

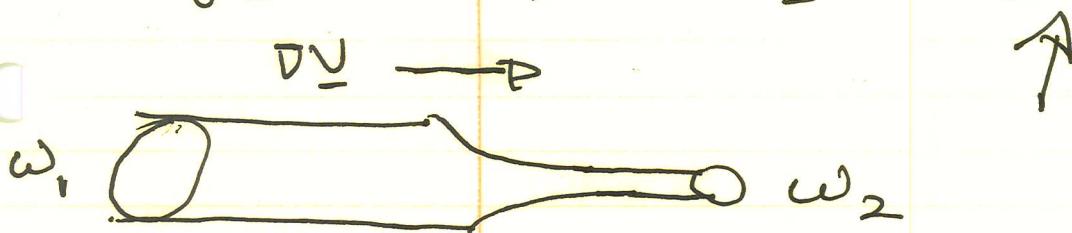
# Fluids in Flatland - A Short Introduction

~ apologies to Edwin Abbott

- A Quick Look: (i.e. Why Notes) (Sneak Preview)
- what physical process underpins K4I Cascade, etc.
  - vortex tube stretching

$$\text{i.e. } \partial_t \underline{\omega} = \nabla \times (\underline{v} \times \underline{\omega}) + r \nabla^2 \underline{\omega}$$

$$\partial_t \underline{\omega} + \underline{v} \cdot \nabla \underline{\omega} - r \nabla^2 \underline{\omega} = \underline{\omega} \cdot \nabla \underline{v}$$



$$- 2D \quad \underline{\omega} \cdot \nabla \underline{v} = 0 \quad (\nabla \cdot \underline{v} = 0)$$

$$\text{or } \partial_t \underline{\omega} + \underline{v} \cdot \nabla \underline{\omega} = r \nabla^2 \underline{\omega} + \underline{f}_{\text{ext}}$$

$$\partial_t \nabla^2 \phi + \nabla \phi \times \nabla \cdot \nabla^2 \phi = r \nabla^2 \nabla^2 \phi + \underline{f}_{\text{ext}}$$

, , two conserved quadratic invariants:

$$\int \frac{v^2}{2} \rightarrow \text{energy}$$

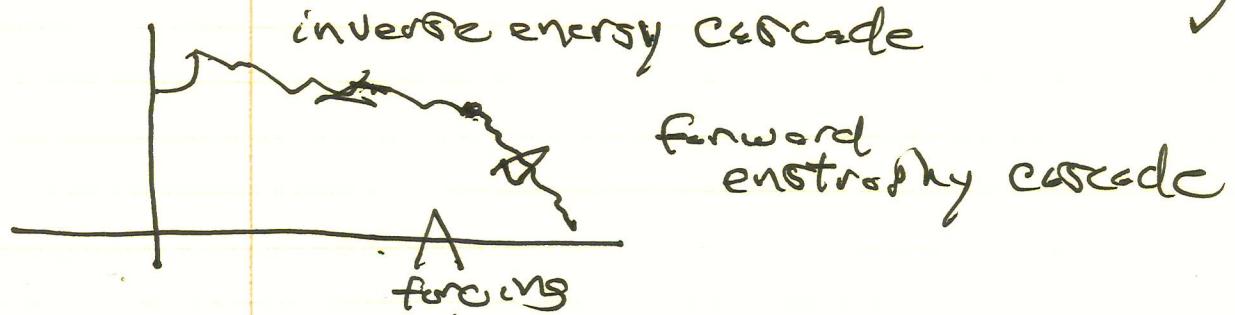
$$\int \frac{\omega^2}{2} \rightarrow \text{enstrophy} (\sim \text{mean squared vorticity})$$

### Refs:

- R. Salmon : Notes on EFD
  - G. Vallis : Atmospheric and Oceanic Fluids
- + Posted Notes from EFD Module
- + Posted References (both Module & Lectures)
- N.B. Boffetta and Ecke review especially recommended.

Key element in 2D turbulence is constraint imposed on dynamics by dust conservation law.

Upshot: Dual Cascade (Kraichnan '67)



dust self-similarity regions,

$$\rightarrow \text{and } \partial_t \underline{\omega} + \underline{v} \cdot \nabla \underline{\omega} + \underline{\omega} \cdot \nabla \underline{v} = \nu \nabla^2 \underline{\omega} + f$$

Why 2D ?  $\rightarrow$  Constrained Dynamics.

= Recall Taylor-Proudman Theorem

$\rightarrow$  in rotating fluid,  $(\underline{\omega} + 2\underline{\Omega})/\rho$  is frozen in.

$\rightarrow \Omega \gg$  other rates

$$2\Omega \partial_z v \approx 0$$

$\Rightarrow$  ~2D dynamics

Immediately realize that

$\sim 2D$  dynamics  $\Leftrightarrow$

characteristic of  
 $\sim 2D$  dynamics

$$Ro = v/L\Omega < 7$$

Rossby  
#

- $(v/\ell) \rightarrow$  other ratio  
i.e.  $v/\Omega \propto 2\ell \times v$
- contract Re

$\Rightarrow$  favors slow, large scale motion in (thin) rotating system  
i.e. atmosphere, ocean, etc.

Ways to 2D-ize:

- rotation ,  $Ro = v/L\Omega$

- stable stratification

$$\# \sim v/L N$$

$$N^2 = g \frac{\partial \phi}{\partial z}$$

- strong magnetic field

$$Ro \rightarrow v/L \Omega_{ci} \Rightarrow \text{Hasegawa-Mima model}$$

cyclotron frequency

## - Low $R_o$ dynamics

Given  $R_o < 1$ , have fundamental relation between pressure and vorticity  
 $\rightarrow$  includes centrifugal

$$\frac{d\mathbf{v}}{dt} = -\nabla \left( \frac{P^*}{\rho} \right) - 2\Omega \times \mathbf{v}$$

$R_o < 1 \Rightarrow$  Geostrophic balance

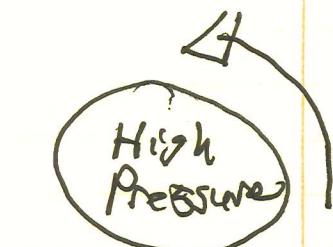
$$\mathbf{v} \approx -\nabla \left( \frac{P}{\rho} \right) - 2\Omega \times \mathbf{v}$$

$$\Rightarrow \mathbf{v}_\perp = \Omega \times \nabla \left( \frac{P^*}{\rho} \right) / \Omega^2$$

