Now, $D(\underline{k}, \omega) = 0$

$$\frac{\partial D}{\partial \underline{k}} \cdot d\underline{k} + \frac{\partial D}{\partial \omega} d\omega = 0$$

$$\Rightarrow \frac{d\omega}{d\underline{k}} = -\frac{\partial D/\partial \underline{k}}{\partial D/\partial \omega}$$

Now, $D = 1 + \frac{2\pi G \Sigma_0 |k|}{\overline{\omega}^2 - \kappa^2 - k^2 c_s^2}$

$$-\partial D/\partial k_r = \frac{-2\pi G \Sigma_0 |k| (2 k c_s^2)}{(\overline{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}$$

$$\partial D/\partial \omega = \frac{-2\overline{\omega}(2\pi G \Sigma_0 |k|)}{(\overline{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}$$

$\therefore$

$$v_{gr,r} = -\mathrm{sgn}(k)\,(\pi G \Sigma_0 - |k_r| c_s^2)/(\omega - m\Omega)$$

so

$$v_{gr} \sim (-\text{sgn}\, k)\,(\pi G \Sigma_0 - |k_r| c_s^2)\,(\omega - n\Omega)^{-1}$$

$(-\text{sgn}\, k)$	$(\pi G \Sigma_0 - \lvert k_r \rvert c_s^2)$	$(\omega - n\Omega)^{-1}$	
⊖ trailing	⊕ long wave	$r > r_{co}$	⊕
⊕ leading	⊖ short wave	$r < r_{co}$	⊖

$$k \lessgtr \frac{\pi G \Sigma_0}{c_s^2}$$

Thus:

<u>long trailing spiral</u>

→ propagates inward for $r > r_{co}$

→ propagates outward for $r < r_{co}$

long trailing spiral always propagates toward co-rotation

<u>short trailing spiral</u>

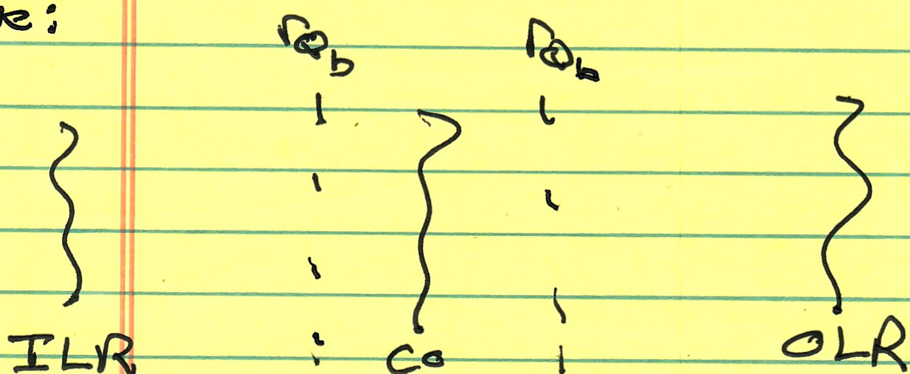
→ propagates outward for $r > r_{co}$

→ propagates inward for $r < r_{co}$

⇒ short trailing spiral always propagates away from co-rotation.

This brings us to Amplifying Spirals:
(WASER) — over-refraction

have:



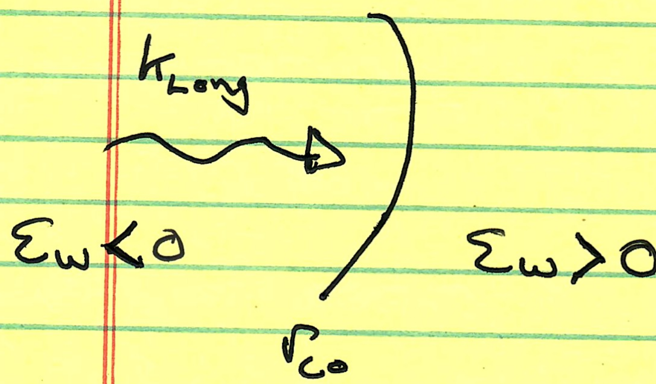
$r < r_{co}$ $\quad\quad$ $r > r_{co}$

$\Sigma\omega < 0$ $\quad\quad$ $\Sigma\omega > 0$

Long ⇒ always propagates toward r_{co}

Short ⇒ always propagates away from r_{co}.

