→ Collisional Regime

Here: $\boxed{l_{ac} < l_{mfp} < l_c}$

(short mean free path)

Point: → $l_{mfp} < l_c$ ⇒ particle random walks parallel and undergoes many kicks in l_c. So parallel motion is diffusive.

→ perpendicular motion is continuous coarse graining/spreading, at

$D_\perp \sim \rho_e^2 \nu_{ee} \sim \rho_e^2 \frac{V_{the}}{l_{mfp}}$

dnev.

So, can write:

$\langle \delta r^2 \rangle \sim D_M \, l_{c,d}$

parallel correlation length (d signifies diffusive regime)

but also note that parallel motion is diffusive, so:

but time set by

$\chi_\parallel / l_{c,\delta}^2 \sim 1/t$

$\Rightarrow \dfrac{\langle \delta r^2 \rangle}{t} \sim \dfrac{\chi_\parallel}{l_{c,\delta}^2} D_M l_{c,\delta}$

collisional CT

$l_{c,\delta} > l_{mfp}$

$\sim D_M \dfrac{\chi_\parallel}{l_{c,\delta}} \sim \boxed{D_M \chi_\parallel / l_{c,\delta}}$

$\sim D_M v_{the} \underline{l_{mfp}/l_{c,\delta}} < \chi_{\text{collisionless}}$

$\boxed{\chi_\perp = D_M \dfrac{\chi_\parallel}{l_{c,\delta}}}$ $\rightarrow$ [perpendicular heat conductivity in collisional regime.

$k_\parallel^2 \chi_\parallel \sim D \dfrac{\partial^2}{\partial x^2}$

$\dfrac{\chi_\parallel}{l_c^2} \sim \dfrac{D}{\delta^2}$

$l_{c,\delta} \sim \delta \left(\dfrac{\chi_\parallel}{D_\perp}\right)^{1/2}$

$\boxed{\text{Now what is } l_{c,\delta}\text{?}}$

Notice $l_{c,\delta}$ is set by competition between 2 processes:

① $\bigcirc \rightarrow \square$ width δ increases due diffusion (coarse graining)

$l_{c,\delta} \sim \delta (\chi_\parallel / D_\perp)^{1/2}$

$\delta \sim l_{c,\delta} (D_\perp / \chi_\parallel)^{1/2}$

so $$(d\delta)^2 \simeq (D_\perp dt)$$

$$d\delta \sim (D_\perp dt)^{1/2}$$

but

$$\chi_\parallel/(dL)^2 \simeq 1/dt$$

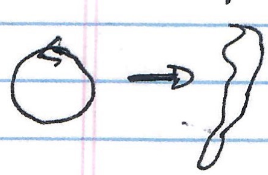
$\Rightarrow$

$$d\delta \sim \left(\frac{D_\perp (dL)^2}{\chi_\parallel}\right)^{1/2}$$

$$\boxed{d\delta \sim \left(D_\perp/\chi_\parallel\right)^{1/2} dL} \checkmark$$

as before.

② width shrinks, due stochastic instability and area conservation:



$$d\delta/dL = -\delta/\ell_c$$ (exponential decay)

then balance at:

$$d\delta \sim (D_\perp/\chi_\parallel)^{1/2}\, dL \sim \frac{\delta}{l_{c\delta}}\, dL$$

smearing ← vs → thinning

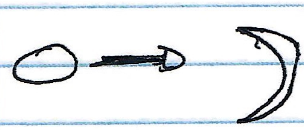
$$\boxed{\delta \sim l_c (D_\perp/\chi_\parallel)^{1/2}}$$ ✓

N.B.: Can select δ from:

$$\partial_t T - \chi_\parallel \nabla_\parallel^2 T - D_\perp \nabla_\perp^2 T = 0$$

$$\Rightarrow \frac{\chi_\parallel}{l_c^2} \sim \frac{D_\perp}{\delta^2}. \qquad \delta \sim l_c (D_\perp/\chi_\parallel)^{1/2}$$