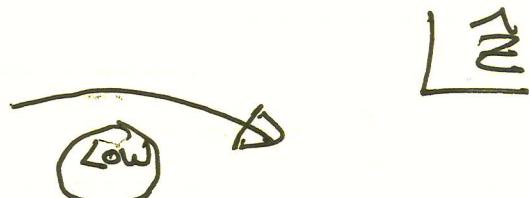
$$\boxed{\frac{\mathbf{v}_\perp}{\Omega} = -\frac{\nabla \left( \frac{P^*}{\rho} \right) \times \hat{\mathbf{z}}}{\Omega}}$$

$$\frac{P^*}{\rho} \Leftrightarrow$$

Pressure - age - stream - function :



clockwise  
counter-clockwise  
about

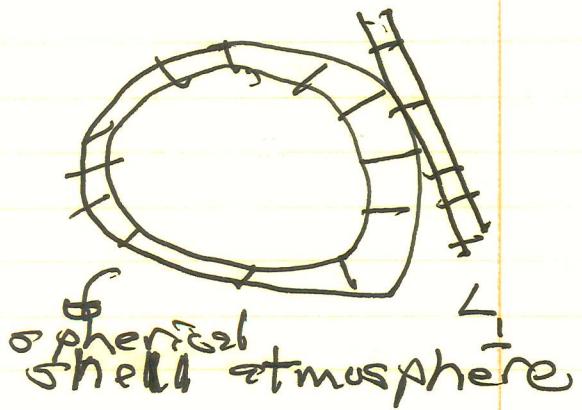


Fluid rotation

about Low } pressure cells.  
High }

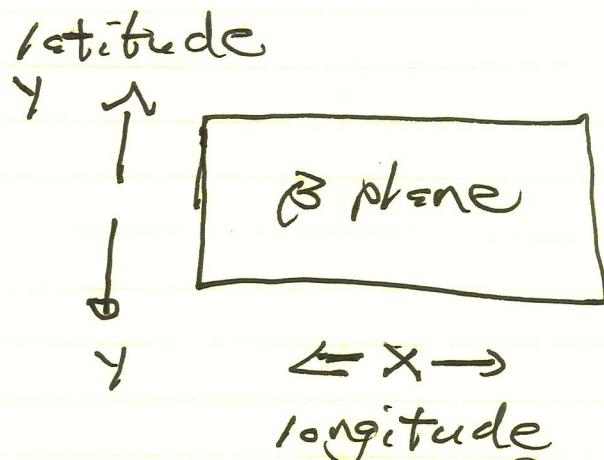
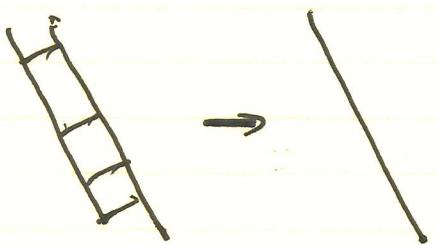
-  $\beta$ -plane Model

→ Quick derivation

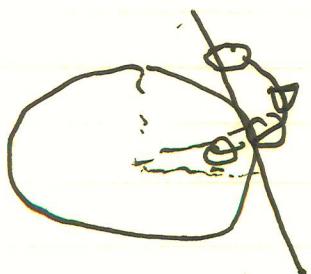


- $\beta$  plane tangent to spherical shell atmosphere
- strong stable stratification on  $\beta$

so describe/approximate dynamics  
in 2D plane tangent to sphere  
i.e.  $\beta$ -plane



Now, consider displacement of fluid/vortex element:



- $\omega + 2\Omega$  frozen in
- $C = \int da \cdot (\omega + 2\Omega)$   
circulation conserved.

Point: displacing fluid element  
causes change in

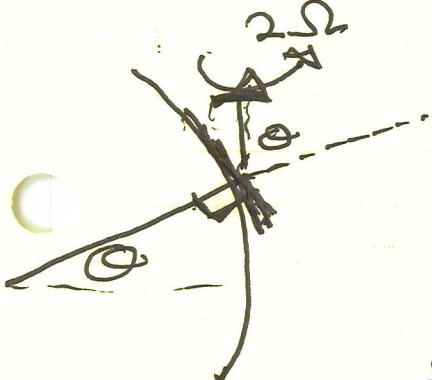
$$\int \rho d\zeta \cdot \underline{\Omega} \sim \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} \sim \cos \theta_p$$

$\theta_p$   
polar

there must be a change in  
fluid vorticity to conserve circulation

since planetary vorticity piece of  
circulation changed by displacement

$A \equiv$  area of vortex



$$\frac{dG}{dt} = 0$$

$$\frac{d}{dt} (A\omega + A2\Omega \sin \theta) = 0$$

projection factor

⇒

$$\frac{d\omega}{dt} = -2\Omega \cos \theta \frac{d\theta}{dt}$$

$$= -\frac{2\Omega}{R} \cos \theta \frac{d}{dt} (R\theta)$$

$$= -\Omega v_y$$

$$\Omega = \frac{2\Omega \cos \theta}{R} \rightarrow \text{gradient in coriolis force}$$

Of course  $\frac{d}{dt}(R\theta) = \frac{d}{dt}y = v_y$

$$\frac{d\omega}{dt} = -\beta v_y$$

+ add dissipation, forcing

Charney

$$\partial_t \omega + \underline{v} \cdot \nabla \omega + u \omega = v \nabla^2 \omega + f$$

$$\frac{d}{dt} = \partial_t + \underline{v} \cdot \nabla \quad z \perp \beta \text{ plane}$$

$$\underline{v} = -\frac{\nabla P}{2\Omega} \times \hat{z} \rightarrow \nabla \phi \times \hat{z}$$

$$\omega = \Omega^2 P / 2\Omega$$

$$R \rightarrow \infty, \beta \rightarrow 0$$

$$\begin{aligned} \partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi + u \nabla^2 \phi \\ = v \nabla^2 \nabla^2 \phi + f \end{aligned}$$

2D fluid equation.

→ simplest incarnation of "2D Fluid"  
 $\hookrightarrow$  2D turbulence (eddies)

→  $\beta$ -plane equation is next  
 simplest  $\Rightarrow$  surface waves  
 eddies, zonal flows.

Observe:

Can re-write 2D inviscid equation as

$$\partial_t \omega_2 + \{ \omega_2, H \} = 0$$

$$H = \phi$$

Conservative Hamiltonian evaluated

Similar to Liouville or Vlasov equation:

$$\partial_t f + \{ f, H \} = 0$$

$$H = \frac{\rho^3}{2m} + \text{let } \phi \quad , \quad + \text{Poisson's equation}$$

$$\partial_t F + \mathbf{v} \cdot \nabla F + \frac{2}{M} E \cdot \nabla_{\mathbf{v}} F = 0$$

i.e.  $\omega_2 \leftrightarrow f \Rightarrow \begin{cases} \text{conserved (phase} \\ \text{space) density.} \end{cases}$

which brings us to:

$\{ \text{Potential Vorticity} \}$

Observe can write equations in conservative form, i.e.

$$\frac{d}{dt} \underline{\omega} = 0$$

(pure 2D)

$$\frac{d}{dt} (\underline{\omega} + \beta \underline{y}) = 0 \quad (\beta\text{-plane})$$

$\frac{d}{dt}$   
fluid  
vorticity

planetary vorticity  
(f.o. in expansion)

