

Physics 239Lecture 19Spiral Density Waves II.

Recall:

→ spiral structure } $\Rightarrow$ density wave $\Leftrightarrow$ i.e. visible spirals are @ standing density waves in differentially rotating disks. $\Leftrightarrow$ ubiquitous→ In WKB, $k_r > k_\theta$ (tight winding limit), density waves satisfy:

Lin-Shu Disp. Relation

$$\boxed{(\omega - m\Omega)^2 = k^2 - 2\pi G |k_r| \Sigma_0 + k_r^2 c_s^2}$$

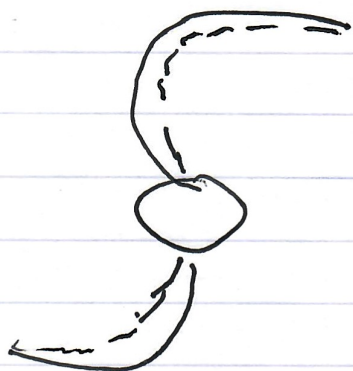
$\underbrace{\hspace{10em}}_{\Phi}$

Shu Vol 2 - 11~ akin Toomre, with $\omega \rightarrow \omega - m\Omega = \bar{\omega}$.

~ quasi-marginal waves

~ much more complex than Toomre

Why buy all this?



M57
radio observations

- synchrotron emission maximal inside optical arms.
- synch $\leftrightarrow$ B → magnetic compression
[in phase with density]
- suggests $\nabla \rightarrow$ shocking gas, displaced slightly from maximal compression

$$\Rightarrow \boxed{\frac{|k|}{k_T} = \frac{2}{Q^2} \left[1 \pm (4 - Q^2(1 - v^2))^{1/2} \right]}$$

$$Q^2 = \kappa c_s / \pi G \Sigma_0$$

$$k_T = K^2 / \lambda \pi G \Sigma_0$$

$$v^2 = \bar{\omega}^2 / K^2$$

Troune

$Q > 1 \rightarrow \underline{\text{stable}}$

Sci

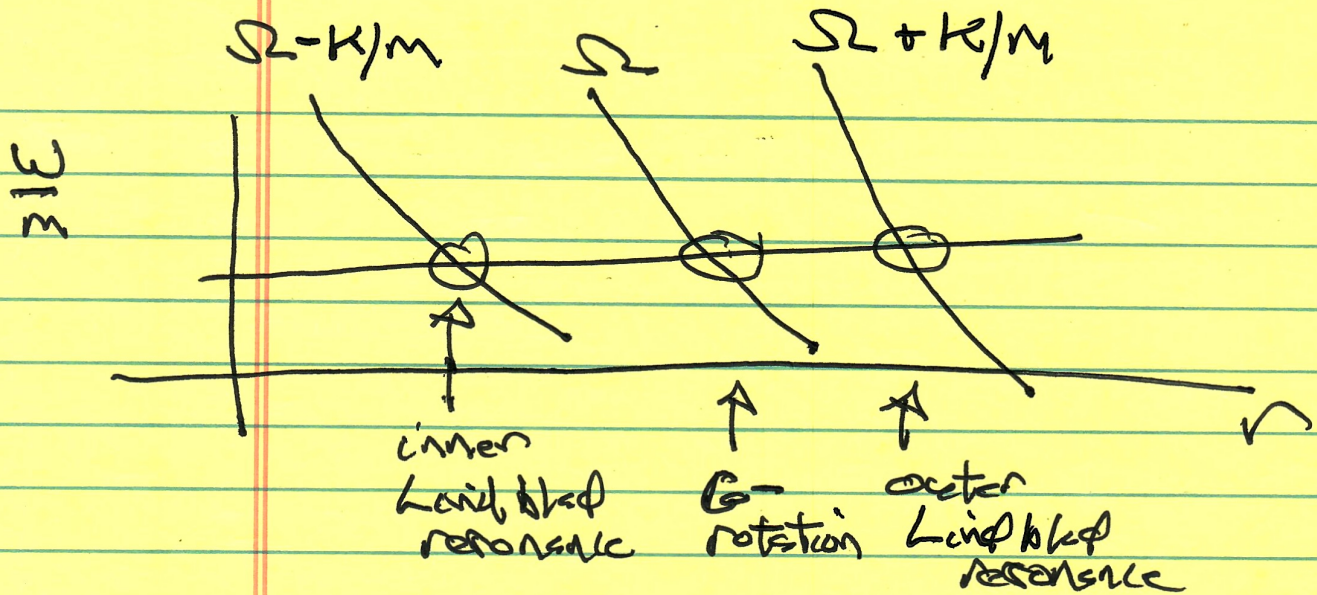
$$- v = 1 \text{ for } \omega = m\Omega \pm K$$

Lindblad resonances

$\Rightarrow$ star/gas sees DC field

$\leftrightarrow$ sites of emission/absorption of spirals (kinetics relevant).

$$- v = 0 \rightarrow \text{co-rotation.}$$



$-k \rightarrow + \rightarrow$ short wave (sensitive to stellar vs gal)
 $- \rightarrow$ long wave (more robust)