Finally, need ~~[illegible]~~ correlation length $l_{c\delta}$ for chunk size δ. Assume set by Fig



$$k_\theta^{-1} \sim \delta e^{z/l_c}\Big|_{l_{c\delta}} \simeq \delta e^{l_{c\delta}/l_c}$$

$$\boxed{l_{c\delta} \sim l_c \ln\left(1/k_0\delta\right)}$$

$$l_{c\delta} \simeq l_c \ln\left(\chi_{\parallel}/D_{\perp}\right)^{1/2}$$

$$\boxed{l_{c\delta} \sim l_c \ln\left(\left(\chi_{\parallel}/D_{\perp}\right)^{1/2} / k_0 l_c\right)}$$

$$\Rightarrow \boxed{\chi_{\perp} \simeq D_M \, \chi_{\parallel} / l_{c\delta}}$$

Apart from log factor:

$$\chi_{\perp} \simeq v_{Th} D_M \underbrace{\left(l_{mfp}/l_c\right)}_{<1}$$

$\Rightarrow$ reduced relative to collisionless value.

Lesson: – collisions reduce ($l_{mfp} < l_c$) reduce χ_{eff} relative to "Collisionless case"

– interplay of perp and parallel diffusion

– again, critical to knock particle off field line.

Now, the above calculation requires thought. Its much more convenient to crank [redacted] mindlessly.

⇒ Hydro approach: Kadomtsev and Pogutse (not mindless, but systematic)

Consider heat flux along wiggling fields i.e.

$$\underline{q} = -\chi_{\parallel} \nabla_{\parallel} T \, \hat{b} \quad - \chi_{\perp} \nabla_{\perp} T$$

parallel conduction

perp. conduction

tricky in codes

$\chi_{\parallel} \gg \chi_{\perp}$

Here: $\underline{b} = \underline{b}_0 + \tilde{\underline{b}}$

$\underline{b}_0 \downarrow$ unperturbed; $\tilde{\underline{b}} \to$ fluctuating

$$\nabla_{\parallel} = \partial_z + \underbrace{\tilde{\underline{b}} \cdot \underline{\nabla}_{\perp}}_{\text{piece along wiggling line}}$$

So, seek mean radial heat flux

$$\langle q_r \rangle = -\chi_{\parallel} \overset{①}{\langle \tilde{b}_r^2 \rangle} \partial_r \langle T \rangle - \chi_{\parallel} \overset{②}{\langle \tilde{b}_r \partial_z \tilde{T} \rangle} - \chi_{\parallel} \overset{③}{\langle \tilde{b}_r \tilde{b}_r \partial_r \tilde{T} \rangle} - \chi_{\perp} \nabla_r \langle T \rangle$$

① ② usual quadratic

③ → cubic

Now $\dfrac{③}{②} \sim \dfrac{\cancel{\chi_{\parallel}} \tilde{b}_r \cancel{\tilde{b}_r \tilde{T}} / \Delta r}{\cancel{\chi_{\parallel}} \cancel{\tilde{b}_r \tilde{T}} / l_{ac}}$

$$\sim \frac{\tilde{b}_r l_{ac}}{\Delta r} \sim \underline{Ku}$$

so cubic nonlinearity dominates
for $Ku > 1$.

$Ku < 1 \Rightarrow$ drop cubic.

To compute $\langle Q_r \rangle$, need
- retain ① (usual), and ②
- iterate for $\tilde{T}$ using

$\nabla \cdot Q = 0$ i.e. std QLT.

Thinking (gasp!) first:

$$\langle Q_r \rangle \cong -\chi_{\parallel} \left[\langle \tilde{b}_r^2 \rangle \partial_r T + \langle \tilde{b}_r \partial_z \tilde{T} \rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

$$\cong -\chi_{\parallel} \left[\langle \tilde{b}_r \, \underline{b} \cdot \nabla T \rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

linearization:

$\underbrace{\tilde{b}_r \partial_r \langle T \rangle + \partial_z \tilde{T}}$

$\therefore$

Point: - need non-zero $b \cdot \nabla T$ fluctuation to drive heat flux

must have variation of T along line.