$\underline{\omega} + \beta \underline{y} \equiv$  simple example of  
potential vorticity (PV)

- generalized vorticity akin  
to phase space density

GFD = the study of fluids with  
PV  
= "The Fluid Dynamics of PV"

More generally on PV:

- recall for rotating fluid:

$$\frac{d}{dt} \left( \frac{\underline{\omega} + 2\underline{\Omega}}{\rho} \right) = \frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \nabla \underline{V}}{\rho}$$

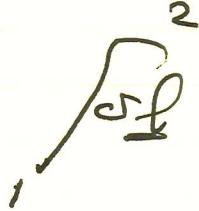
ahm.

$$\frac{d}{dt} \underline{\delta l} = \underline{\delta l} \cdot \nabla \underline{V}$$

same eqn  $\rightarrow \frac{\underline{\omega} + 2\underline{\Omega}}{\rho}$  frozen in

Now, consider conserved scalar field:  $\psi$

$$\frac{d}{dt} \psi = 0$$



$$\frac{d}{dt} (\psi_1 - \psi_2) = 0$$

$$d\psi = \nabla \psi \cdot d\ell$$

or

$$\frac{d}{dt} (\nabla \psi \cdot d\ell) = 0$$

$$\text{and } d\ell \leftrightarrow \frac{\underline{\omega} + 2\underline{\Omega}}{\rho}$$

so if  $\nabla \psi$  satisfies,  $\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}$  must satisfy

⇒

$$\boxed{\frac{d}{dt} \left( \frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \nabla \psi}{\rho} \right) = 0}$$

along  
trajectories

→ general statement  
of PV conservation

$$\boxed{g = \frac{\underline{\omega} + 2\underline{\Omega} \cdot \nabla \psi}{\rho}}$$

$\rho_V$   
(general)  
any  $\underline{\nabla} \psi$

Defn:

$$\rho = \rho_0 + \tilde{\rho}$$

$$\tilde{\pi} = (\kappa_1 \tilde{\phi} / T) \rho_0$$

$$\nabla \psi = \hat{z}$$

- PV conservation  $\leftrightarrow$  particle re-labeling symmetry  
(i.e. particles can be re-labeled without changing thermodynamic state)

N.B. If consider finite thickness shell

$$I = \nabla^2 \phi + \beta y + \underbrace{\frac{f_0^2}{\rho} \partial_z \left( \frac{\rho}{N^2} \partial_z \phi \right)}$$

$f_0 = 2 \Omega \sin \theta$  - rotation

$N^2 = g / L_\theta$  - buoyancy

Relevance of finite thickness?

Scale!  $\rightarrow 1/L_1^2$  vs  $\frac{f_0^2}{N^2 H^2}$   
 $\hookrightarrow$  layer thickness  
 $\text{(deflection radius)}^{-2}$   
 $\sim 1/L_{\text{lat}}^2$

C  $\approx$

$$\gamma L^2 \sim \gamma L_d^2$$

→ relative vorticity  
and deformation  
effects contribute  
equally

{ ~ 100 km ocean  
~ 1000 km atmosphere

$L < L_d \rightsquigarrow \beta\text{-plane}$ .

## → 2D Turbulence

- issues: conservation of energy, enstrophy
- trends in constrained spectral evolution
- self-similarity ranges, inverse cascade
- fate of energy

Issues:

- 2D turbulence is the generic problem of GFD

- $R \rightarrow 0$

$$\partial_t \nabla^2 \phi + \nabla \phi \times \vec{z} \cdot \nabla \nabla^2 \phi - \nu \nabla_{\perp}^2 \nabla_{\perp}^2 \phi + \mu \nabla_{\perp}^2 \phi = f$$

+  
 drag scale  
 invariant damping  
 → control large scale

any scale

2 inviscid invariants:

$$\langle (\nabla \phi)^2 \rangle \rightarrow \text{energy}$$

$$\langle (\nabla^2 \phi)^2 \rangle \rightarrow \text{enstrophy}$$

N.B.:

- in 3D, enstrophy produced:

$$\frac{d}{dt} \langle \omega^2 \rangle \sim \langle \underline{\omega} \cdot (\underline{\omega} \cdot \nabla \underline{v}) \rangle$$

$$\langle D(k) \rangle \sim k^2 k^{-5/3} \sim k^{1/3}$$

- 2D,  $\underline{\omega} \cdot \nabla \underline{v} \rightarrow 0$

$\rightarrow$  all powers  $\int d^2x \omega^n$  conserved

$\int d^2x \tilde{\omega}^n \rightarrow \langle \omega^2 \rangle$  conserved on finite box

$\therefore$  story incompatible with  $k^{4/3}$

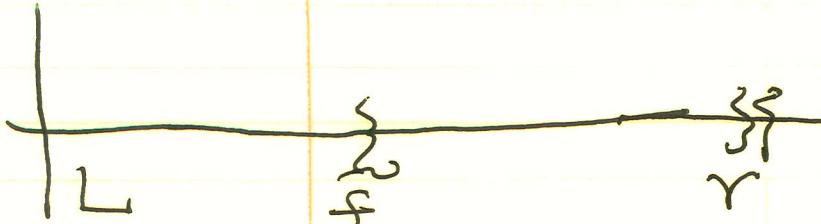
Problem of 2D Fluid:

- given forcing at any scale  $b_f$  s.t.

$$L \geq b_f > b_r$$

$\rightarrow$  how does dual conservation of  $E, S$  constrain transfer?

$\rightarrow \sigma_{\text{eff}} - \sigma_{\text{inertia}} \text{ ranges?}$



85

1.8. u( )

bounding polygons are as-

$\langle \text{V0}, \text{V1}, \text{V2} \rangle$  or  $\langle \text{V2}, \text{V3}, \text{V0} \rangle$

with the  $\text{N}^{\text{st}}$  +  $\langle \text{V0}, \text{V1} \rangle$

$\text{O} \leftarrow \text{V0}, \text{V1}, \text{V2} =$

previous  $\langle \text{V0}, \text{V1} \rangle$  must be re-

positioned  $\langle \text{V0} \rangle \leftarrow \langle \text{V0}, \text{V1} \rangle$   
and still .

