Physics 239 — Lecture 20

Spiral Density Waves III

Recall:

→ Spiral structure — Density wave

→ WKB:

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G|k_r|\Sigma_0 + k_r^2 c_s^2$$

~ quasi-marginal waves, $Q \sim 1^+$ — $\left.\begin{matrix} \text{Massive} \\ \text{disks} \end{matrix}\right\}$

$$\frac{|k|}{k_T} = \frac{2}{Q^2}\left[1 \pm \left(1 - Q^2(1-\nu^2)\right)^{1/2}\right]$$

$Q = \kappa c_s / 2\pi G\Sigma_0$

$k_T = \kappa^2 / 2\pi G\Sigma_0$

$\nu^2 = \bar{\omega}/\kappa^2$

$\left\{\begin{matrix} \nu = 1 \\ \omega = m\Omega \pm \kappa \\ \text{Lindblad resonances} \end{matrix}\right.$

↓

excitation by external torque
- galaxy
- planet
- moon.

$\Rightarrow$ $\phi(r) = m\theta + g(r,t)$

$d\phi = m\,d\theta + \frac{\partial g}{\partial r} dr$

$d\phi = 0$

$\frac{d\theta}{dr} = -\frac{\partial g/\partial r}{m} = -\frac{k_r}{m}$

$\left.\begin{array}{l} \frac{d\theta}{dr} < 0 \Rightarrow \text{trailing} \\ \quad\;\; > 0 \Rightarrow \text{leading} \end{array}\right\}$ spiral

$\Rightarrow$ Gravitational Couple of spiral wave (in $\to$ out)

$C(r) = \int ds\; r|\hat{\phi}_0|^2 g'(r)\, m > 0$, $\left\{ \begin{array}{l} \text{to have} \\ \text{inner disk} \\ \text{spin-up} \\ \text{outer disk} \end{array} \right.$

$g'(r) > 0 \Rightarrow$ trailing spiral

n.b. $\Leftrightarrow$ sign Reynolds stress

$\Rightarrow$ Wave Energy and Momentum..!

→ Dynamics of Wave Growth and Maintenance.

Essence of problem:

- spirals are waves → stable ($Q = 1+$)
  so how excited, propagate, amplify.

- How do spirals transport angular momentum, aid in relaxing galaxy

⇒ Wave Kinetics for spirals ?!

Now:

ⓐ
- wave energy and angular momentum

ⓑ
- negative energy spiral wave → Physics!

[N.B.: Negative energy waves easily amplified by:
  - dissipation
  - coupling to positive energy wave.

[Negative energy waves almost as much fun as instabilities...]

(c) - Eikonal theory and wave kinetics for spirals $\Rightarrow$ Wave Action evolution

For wave energy, angular momentum:

- consider adiabatic switch-on of external perturbation, from $t = -\infty$

$$\frac{dE_W}{dt} = \int_0^{2\pi} d\phi \int_0^{\infty} r\,dr \, \tilde{\sigma} \frac{\partial \tilde{\phi}_{ext}}{\partial t}$$

$$\frac{dE}{dt} = \frac{\partial \tilde{\phi}_{ext}}{\partial t}$$

$$E_{wave} = \int_{-\infty}^{t} dt' \int_0^{2\pi} d\phi \int_0^{\infty} r\,dr \, \tilde{\sigma} \frac{\partial \tilde{\phi}}{\partial t}$$

likewise $\qquad r \frac{1}{r}\frac{\partial}{\partial \phi}$

$$L_{\theta\,wave} = \int_{-\infty}^{t} dt' \int_0^{\infty} r\,dr \int_0^{2\pi} d\phi \left(-\tilde{\sigma} \frac{\partial \tilde{\phi}_{ext}}{\partial \phi} \right)$$

Now, need relate external perturbation to self-consistent field.