$\Rightarrow$ - i.e. temperature can't be constant along line, to drive parallel heat flux!

- $\underline{\nabla} \cdot \underline{q} = 0 \Rightarrow$ result must imply $\underline{\chi_\perp}$ dependence!

b)

$$\langle q_r \rangle = -\chi_{\parallel} \left[\langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle + \langle \tilde{b}_r \partial_z \tilde{T} \rangle \right] - \chi_\perp \nabla_\perp \langle T \rangle$$

$$\boxed{\nabla \cdot \underline{q} = 0}$$

$$\Rightarrow \quad \nabla_\parallel \tilde{q}_\parallel + \nabla_\perp \cdot \tilde{q}_\perp = -\chi_\parallel \, \partial_z \tilde{b} \, \partial \langle T \rangle / \partial r$$

i.e.

$$q = -\chi_{\parallel}\left[(\partial_z + \tilde{\underline{b}}\cdot\underline{\nabla})(T_0+\tilde{T})(\underline{b}_0+\tilde{\underline{b}}) \right] - \chi_{\perp}\nabla_{\perp} T$$

so

$$-\chi_{\parallel}\partial_z^2\tilde{T} - \chi_{\perp}\nabla_{\perp}^2\tilde{T} = -\chi_{\parallel}\partial_z\tilde{b}\frac{\partial\langle T\rangle}{\partial r}$$

$\Rightarrow$

$$\tilde{T}_k = \frac{-\chi_{\parallel} i k_z \tilde{b}_k \, \partial\langle T\rangle/\partial r}{\left(\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2\right)} \qquad \text{for } T$$

so

$$-\chi_{\parallel}\langle\tilde{b}^2\rangle\frac{\partial\langle T\rangle}{\partial r} - \chi_{\parallel}\langle\tilde{b}\,\partial_z\tilde{T}\rangle$$

$$= -\chi_{\parallel}\sum_{k}\left(-\frac{\chi_{\parallel}k_{\parallel}^2|\tilde{b}_k|^2}{\chi_{\parallel}k_z^2+\chi_{\perp}k_{\perp}^2} + |\tilde{b}_k|^2\right)\frac{\partial\langle T\rangle}{\partial r}$$

$$= -\chi_{\parallel}\frac{\partial\langle T\rangle}{\partial r}\sum_{k}\left(\frac{\cancel{-\chi_{\parallel}k_{\parallel}^2}}{\chi_{\parallel}k_z^2+\chi_{\perp}k_{\perp}^2} + \frac{\cancel{\chi_{\parallel}k_{\parallel}^2} + \chi_{\perp}k_{\perp}^2}{\chi_{\parallel}k_z^2+\chi_{\perp}k_{\perp}^2}\right)$$

so

$$\boxed{\langle q_r \rangle_{NL} \cong -\chi_{\parallel} \frac{\partial \langle T \rangle}{\partial r} \sum_{\underline{k}} \frac{\chi_{\perp} k_{\perp}^2 |b_{\underline{k}}|^2}{\chi_{\parallel} k_{\parallel}^2 + \chi_{\perp} k_{\perp}^2}}$$

$\boxed{\text{Note explicit dependence on } \chi_{\perp}\text{!}}$

$$q \sim (\chi_{\parallel} \chi_{\perp})^{1/2} \sim D_B$$

so

$$\langle q_r \rangle_{NL} \cong -\cancel{\chi_{\parallel}} \frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} \int dk_z \frac{\chi_{\perp} k_{\perp}^2 \langle \tilde{b}_{\underline{k}}^2 \rangle}{\cancel{\chi_{\parallel}} \left(k_z^2 + \frac{\chi_{\perp} k_{\perp}^2}{\chi_{\parallel}} \right)}$$