With new additions proto .

is being used to render

the  $\text{V}$  also you try to make sure -  
 $\text{v}_1 < \text{v}_2 \leq \text{v}_3$

SL 3 The previous looks good and is  
probably worth doing

if we're simplifying things a



## Theoretical "clues":

- consider 3 modes, interacting  
(3 to conserve quadratic energy)

$$\begin{array}{ccc}
 & 1 & 2 & 3 \\
 k_1^2 & k_2^2 & k_3^2 \\
 \text{LHS} & | & \text{---} & | & \text{RHS} \\
 & 1 & 2 & 3 & \\
 & & & & \\
 & & & & k_1^2 < k_2^2 < k_3^2 \\
 & & & & k_2^2 \leftrightarrow k_3^2
 \end{array}$$

Conservation:

$$E_2 = E_1 + E_3$$

$$\Omega_2 = \Omega_1 + \Omega_3$$

$$\text{but } \Omega(k) = k^2 E(k)$$

$$\begin{cases}
 E_2 = E_1 + E_3 \\
 k_2^2 E_2 = k_1^2 E_1 + k_3^2 E_3
 \end{cases}$$

$$\therefore E_1 = \left( \frac{k_3^2 - k_2^2}{k_3^2 - k_1^2} \right) E_2 \rightarrow E_1 \sim E_2$$

$$E_3 = \left( \frac{k_2^2 - k_1^2}{k_3^2 - k_1^2} \right) E_2 \rightarrow E_3 \sim \frac{k_2^2}{k_3^2} E_2$$

(1)

Linearization ( )

Extraction from  $\mathcal{E}$  reduced  
( $\mathcal{E}$  to the structures around  $\mathcal{E}$ )



$$E \geq H_1 + H_2$$

$$H_1 = H_2$$

Iteration ( )

$$H_1 + H_2 = n H$$

$$H_1 + H_2 = n H_2$$

$$(n)H^2 \approx (n)H - nH$$

$$H_1 + H_2 = H$$

$$H_1 + H_2 = H$$

$$H_1 + H_2 + \frac{1}{2} \left( \frac{\partial H_1}{\partial H_2} + \frac{\partial H_2}{\partial H_1} \right) = H$$

$$H_1 + H_2 + \frac{1}{2} \left( \frac{\partial H_1}{\partial H_2} + \frac{\partial H_2}{\partial H_1} \right) = H$$

as  $\Omega(k) = k^2 E(k)$

$E_1 \sim E_2 \rightarrow$  energy transferred  
to large scale mode!?

$\Omega_3 \sim \Omega_2 \rightarrow$  enstrophy transferred  
to small scale mode!?

~ suggests: energy accumulates at large scale,  
enstrophy accumulates at small scale,

$\Rightarrow$  2 self-similar transfer ranges in 2D  
= turbulence!?

N.B. Analogy: Asymmetric Top.

(restricted)

Conserv:  $\sum_i L_i^2 = L^2$

$$E = \sum_i L_i^2 / 2I_i$$

$$L^2 = L_1^2 + L_2^2 + L_3^2$$

$$E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

$I \propto 1/k^2$ , etc.

and energizing intermediate axis  
 $\rightarrow$  decay to 1, 3 etc.

$$(W) E^{\text{in}} = (W) S_L \quad 20 \quad )$$

for  $E^{\text{out}}$  start  $\times 10^3$   $\text{J} =$   $10^3 \times 10^3$

to calculate  $E^{\text{out}}$   $\text{J} = 10^3 \times 850$

point 3 (3) to calculate  $E^{\text{out}}$   $\text{J}$   $=$   $10^3 \times 1000$   $\text{J}$   $=$   $10^6 \text{ J}$

$$\text{the net energy output } 20(1000 - 850) \text{ J} = 400 \text{ J}$$

net output  $\text{J} = 10^6 \text{ J} + 400 \text{ J} = 10^6 \text{ J} + 0.4 \times 10^6 \text{ J} = 1.04 \times 10^6 \text{ J}$

$$\frac{10^6}{10^3} = 1000 \text{ W} \quad (\text{output})$$

$$1000 \text{ W} = 1 \text{ kW}$$

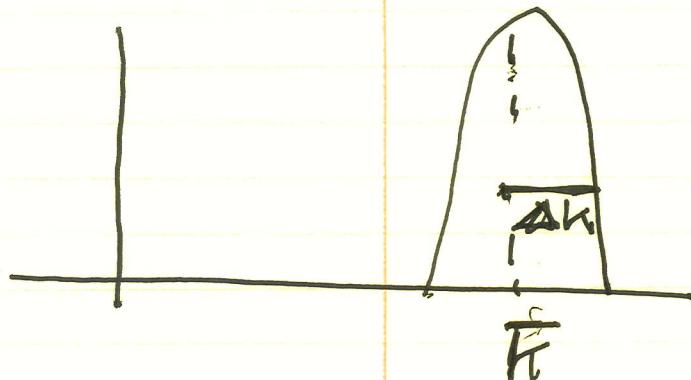
$$\frac{10^6}{10^3} + \frac{400}{10^3} + \frac{0.4 \times 10^6}{10^3} = 1.04 \text{ kW}$$

$$\frac{10^6}{10^3} + \frac{400}{10^3} + \frac{0.4 \times 10^6}{10^3} = 1.04 \text{ kW}$$

$$\text{net } E^{\text{out}} = 1$$

two components existing here  
to end of year one

→ But many D.O.F.'s ...



(Rhines:  
after the fact)

Consider a spectral 'slug' of turbulence, initialized.

How will  $\bar{k}$  evolve, given  $\langle (\Delta k)^2 \rangle > 0$ ?  
i.e., assume spectrum spreads ...

N.B. Does  $\langle \Delta k^2 \rangle$  exist?

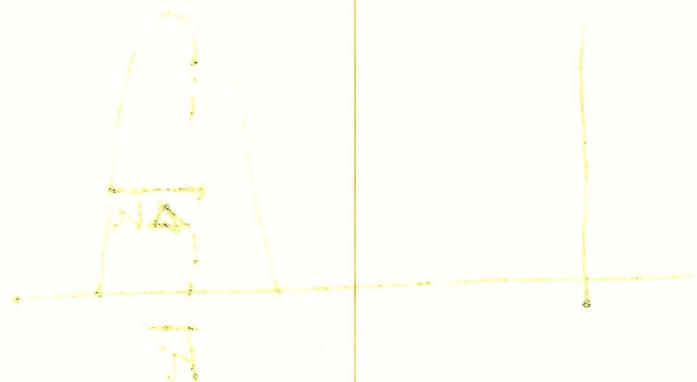
$$\begin{aligned}
 \langle (\Delta k)^2 \rangle &= \int dk (k - \bar{k})^2 E(k) / \int dk E(k) \\
 &= \int dk (k^2 - 2k\bar{k} + \bar{k}^2) E(k) / \int dk E(k) \\
 &= \int dk (k^2 E(k) - 2k\bar{k} E(k) + \bar{k}^2 E(k)) / \int dk E(k) \\
 &= (\Omega_0 - 2\bar{k}^2 E_0 + \bar{k}^2 E_0) / E_0 \\
 &= \Omega_0 / E_0 - 2\bar{k}^2 \\
 &= (\Omega_0 / E_0) - \bar{k}^2
 \end{aligned}$$

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2014

... v̂ H D from TWS a )

is constant  
in time



The right part is referred  
to as induction, or inductance

such values are called self  
inductance and common L.