in plasma: (recall basic EM)

$$\nabla^2\phi = -4\pi\Big(\underbrace{\tilde{\rho}_{ind}}_{\text{polarization charge.}} + \tilde{\rho}_{ext}\Big)$$

so

$$\underbrace{\epsilon(k,\omega)}_{\text{dielectric}}\,\phi_{k,\omega} = \tilde{\rho}^{\,ext}_{k,\omega}\,\frac{4\pi}{k^2}$$

$$\underbrace{\epsilon}_{\substack{\text{dielectric}\\ \text{i.e. dynamic}\\ \text{collective response}}} = 1 + 4\pi\,\underbrace{\chi(k,\omega)}_{\substack{\text{susceptibility}\\ \text{relates } \rho,\ \phi}}$$

For spirals (and self-gravitating matter, in general), do the same!

$$\nabla^2\phi = 4\pi G\Sigma\,\delta(z)$$

$$\nabla^2\phi - 4\pi G\Sigma_0 \frac{\tilde{\Sigma}^{ind}}{\Sigma_0}\delta(z) = 4\pi G\Sigma_0 \frac{\tilde{\Sigma}_{ext}}{\Sigma_0}\delta(z)$$

$$\int_{-\epsilon}^{+\epsilon} dz \Rightarrow$$

$$\underbrace{-2|k|\tilde{\phi}_k}_{2D} - 4\pi G\Sigma_0\left(\frac{\tilde{\Sigma}_k}{\Sigma_0}\right) = 4\pi G\Sigma_0\frac{\tilde{\Sigma}_{ext}}{\Sigma_0}$$

Now,

$$\frac{\tilde{\Sigma}_{k,\omega}}{\Sigma_0} = \underbrace{\chi_{grav}(k,\omega)}_{\text{\underline{gravitational} susceptibility}}\tilde{\phi}_{k,\omega}$$

so

$$\boxed{\left[|k| + 2\pi G\Sigma_0\hat{\chi}(k,\omega)\right]\tilde{\phi}_{k_0} = -2\pi G\Sigma_0\frac{\tilde{\Sigma}_{ext}}{\Sigma_0}}$$

$$\boxed{\underbrace{D(k,\omega)}_{\epsilon^{eff}_{k,\omega}} = 1 + \frac{2\pi G\Sigma_0}{|k|}\hat{\chi}(k,\omega) \qquad \hat{\chi} = \delta(\tilde{\Sigma}/\Sigma_0)/\delta\phi}$$

$\rightsquigarrow$ effective dielectric (gravitational screening)

For $D(k,\omega)$ relate $\tilde{\sigma}/\sigma_0$ to $\tilde{\phi}$

have, in tight winding:

$$-i\bar{\omega}\tilde{v}_r - 2\Omega\tilde{v}_\theta = -ik_r\left(c_s^2\,\tilde{\sigma}/\sigma_0 + \tilde{\phi}\right)$$

$$-i\bar{\omega}\tilde{v}_\theta + \frac{\tilde{v}_r}{r}\frac{\partial}{\partial r}(r^2\Omega) = 0$$

$$-i\bar{\omega}\,\tilde{\sigma}_{k,\omega} + ik_r\sigma_0\tilde{v}_r = 0$$

so

$$\frac{\tilde{\sigma}}{\sigma_0} = k_r^2\tilde{\phi}\Big/\bar{\omega}^2 - \kappa^2 - k_r^2c_s^2$$

$\Rightarrow$

$$\boxed{\begin{gathered}\chi(k,\omega) = k_r^2\big/\left(\bar{\omega}^2 - \kappa^2 - k_r^2c_s^2\right) \\[2ex] D(k,\omega) = 1 + \frac{2\pi G\sigma_0|k_r|}{\bar{\omega}^2 - \kappa^2 - k_r^2c_s^2}\end{gathered}}$$

Check: $D(k,\omega) = 0$

$$\bar{\omega}^2 = \kappa^2 + k_r^2 c_s^2 - 2\pi G \Sigma_0 |k_r|$$

recover Lin-Shu ✓

Exercise: Extend to kinetics, using known reduction factor.