→ "Over-Refraction"

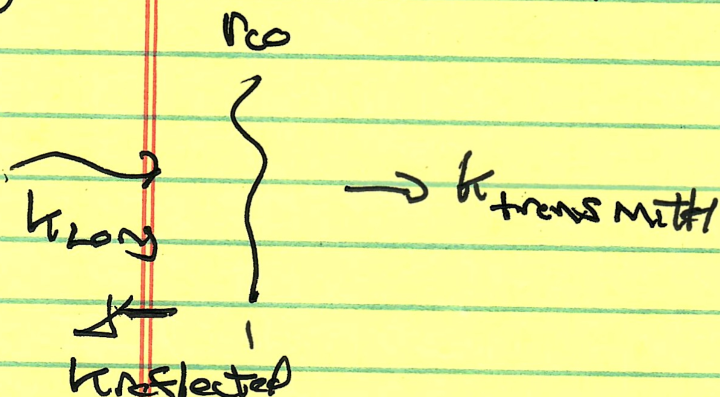


$Q \cong 1$

→ ignore Q barrier (simplicity)

Long to co.

so, ala' freshman physics:



→ so clearly reflected / transmitted wave must

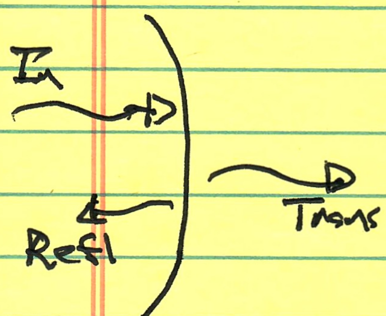
be short, as propagate away from co-rotation

→ Long → short conversion at co-rotation

→ Now must conserve wave energy density flux accross co-rotation:

$$V_{gr}^{L}\,\Sigma^{L}_{IN} = V_{gr}^{S}\,\Sigma^{S}_{Refl} + V_{gr}^{S}\,\Sigma^{S}_{Trans}$$

$r < r_{co}$ $\qquad$ $r > r_{co}$



Now:

- if $\Sigma_w > 0$ (as in freshmen physics....), $\Sigma^{S}_{Refl} < \Sigma^{L}_{IN}$ necessarily, as energy lost to transmitted wave

but → the point!:

→ if $\Sigma_w < 0$ → negative energy incoming wave!

$\Rightarrow$

$$V_{gr}^{L}\,\mathcal{E}_{IN}^{L} = V_{gr}^{S}\,\mathcal{E}_{Refl}^{S} + V_{gr}^{S}\,\mathcal{E}_{Trans}$$

$$\mathcal{E}_{refl}^{S} = \left(\frac{V_{gr}^{L}}{V_{gr}^{S}}\right)\overset{<0}{\mathcal{E}_{IN}^{L}} - \mathcal{E}_{Trans}^{S}$$

$\ominus$ (as $\mathcal{E}_{IN}$ is NEW) $\qquad \oplus$

so

$$\mathcal{E}_{refl}^{S} < 0$$

then for $V_{gr}^{L} \sim V_{gr}^{S}$,

$$\boxed{|\mathcal{E}_{refl}^{S}| > |\mathcal{E}_{IN}^{L}|}$$

i.e. Magnitude of wave energy must increase!

i.e. $\mathcal{E}_\omega = \left(\frac{\partial D}{\partial \omega}\right) \omega |A|^2$

$|A|$ must increase!

→ Life is more interesting than in freshman physics!

→ reflected signal amplified!

→ negative energy wave crucial!

→ N.B.:

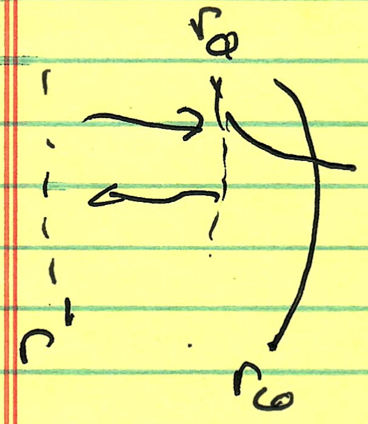
- Energy source is differential rotation

- Not exponential growth, as in linear instability

Now what of Q barrier?

- if tunneling, then weak amplification at each scattering

- but if, can reflect from $r < r_{co}$, can build up standing wave → WASER oscillator



$r_{ILR} < r' < r_{co}$

&

- with Q barrier:

$$\Sigma_0 \qquad \Sigma_R \quad \Sigma_T$$

before

$$-1 = -2 + 1$$

now:

$$-1 = -(1+G) + \epsilon$$

tunneling reduces [over reflected]

G = tunneling factor

$$\epsilon \sim \exp\left[-2\int_{Q_-}^{r_{co}} dr\, |k_r|\right]$$

→ Need find an inner reflection point:

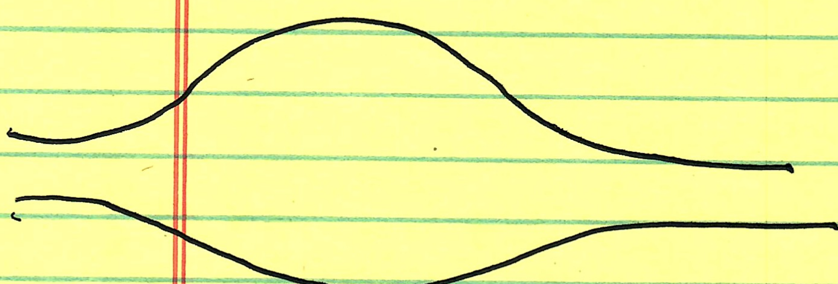
– recall:

$$v_{gr} = -\mathrm{sgn}\,k\,(\pi G \Sigma_0 - k_r c_s^2)/\omega$$

$$k_r > \frac{\pi G \Sigma_0}{c_s^2} \equiv k_T \quad \rightarrow \text{short wave}$$

acoustic term dominates for short wave.