? since  $\langle \text{H} \rangle = 0$  with

$$\text{H} = \frac{1}{\mu_0} \left( \frac{V}{R} + \frac{V}{R-L} \right) = \langle \text{H} \rangle$$

$$\langle \text{H} \rangle = \frac{1}{\mu_0} \left( \frac{V}{R+L} - \frac{V}{R} \right)$$

$$\langle \text{H} \rangle = \frac{1}{\mu_0} \left( \frac{V}{R+L} - \frac{V}{R} \right) =$$

$$\frac{V}{\mu_0 R} - \frac{V}{\mu_0 R} =$$

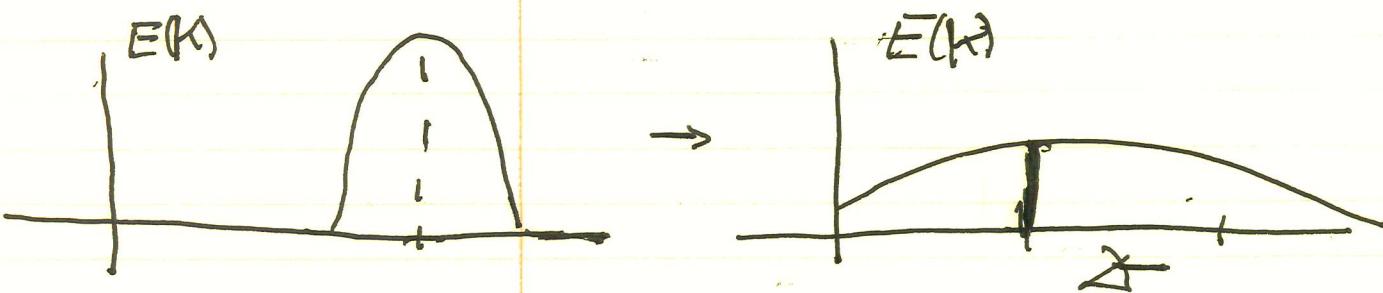
$$\frac{V}{\mu_0 R} - \frac{V}{\mu_0 R} =$$

Here:  $\int dk E(k) = E_0 \rightarrow \text{const}$

$$\int dk k^2 E(k) = \Omega_0 \rightarrow \text{const}$$

$$\int dk k E(k) = \bar{k} E_0 \rightarrow \text{defines centroid}$$

$\hat{\omega}_t \hat{\omega} \langle \Delta k^2 \rangle > 0 \Rightarrow \hat{\omega}_t \bar{k} < 0$



- spectrum broadens but also shifts toward large scales
- energy content shuffled / coupled to larger scale
- suggestive of inverse energy cascade
- similar story for enstrophy  $\Rightarrow$  forward cascade!

∴ Enter the Dual Cascade!

Time  $\leftarrow$   $\alpha_1 = \text{ReLU}(w_1)$  :  $w_1 \in \mathbb{R}^n$

Time  $\leftarrow$   $\alpha_2 = \text{ReLU}(w_2)$

Final output  $\rightarrow \tilde{\alpha} = \text{ReLU}(w_3)$

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \times \mathbb{C}$

$w_1$

$w_2$



With this idea of convolutional networks -  
we can do feature extraction -

which basically extracts features -  
like edges

and then applies softmax to distinguish &  
to have the interpretations of what's happening at  
each location

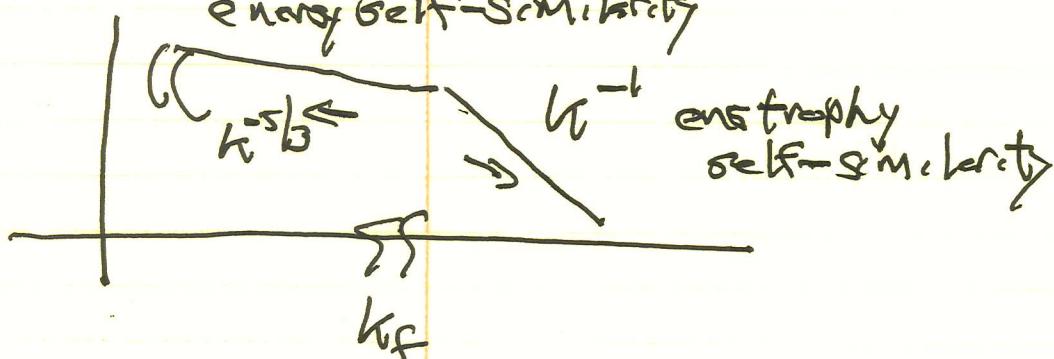
It exploits local and non-local info

Dual cascade (Kraichnan '67) =

from forcing, system supports  $\geq$  self-similarity ranges:

- forward enstrophy range / cascade ( $k > k_f$ )
  - no forward energy flux
  - no ~~energy~~ dissipation by ~~viscosity~~ ( $Re \rightarrow \infty$ )
- inverse energy cascade
  - no inverse energy flux
  - no viscous power dissipation
  - damping by drag, etc.
  - not stationary

Cascade = range self-similar transfer  
energy self-similarity



$$\dot{N} = \frac{d}{dt} \langle \omega^2 \rangle \sim (\psi/k_f)^3$$

$$\dot{E} = \frac{d}{dt} \langle v^2 \rangle \sim V_F^3/k_f \quad \rightarrow \begin{matrix} \text{forcing rate} \\ \text{not-dissipation} \end{matrix}$$

is first mentioned) it does not ( )

then it is known that there exist next  
two other solutions

exists several solutions for which -  
 $(2N < N)$

most years know not an  $\Leftrightarrow$   
and most of the time we do  
 $\Leftrightarrow$   $\Rightarrow$   $\Leftrightarrow$   $\Rightarrow$

exists no real solutions -

most years obtain an  $\Leftrightarrow$   
but with small errors on  $\Leftrightarrow$   
the right of the graph  $\Leftrightarrow$   
 $\Rightarrow$  no solution for

and from this - the right is off the  
graph - the left is off the graph



$$f(g(x)) = g(f(x)) \underset{x=x_1}{=} 0 = f(x_1)$$

then obtain -

$$g(f(x)) = f(g(x)) \underset{x=x_2}{=} 0 = g(x_2)$$

of course  $\kappa_F^2 E \sim M$ .