Observation: Dissipation ? → What happens ?

n.b. need for $\int_{-\infty}^{t} dt$ to converge.

$$\epsilon(k,\omega) = \epsilon_r(k,\omega) + i\,\underbrace{\epsilon_{Im}(k,\omega)}_{\text{small}}$$

or $D(k,\omega)$

$\epsilon(k,\omega) = 0 \Rightarrow$

$$\epsilon_r(k,\omega) + \underbrace{(\omega - \omega_k)}_{i\gamma_k} \left.\frac{d\epsilon_r}{d\omega}\right|_{\omega_k} = -i\,\epsilon_{Im}(k,\omega_k)$$

so $\epsilon_r = 0$

$$\gamma_k = -\frac{\epsilon_{IM}}{\partial\epsilon/\partial\omega|_{\omega_r}}$$

$\partial\epsilon/\partial\omega > 0 \Rightarrow \epsilon_{IM}$ $> 0 \rightarrow$ damping (usual)
$< 0 \rightarrow$ growth

$\partial\epsilon/\partial\omega < 0 \Rightarrow \epsilon_{IM} > 0 \Rightarrow$ growth!
$\epsilon_{IM} < 0 \rightarrow$ damping

flip
$\Downarrow$
negative energy wave.

$\Rightarrow$ suggests that $\partial D/\partial\omega$ is crucial to spiral wave dynamics.

Sign $\rightarrow$ $\oplus$ / $\ominus$ energy wave.

so back to wave energy....

$$\frac{dE_w}{dt} = \int_0^{2\pi} d\phi \int_0^{\infty} r\,dr\; \tilde{\sigma}\, \frac{\partial\tilde{\phi}_{ext}}{\partial t}$$

but $\quad D(k,\omega)\hat{\phi}_{k,\omega} = \hat{\phi}^{ext}_{k,\omega}$

Near or at collective resonance (i.e. wave):

for now

$$D(k,\omega) = \cancel{D_r(k,\omega_k)} + (\omega-\omega_k)\left.\frac{\partial D}{\partial \omega}\right|_{\omega_k} + i\,\hat{D}_{I(\,)}$$

so

$$\underbrace{(\omega-\omega_k)}_{i\gamma_k}\left.\frac{\partial D}{\partial \omega}\right|_{\omega_k}\hat{\phi}_{k,\omega} = \hat{\phi}^{ext}_{k,\omega}$$

$$\frac{\partial\hat{\phi}^{ext}}{\partial t} = -i\omega_k\hat{\phi}_{ext} = \gamma_k\,\omega_k\left.\frac{\partial D}{\partial \omega}\right|_{\omega_k}\tilde{\phi}_{k,\omega}$$

Now, from Poisson eqn.:

$$\tilde{\sigma}_{k,\omega} = -\frac{|k|}{2\pi G}\tilde{\phi}_{k,\omega}\exp\left[i(\underline{k}\cdot\underline{x}-\omega t)e^{\gamma_k t}\right]$$

$$\frac{d\mathcal{E}_\omega}{dt} = \frac{\mathrm{re}}{2}\left(\sigma^* \frac{\partial\phi}{\partial t}\right)$$

$$\mathcal{E}_\omega = \int_{-\infty}^{t} dt\, e^{2\gamma_k t}\left(\frac{-|k|}{4\pi G}\right)\omega_k \frac{2\gamma_k}{2}\left.\frac{\partial D}{\partial \omega}\right|_{\omega_k}|\tilde{\phi}_{k,\omega}|^2$$

$$= \int_{-\infty}^{t} dt' (2\gamma_k) e^{2\gamma_k t}\left(\frac{-|k|}{8\pi G}\right)\omega_k \left.\frac{\partial D}{\partial \omega}\right|_{\omega_k}|\tilde{\phi}_{k,\omega}|^2$$

$\Rightarrow$

$$\boxed{\mathcal{E}_\omega = \left(\omega_k \left.\frac{\partial D}{\partial \omega}\right|_{\omega_k}\right)\left(\frac{-1}{8\pi G}\, k^2 |\tilde{\phi}_k|^2\right)}$$

$\downarrow$
wave energy density

Obvious counterpart of:

$$\boxed{\mathcal{E}_{\omega_k} = \omega_k \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}}$$

i.e. field energy + media energy.

