$

$$= (1-p)^2 p \frac{d}{dp} (p/1-p)$$

$$= p \; !$$

Probability occupied.

N.B. Standard trick in Statistical Mechanics → relate index to derivative.

Now: define

w_s = Probability of cluster to which arbitrary site belongs, contains s sites

$$w_s = \frac{n_s s}{\sum_s n_s s} \rightarrow \text{fraction occ.} \; f$$

so average cluster size:

$s^p n_s$

$$\bar{s} = \sum_s s\, w_s = \frac{\sum_s n_s s^2}{\sum_s n_s s}$$

(why moments scaling of some concern).

so

$\frac{d^2}{dp^2}$

$$\bar{s} = \frac{\sum_s (1-p)^2 p^s s^2}{\sum_s (1-p)^2 p^s s} = \boxed{\frac{1+p}{1-p}} = \bar{s}$$

after some trickery

$$\bar{s} = \frac{p+1}{1-p}$$

obviously, $\bar{S} \to \infty$ $\quad p + p_0 = 1.$

if $p = 1 - \delta$

$\bar{S} \sim \frac{2}{\delta}$!

Coming → further consideration of scaling theory.

1D Percolation