- Forward  $\rightarrow$  Enstrophy

$$\frac{\Omega(\ell)}{T(\ell)} \sim M$$

$$\frac{v(\ell)}{\ell} \sim T(\ell)^{-1}$$

$$\sim \omega(\ell)$$

$$\omega(\ell) \sim M$$

$$\Omega(\ell) \sim \omega(\ell)^2$$

$$\omega^3(\ell) \sim M^{2/3}$$

but  $\omega^3(\ell) \sim k \Omega(k)$

$$\Rightarrow \boxed{\begin{aligned} \Omega(k) &\sim M^{2/3}/k \\ E(k) &\sim \Omega(k)/k^2 \sim M^{2/3}/k^3 \end{aligned}}$$

energy spectrum in enstrophy range

- no forward energy flux in enstrophy range

- observe  $1/\gamma_\ell \sim M^{1/3} \rightarrow \text{const, here}$

vs  $k^{-4/3}$

$1/\gamma_\ell \sim \frac{C}{\ell^{2/3}} \rightarrow$  faster for smaller

प्राचीन

सामाजिक दृष्टि

विवरणीय और प्राचीन -

प्राचीन  
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(प्राचीन एवं प्राचीन तथा )

प्राचीन

प्राचीन

प्राचीन या प्राचीन

प्राचीन विवरणीय वा प्राचीन विवरणीय

प्राचीन विवरणीय वा प्राचीन विवरणीय

प्राचीन विवरणीय

प्राचीन

प्राचीन विवरणीय

प्राचीन

- ⇒ tip-off that since all scales transfer at some rate, non-local transfer of enstrophy can occur
- ⇒ Corrections [Logarithmic due to straining]

### - Inverse Energy

$$\epsilon = v(\ell)^2 \frac{v(\ell)}{\ell}, \text{ as before}$$

but  $\epsilon \equiv$  energy stirring rate

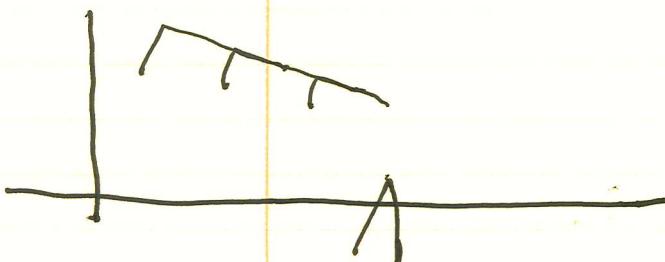


$$E(k) = \epsilon^{2/3} k^{-5/3} \Rightarrow \text{inverse energy cascade range}$$

→ akin 3D, but unstable

→  $\langle T(\ell) \rangle \sim \frac{v(\ell)}{\ell} \rightarrow$  cascade slows  
as longer scales approached.

→ not stationary state



largest scale slowest,  
keeps evolving

→ eventually encounters drag,  
boundary, etc,

R.

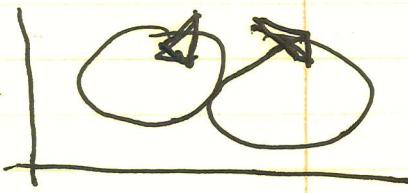
Soil in the section tent - first is  
sand, then loam, then clay loam.  
These three categories do not have  
sharp boundaries. In practice one  
category overlaps the other.

Soil profile -  
consists of  $\text{O}_{\text{H}} \text{ (O)H} = \text{H}$   
 $\text{O}_{\text{H}} \text{ (O)H} = \text{H}$   
and  $\text{O}_{\text{H}} \text{ (O)H} = \text{H}$   
and  $\text{O}_{\text{H}} \text{ (O)H} = \text{H}$

Soil profile consists of  
topsoil horizon or  $\text{O}_{\text{H}} \text{ (O)H} = \text{H}$  +  
subsoil horizon which is  
subject to penetration +  
desorption and adsorption



Soil contains substances +  
water, air, etc.



→ straining  
(non-local) effect  
in scale?

→ structure at  
large scale.

→ no curved enstrophy flux  
in energy range.

→ no forward energy flux,  
Pd by viscosity  $\rightarrow 0$ .

$\frac{1}{t}$   
dissipated power

→ So, where does the energy go?  
- Rotation

- boundary effects

- straining on small scales

(Boffetta et al.  
posted)

→  $\langle \delta u(k) \rangle = +3/2 \propto k$

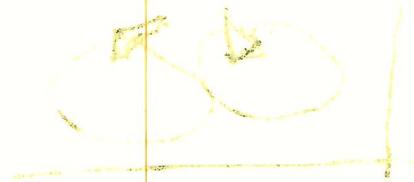
~ analogue of  $4/5$  for inverse  
energy range

~ + for  $k > k_c$

~  $\propto$  here is stirring rate.

→ (static) vorticity contours in  
inverse energy range exhibit  
statistics of percolation cluster!

→ No "conventional" intermittency in inverse  
range. Deviations from Gaussian occur



• Wants on  
- What's (Wanting)  
- What (Want)

• Belonging or  
- Who (Want)

• Wants belong on &  
- Who wants in

• Wants belong on &  
- Who - wants to get  
new fruits of oasis

• Wants belong on &  
- Who wants -

positive want -

negative want -

• Wants belong on &

• Wants belong on & belong -

- Who wants to get -

- Who wants to get -

- Who wants to get -

• Wants belong position (Want) & (

- Who wants to get position

- Who wants position for position

- Who wants position located on &  
- Who wants position located in area

→ What of particle dispersion?

Revisiting Richardson:



$h_{1,2} \rightarrow$  energy range

$$\frac{dh_{1,2}}{dt} = V(h_{1,2}) = \epsilon^{1/3} l^{1/3}$$

$$h_{1,2}^3 \sim \epsilon t^3; \text{ as before}$$

$h_{1,2} \rightarrow$  enstrophy

$$\frac{df}{dt} = v(l) = [\Omega(l)]l = \gamma^{1/3} l$$

⇒ separation grows exponentially  
in enstrophy range

N.B. - Dual cascade used to justify  
selective decay - minimum  
enstrophy

$$\begin{aligned} \Omega &= \int d^3x (V^2 \phi)^2 + \lambda \int d^3x (\nabla \phi)^2 \\ \delta \Omega &= 0. \end{aligned}$$

Geometrische Struktur bestimmen  $\rightarrow$  )  
z. B. quadratische Funktionen

z. B. quadratische Funktionen



Open Forme  $\rightarrow$   $y =$

$$y = ax^2 + bx + c$$

Einheitsform  $\rightarrow$   $y = a(x - p)^2 + q$

$$y = a(x - p)^2 + q \quad \text{Wurzeln } \rightarrow \text{Nullstellen } \rightarrow \text{Werte}$$

Wurzelausdrücke  $\rightarrow$  Lösungsmethode  
Nullstellen  $\rightarrow$  Wurzeln  $\rightarrow$  Nullstellen

Wurzeln mit dem Bruch  $\frac{1}{\sqrt{a}}$  bilden  
Koeffizienten  $\rightarrow$  gleich ausmultiplizieren  
Multiplikation

$$\frac{1}{\sqrt{a}} \cdot \sqrt{a} + \frac{1}{\sqrt{b}} \cdot \sqrt{b} = 1$$

$\rightarrow \beta$ -Plane: Turbulence Waves  
Flows

Recall:

$$\partial_t \nabla^2 \phi + \nabla \phi \times \vec{\Sigma} \cdot \nabla \nabla^2 \phi - \nu \nabla^2 \nabla^2 \phi + \mu \nabla^2 \phi = -\beta V_y \tilde{f}$$

Ignoring:  $\nu, \mu, \tilde{f}$

$$\partial_t \nabla^2 \phi + \nabla \phi \times \vec{\Sigma} \cdot \nabla \nabla^2 \phi = -\beta V_y$$

$\Rightarrow$  waves

$$\omega_k = -\beta k_x / k^2, \quad V_{0y} = \frac{2\beta k_x k_y}{(k^2)^2}$$

$\rightsquigarrow$  Rossby wave

and

$\Rightarrow$  flows

how does large scale  
order emerge?

$$\left. \begin{aligned} k_x &\rightarrow 0 \\ k_y &\text{ finite} \\ \omega_k &\rightarrow 0 \end{aligned} \right\}$$

zonal mode



Jets, belts, jet stream



2 new players  $\rightarrow$  waves, flows.