As $\frac{\partial D}{\partial \bar{\omega}} = \frac{-2\pi G \sigma_0 |k| \bar{\omega}}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}$

$$\boxed{\mathcal{E}_w = \frac{\omega_R \bar{\omega}_R}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2} \sigma_0 \left(\frac{|k \tilde{\phi}_a|^2}{2}\right)}$$

$\Rightarrow$ wave energy density

Obvious: - $\bar{\omega}_R < 0$ for $r < r_{co}$

$\mathcal{E}_w < 0$ $\Rightarrow$ spiral is negative energy inside co-rotation

- $\bar{\omega}_R > 0$ for $r > r_{co}$

$\mathcal{E}_w > 0$

Similar procedure $\Rightarrow$

$$\boxed{\mathcal{L}_\theta = \frac{m\bar{\omega}\,\sigma_0}{(\omega^2 - \kappa^2 - k^2 c_s^2)^2}\left(\frac{k^2|\hat{\phi}_k|^2}{2}\right)}$$

$\downarrow$

wave angular momentum density

Note:

- $\mathcal{E}_{\text{wave}} = (\omega/m)\,\mathcal{L}_{\text{wave}}$

so $\quad \mathcal{E} = \omega N \qquad \rightarrow \quad P_\theta = k_\theta N$

semi-classical $\quad \underline{P} = \underline{k} N \qquad\qquad \mathcal{L}_\theta = m N$

and $\quad \mathcal{E}_w / \mathcal{L}_w = \omega/m$

$$\left(\frac{\mathcal{E}_w}{\mathcal{L}_w}\right) > 0 \quad \text{for} \quad \frac{\omega}{m} > \Omega \qquad \rightarrow \quad r > r_{co}$$

$$\left(\frac{\mathcal{E}_w}{\mathcal{L}_w}\right) < 0 \quad \text{for} \quad \frac{\omega}{m} < \Omega \qquad \rightarrow \quad r < r_{co}$$

$$N = \frac{\bar{\omega}_k}{(\bar{\omega}^2 - \kappa^2 - k^2 c_s^2)^2} \, \sigma_0 \left(\frac{k^2 |\tilde{\phi}_k|^2}{2} \right)$$

Wave Action Density $\Rightarrow$ $N = \mathcal{E}/\omega$

(Adiabatic invariant) $\quad \frac{dN}{dt} = 0.$

$\Rightarrow$ wave kinetics ...

What does this mean?

- Wave Energy relative to certain frame $\Rightarrow$ not Galilean invariant

- medium active (differential rotation) $\Rightarrow$ supports energy, possibly accessible.

ie Plasma $\quad \epsilon = 1 - \frac{\omega_p^2}{\omega^2} \quad +$

$\rightarrow$ (+) energy waves

Beam-Plasma $\quad \epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kV_0)}$ $\quad$ beam velocity

$\rightarrow$ can be (−) energy.

- Positive energy wave excited by adding energy to wave
- Negative energy wave excited by extracting energy

→ destabilized by damping.

Thus any damping/dissipation of a spiral wave would dissipate:

→ a ⊕ ammount of $\begin{pmatrix} \mathcal{E} \\ \mathcal{L}_{\Theta} \end{pmatrix}$ for $r > r_{co}$

→ a ⊖ ammount of $\begin{pmatrix} \mathcal{E} \\ \mathcal{L}_{\Theta} \end{pmatrix}$ for $r < r_{co}$

but:

→ ⊕ wave $\mathcal{L}_{\Theta}$ dissipated → positive wave angular momentum transferred to medium → medium angular momentum increases.

→ $\ominus$ wave $\underset{\wedge}{\int_0}$ dissipated ⇒ negative wave angular momentum transferred to medium ⇒ medium angular momentum decreases.