$$= -\frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} \int dk_z \frac{\chi_{\perp} k_{\perp}^2 \langle \tilde{b}^2 \rangle_{\underline{k}}}{\left(\frac{k_z^2}{(\chi_{\perp}/\chi_{\parallel}) k_{\perp}^2} + 1 \right) \left(\frac{\chi_{\perp}}{\chi_{\parallel}} k_{\perp}^2 \right)}$$

$$= -\frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} \frac{k_{\perp}^2 (\chi_{\parallel} \chi_{\perp})^{1/2}}{\sqrt{k_{\perp}^2}} \langle \tilde{b}^2 \rangle_{k} l_{ac}$$

auto correlation

autocorrelation l_{ac} enters via normalization:

$\Rightarrow$

$$\langle q_r \rangle_{NL} \cong -\sqrt{\chi_\parallel \chi_\perp}\, \langle \tilde{b}^2 \rangle\, l_{ac} \sqrt{\overline{k_\perp^2}}\, \frac{\partial \langle T \rangle}{\partial r}$$

Note: – need $\nabla_\parallel \tilde{T} \neq -\tilde{b}_r\, \partial \langle T \rangle / \partial r$

$(\underline{B} \cdot \underline{\nabla} T \neq 0)$ for $\perp$ heat flux

– $\langle \tilde{b}^2 \rangle\, l_{ac} \sim D_M$

$\sqrt{\overline{k_\perp^2}} \sim 1/\Delta_\perp$

so

$$\langle q_r \rangle \cong -\chi_{\perp\,\mathrm{eff}}\, \partial \langle T \rangle / \partial r \;-\; \chi_\perp\, \partial \langle T \rangle / \partial r$$

$$\boxed{\begin{aligned} \chi_{\perp\,\mathrm{eff}} &\cong \sqrt{\chi_\parallel \chi_\perp}\, \frac{D_M}{\Delta_\perp} \\ &\simeq \frac{D_B D_M}{\Delta} \end{aligned}}$$

$\left(\chi_\parallel \chi_\perp \sim \dfrac{v_{th_e}^2}{\nu}\, \rho_e^2 \nu \sim D_B \right)$

$$\boxed{\chi_{\perp\,\mathrm{eff}} \simeq \frac{D_B\, D_M}{\Delta_\perp}}$$

- $\chi_{\perp\,\mathrm{eff}}$ scales with Bohm, not Spitzer ($\chi_{\parallel}$) ✓

- kicking off line important, again.

$$\chi_\perp \sim D_M \frac{\chi_{\parallel}}{l_c} \sim v_{Th} D_M \frac{l_{mfp}}{l_c}$$

To compare R&R:

$$\chi_\perp \simeq \sqrt{\chi_{\parallel}\chi_\perp} \frac{\langle \tilde{b}^2 \rangle}{\Delta_\perp} l_{ac}$$

what is $\Delta_\perp$?

Now $$\frac{\chi_{\parallel}}{l_c^2} \sim \frac{\chi_\perp}{\Delta_\perp^2}$$

{diffusion set ⊥ scale.

$$\Delta_\perp \sim l_c \sqrt{\chi_\perp / \chi_{\parallel}}$$

enters spectrum (small layer)

$\Rightarrow$

$$\chi_\perp \sim \sqrt{\chi_{\parallel}\chi_\perp}\, \frac{\langle b^2 \rangle l_{ac}}{l_c\,(\chi_\perp/\chi_{\parallel})^{1/2}}$$

$$\boxed{\chi_\perp \sim \frac{\chi_{\parallel}}{l_c} D_M}$$

✓ same as R+R, to log.

so – modulo $k_\perp$, $\Delta_\perp$; agrees with R+R to within log. factor

– $\chi_\perp \sim v_{th} D_M \dfrac{l_{mfp}}{l_c}$

$\Rightarrow$ covers diffusion in $K_u < 1$ stochastic fields

$\Rightarrow$ Lesson: Take care re: irreversibility!