Numerous questions:

- ②  $\rightarrow$  how do zonal flows form?  
why  $\Rightarrow$  many ways! ✓
- ①  $\rightarrow$  how do {waves} modify, interact/  
with inverse cascade? ✓
- ③  $\rightarrow$  scale of zonal flows? ✓
- ④  $\rightarrow$  implications for atmospheric  
phenomenology

On zonal flows:

Reynolds stress

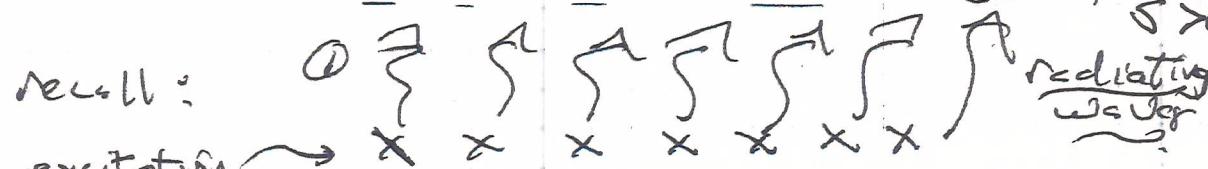


- ZFs ubiquitous
- Flows produced by Momentum transport
- Simplest perspective  $\leftrightarrow$  wave  
propagation!



(Linear) wave propagation  
on account for ZF formation

27.

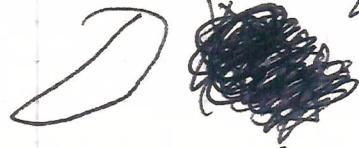
recall: 

excitation (storms etc.)  $\rightarrow$   $\text{SLO}$

Radiation on latitude  $- - - -$

$$S = v_{xy} \tilde{E}_y = 2 \frac{k_x k_y B \epsilon}{(\omega^2)^2} \tilde{y}$$

beach  
(absorber)  
(unreflected current)

① outgoing waves  $\rightarrow k_x k_y > 0$   
 $S \sim (+) \tilde{y}$  

②  $S \sim -\tilde{y} \rightarrow k_x k_y < 0$  

eddy tilt  
change

but

$$\langle \tilde{v}_y \tilde{v}_x \rangle = \sum_k k_x k_y \theta k^2$$

so ①  $\rightarrow \pi_{y,x} < 0$

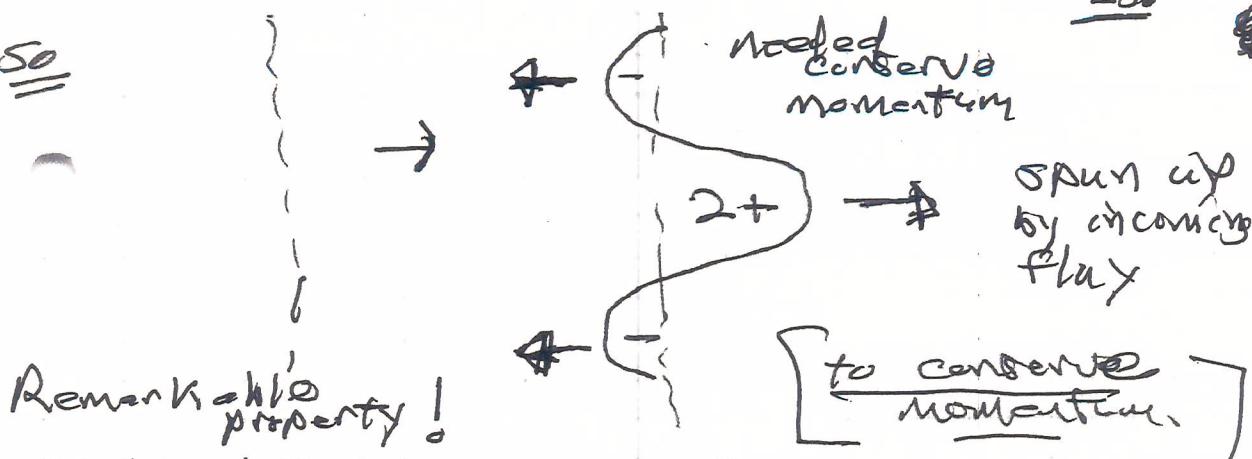
②  $\rightarrow \pi_{y,x} > 0$

points:  
outgoing  
wave energy  
density flux  
generated  
incoming  
momentum  
fluxes



50

280



Remarkable property!

→ beautiful example of:

... "the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum

into this region"

(stirring  $\rightarrow$  spin-up!)

Flows  $\leftrightarrow$  energy stirring

Isaac Held  
('01)

→ Wave mechanism requires separation of excitation and dissipation (break) regions.

→ Requires:

- waves

- vorticity/momentum transport in space

\* → irreversibility  $\rightarrow$  outgoing waves

→ symmetry breaking,  $\Delta$  has direction

- self-forcing/damping



29.

Something  
general

- ⇒ Useful to investigate wave
- theorems for flow production

→ Key observation:  
(inhomogeneous PV mixing)

$$\langle \tilde{U}_y \tilde{Z} \rangle_z \approx \text{const.}$$

$$= \langle \tilde{U}_y \tilde{\sigma}_\phi^2 \rangle_z$$

$$= \langle (\partial_x \phi) (\partial_x^2 \phi + \partial_y^2 \phi) \rangle_x$$

PV Flux

$$\int \tilde{Z} = \partial_y + \tilde{\sigma}_\phi^2$$

Why?

recall essence of  
PV conservation force  
planetary - flow  
vorticity exchange.

but:  $\langle \partial_x \phi \partial_x^2 \phi \rangle = \langle \partial_x \left[ \frac{(\partial_x \phi)^2}{2} \right] \rangle_x = 0$   
symmetry b.

$$\langle \tilde{U}_y \tilde{Z} \rangle_z = - \langle (\partial_x \phi) \partial_y^2 \phi \rangle_x$$

$$= - \partial_y \langle \partial_x \phi \partial_y \phi \rangle_x + \langle \partial_{xy}^2 \phi \partial_y^2 \phi \rangle_x$$

$$\langle \partial_x \left( \frac{(\partial_x \phi)^2}{2} \right) \rangle_x$$

Taylor Identity  $= \partial_y \langle \tilde{U}_y \tilde{U}_x \rangle_x$

$$\boxed{\langle \tilde{U}_y \tilde{g} \rangle_z = \partial_y \langle \tilde{U}_y \tilde{U}_x \rangle_x}$$

(constant 3D) - EP.  
z dropped hereafter.  $\hookrightarrow$  Reynolds force

drives flow!