⇒ Spirals can couple angular momentum from smaller radius to larger radius.

⇒ good for accretion / lowering energy.

But: Need deal with Q barrier.

TBC.

Summary:

	Wave	dissipation	
$r > r_{co}$	$\mathcal{E}_w > 0$ $\mathcal{L}_{0w} > 0$	increases angular momentum of disk	boosts angular momentum $r > r_{co}$
$r < r_{co}$	$\mathcal{E}_w < 0$ $\mathcal{L}_0 < 0$	reduces angular momentum of disk matter	depletes angular momentum $r < r_{co}$

$\Rightarrow$ Spirals transfer angular momentum from $r < r_{co}$ to $r > r_{co}$.

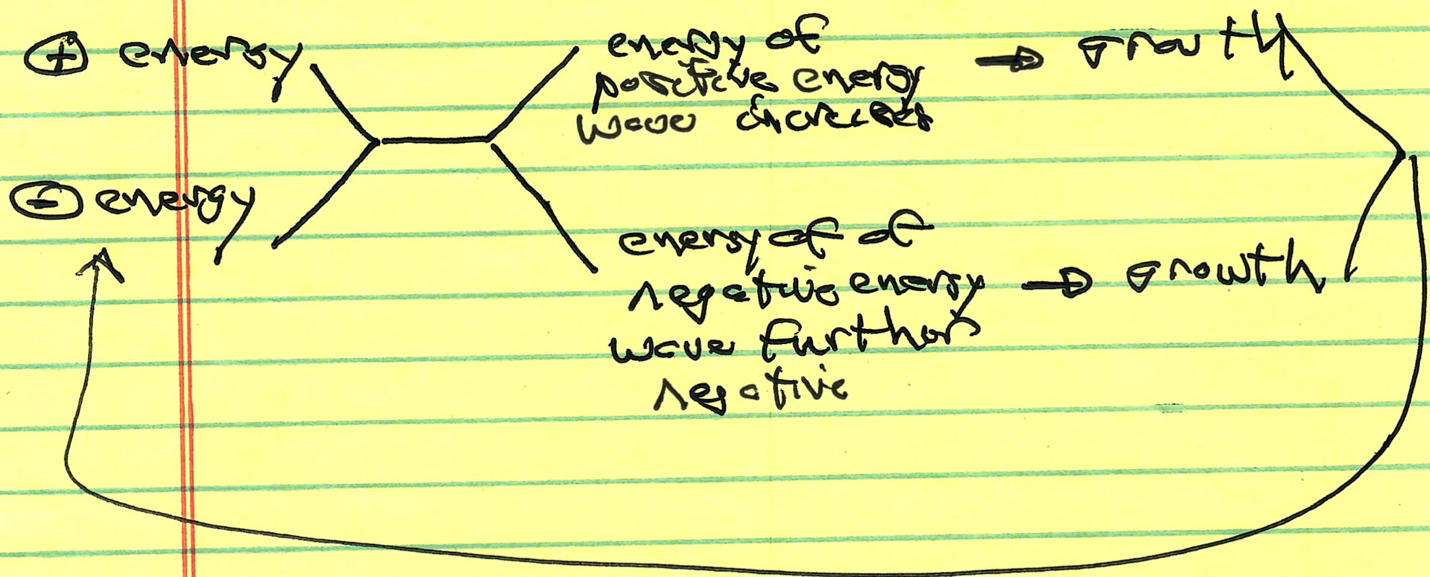
$\Rightarrow$ Spirals – negative energy waves – promote accretion!

(b)

$\Rightarrow$ Now, explore spiral wave amplification $\Leftrightarrow$ how?

$\Rightarrow$ interaction of positive and negative energy waves.

i.e. Feedback loop:



Can interaction be arranged ?

→ Look at propagation !

Convenient to look at via eikonal equations:

$$\frac{d\underline{x}}{dt} = \vec{v}_{gr} = \frac{d\omega}{d\underline{k}}$$

$$\frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial\underline{x}}$$