so if bulge:



$\left|\frac{d\Sigma}{dr}\right|$ rapidly increases toward center

$v_g \rightarrow 0$ when $\pi G \Sigma(r) = k_r c_s^2$

→ defines r_{IRP}

inner reflection point

where $r_{ILR} < r_{IRP} < r_{Q_-}$

at r_{IRP}, $\xrightarrow{\sim}$ inward

propagating short wave converts to

outward propagating long wave.

∴ restarts cycle!

so

WASER feedback loop ⟺ standing spiral amplification mechanism.

i.) long wave incident on

$r_{Q_-} < r < r_{co}$ from $r < r_{co}$

wave is NEW

ii.) over-reflected short wave

back-scattered to $r < r_{co}$, propagating

away from r_{co}, amplified $|A|^2/|A_0|^2 \sim 1+\epsilon$

(iii.) back scattered short wave comes to r_{IRP}, due bulge. Short wave backscattered (no transmission!) as long wave propagating toward r_{co}.

(iv.) return to (i.)

⇒ WASER ⟺ [Negative energy wave is key element.]

N.B.

- much not understood despite long story

- amplification, NL saturation ⇒ Σ profile dynamic heating .....

Next: – Lindblad resonances and spiral interaction there.

→ Goldreich + papers

– Angular momentum exchange at Lindblad resonances

→ Spiral kinetics: Lynden-Bell and Kalnajs.

All good things must come to an end ....

→ Look to proto-planetary disks as venue for future work on spirals.

1) **Basic Scales, Ideas**
   Viscous flow in Cylindrical Geometry
   Viscous Stress

2) **Viscous Stress Model of Accretion in Thin Disk**
   a) Disk structure
   b) Velocity and time scales
   c) Surface density and angular momentum transport
   d) $\dot{M}$, ν, Σ relations, alpha model
   e) Viscous Heating and Luminosity

3) **Dynamics of Accretion**
   a) Relaxation
      - Fixed mass → solid body rotation
      - Accretion → mass to center + 1 particle at large radius end state
   b) Rayleigh
      - 2 particles
      - Interchange at constant angular momentum, each
      - Radial buoyancy
      - Rayleigh Discriminant/Epicyclic Frequency
   c) L-B + P
      - 2 particles → conserve **sum** of angular momentum, mass
      - $\Delta E < 0$ → accretion + angular momentum transport outward → end state
   d) "Donkeys", ala' Lynden-Bell
   e) Elastically coupled particles → to MRI

4) **Crash Course in MHD**
   a) Equations, especially induction
   b) Alfven Theorem, Freezing-in
   c) Stresses, Energetics
   d) Waves, especially Shear Alfven
   e) Magnetic Braking, Torsional Alfven Wave
   f) Virial Theorem for MHD
   g) Partially ionized MHD + Ambipolar Diffusion
   h) Energy Principle (Introduction)

5) **Magnetorotational Instability, Turbulence, Alpha, Saturation**
   a) Basic Physics of MRI, Toy Model of Basic MRI, Connection to Accretion
   b) Detailed Linear Theory MRI — resistivity, partial ionization, viscosity
   c) MRI-Induced Transport:
      - Intro to Mixing Length Theory, Application to MRI

- Alpha scalings and underlying physics issues

d) Open Issues in MRI Turbulence
e) Magnetic Buoyancy in Disks, Parker Instability
f) Magnetized Disk Coronae and Layered Disks, Nanoflares

6) Planetesimal Formation

a) Drag Mechanisms, Epstein-Stokes drag, dust particulate evolution
b) Sedimentation in Turbulence — Fokker-Planck → Subdisk thickness
c) Goldreich-Ward Model, including Toomre stability criterion for subdisk
d) Radial infall
e) Outlook on planetesimal formation

7) Basics of Galactic Dynamics

a) Scales, Phenomena → Collisionless Stellar Dynamics
b) Vlasov-Poisson system
c) Physics of Vlasov Equation
d) Jean Theorems — Weak and Strong
e) Stationary Solutions, mostly spheroidal
f) Jeans Equations, Virial Theorem
g) Landau Damping
h) Violent Relaxation and its foundations in phase mixing

8) Spiral Density Waves

a) Motivation, Lin-Shu Hypothesis
b) Basic Analysis
c) Resonances: Lindblad and Co-Rotation
d) Wave Action and Energy-Momentum Theorems
e) Amplification Mechanisms
f) Spirals and Angular Momentum Transport — return of L-B's "2 particles"
g) Spirals in Vlasov Theory
h) Outlook on spiral structure