⇒ Look at potential enstrophy balance.



- ⇒ Zonally averaged Latitudinal PV flux = zonally averaged  
 $\rho V$  flux = zonally averaged  
 Latitudinal Reynolds force  $\rightarrow$  <sup>driven</sup> flow.

As Reynolds stress controls flow;  
 i.e.

$$\rho \left( \frac{\partial \underline{V}_x}{\partial t} + \underline{V} \cdot \nabla \underline{V}_x \right) = - \nabla p - \cancel{(2 \cancel{\rho} \cancel{\underline{V}} \times \underline{V})_x}$$

<sup>cancel</sup>  
 $\downarrow$   
geostrophic balance

$$\boxed{\frac{\partial \underline{V}_x}{\partial t} = - \nabla p - \cancel{(2 \cancel{\rho} \cancel{\underline{V}} \times \underline{V})_x} + \nu \nabla^2 \underline{V}_x}$$

$$= u \langle \underline{V}_x \rangle$$

then PV evolution } necessarily  
 Potential Enstrophy }  
control Flow.

- ⇒ What are essential to ZF generation:  
 $\left\{ \begin{array}{l} \text{inhomogeneous} \\ \text{- PV mixing / transport in SSEC} \\ \text{- translation symmetry in direction} \\ \text{of the flow.} \end{array} \right.$



Now, consider P.E balance:

B1  
~~Forcing~~

$$\frac{d}{dt} \bar{\zeta} - r^2 \bar{\zeta} = 0$$

$$\frac{\partial}{\partial t} \bar{\zeta} + \nabla \cdot \bar{\zeta} - r^2 \bar{\zeta} = - \bar{v}_y \frac{d \bar{\zeta}}{dy}$$

Potential enstrophy evolution

$$\begin{aligned} & \frac{\partial}{\partial t} \langle \bar{\zeta}^2 \rangle + \partial_y \langle \bar{v}_y \bar{\zeta}^2 \rangle + r \langle (\nabla \bar{\zeta})^2 \rangle \\ & \quad \xrightarrow{\text{Flux of potential enstrophy.}} \quad \uparrow \\ & = - \langle \bar{v}_y \bar{\zeta} \rangle \frac{d \langle \bar{\zeta} \rangle}{dy} \quad D_{12} \rightarrow \text{dissipation} \end{aligned}$$

$\left[ \begin{array}{l} \text{potential enstrophy production,} \\ \text{flux - gradient} \end{array} \right]$

$$\left( \frac{d \langle \bar{\zeta} \rangle}{dy} \right)^T \left[ \partial_t \langle \bar{\zeta}^2 \rangle + \partial_y \langle \bar{v}_y \bar{\zeta}^2 \rangle + r \langle (\nabla \bar{\zeta})^2 \rangle \right]$$

$$= - \langle \bar{v}_y \bar{\zeta} \rangle = - \langle \bar{v}_y \bar{v}^2 \phi \rangle$$

but mean (zonal) flow

$$\begin{aligned} \partial_t \langle \bar{v}_y \rangle &= - \frac{d}{dy} \langle \bar{v}_y \bar{v}_x \rangle = M \langle \bar{v}_x \rangle \\ &= \langle \bar{v}_y \bar{v}^2 \phi \rangle = M \langle \bar{v}_x \rangle \end{aligned}$$



$$\nabla \cdot (\tilde{v}_y \vec{D}^2 \vec{\varphi}) = -(\partial_t \langle v_x \rangle + u \langle \dot{v}_x \rangle)$$

$\therefore$  WAD

$$\partial_t \{ \langle v_x \rangle + \frac{\langle \vec{D}^2 \rangle}{2 \frac{d \langle \vec{v} \rangle}{dy}} \} = - \frac{v \langle \vec{D}^2 \rangle}{\frac{d \langle \vec{v} \rangle}{dy}}$$

Wave  
Activity  
Density

WAD

$$- \frac{\partial_y \langle \tilde{v}_y \vec{D}^2 / 2 \rangle}{\frac{d \langle \vec{v} \rangle}{dy}} \rightarrow \partial_t \langle v_x \rangle$$

pseudomomentum

$$\partial_t \{ \langle v_x \rangle - \left( \frac{-k_x \langle \vec{D}^2 \rangle}{2 k_x \frac{d \langle \vec{v} \rangle}{dy}} \right) \}$$

$$= - u \langle v_x \rangle - \delta \langle \vec{v}^2 \rangle / \frac{d \langle \vec{v} \rangle}{dy}$$

absent

$\begin{cases} - \text{drag} \\ - \text{damping} \\ - \text{mixing (3rd order)} \end{cases} \Rightarrow \begin{cases} \text{Flow forced} \\ \text{to wave} \\ \text{momentum} \end{cases}$

non-acceleration thm.

$\begin{cases} \text{density} \\ \text{(Charney} \rightarrow \text{Deacon)} \\ \text{Thm.} \end{cases}$

ZF's  $\nleftrightarrow$  Wave Momentum Deficit.

Cannot accelerate (or maintain vs drag)  
 zonal flow without changing (dissipating)  
 wave intensity.



Note:  $\frac{-k_x \langle \tilde{z}^2 \rangle}{2 k_x d \langle q \rangle / dy}$

$$\tilde{z} = D\phi + \beta y$$

absent mean flow,

$$\frac{d\tilde{z}}{dy} = \beta !$$

$$\langle \frac{\tilde{z}^2}{2} \rangle = k^2 \varepsilon$$

$$\frac{-k_x k^2 \varepsilon}{2 k_x \frac{d \langle q \rangle}{dy}} = \frac{k_x \varepsilon}{-k_x D} = \frac{k_x \varepsilon}{\omega_n}$$

$\rightarrow$  Action Density

$$= k_x N_n$$

$$= \rho_w \quad \text{i.e. adiabatic invariant}$$

$\Rightarrow$  wave momentum density

$$\partial_t \{ \langle v_x \rangle - p_w \} = -\mu \langle v_x \rangle$$

$$- \sigma \frac{d \langle \tilde{z}^2 \rangle}{d \langle q \rangle / dy} + \dots$$

$$\frac{\mu}{\ell} \sim \frac{v(\ell)}{\ell} \sim e^{\frac{1}{\ell}} \frac{\ell^{-3/2}}{\ell}$$

$$\ell^{2/3} \sim \frac{1}{e^{4/3}} \frac{\epsilon^{1/3}}{\mu}$$

$$\underline{\ell} \sim \frac{\epsilon^{1/2}}{\mu^{3/2}} \ll L$$