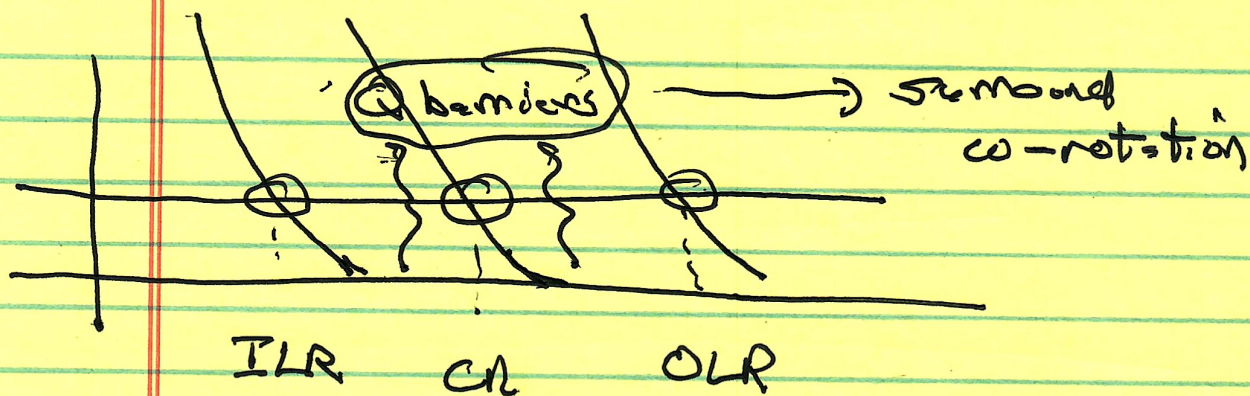
$\rightarrow \quad \nu_{ILR} < \nu < \nu_{OLR} \quad \rightarrow$ Principal Range.

$\rightarrow$ Long wave propagates in Principal Range.

$k_L \rightarrow 0$ at $\begin{cases} ILR \\ OLR \end{cases}$

$\rightarrow$ Q barriers, i.e. for

$Q \sim 1 + \epsilon \Rightarrow$ barriers to propagation at $\nu^2 < \epsilon$



N.B.: $Q \gg 1 \Rightarrow \sqrt{Q} \rightarrow \sqrt{L_R}$
 $\Rightarrow$ no spirals.

de. if Σ_0 low, s.t. $Q \gg 1 \Rightarrow$
 no spiral propagation.

Now:

① - Trailing vs. Leading Spirals

② - Key Pt: Trailing spirals only
 exert gravitational torque
 from inner disk on
 outer disk.

$\Rightarrow$ Transport angular momentum ~~inward~~
 outward.

③

— Wave Energy, Angular Momentum,
Action

④

— Energy Theorems.

Useful: — Shu Vol. 2 → Chapt. 12

— Whitham

— Posted Papers:

Lynden-Bell & Kalnajs
Dewar

— Landau & Lifshitz: Electrodynamics of
Continuous Media
(Wave Energy). (Energy Theor in Media)

— J.D. Murray: Mathematical Biology
(Sparks in chemical
waves)

→ spirals

$$\psi \sim e^{i\phi(r)}$$

phase

i.e. $\psi \sim A e^{i\phi}$

$$i\phi(r) = i(m\theta + g(r, t))$$

$$= i(m\theta + g(r) - \omega t)$$

echoed
of $\hbar\omega$

$$k_r = \frac{\partial g}{\partial r}$$

$$\psi = A e^{i(m\theta + g(r, t))}$$

$$\omega = -\frac{\partial g}{\partial t}$$

$$d\phi = m d\theta + \frac{\partial g}{\partial r} dr$$

$$d\phi = 0 \Rightarrow \text{constant phase}$$

$$\Rightarrow \frac{d\theta}{dr} = -\frac{\partial g / \partial r}{m} = -\frac{k_r}{m}$$

$$\frac{d\theta}{dr} < 0 \rightarrow \text{trailing spiral}$$

$$\Rightarrow k_r / m > 0$$

$d\phi/dr > 0 \rightarrow$ leading spiral (?)

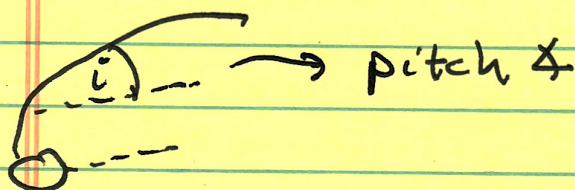


$$\frac{d\phi}{dr} > 0$$

$$\frac{v_{\phi}}{v_r} < 0$$

N.B. Only trailing spirals ~~exert a torque~~ exert a torque on outer part of galaxy/disk.

For pitch spirals:



$$\cot i = r \frac{\Delta\phi}{\Delta r} = r \frac{d\phi}{dr}$$

if $\phi = \phi_0 + \Omega(r)t$

$\Rightarrow$

$\cot i = r \Omega' t \Rightarrow$ wind-up i.e. pitch increases with time.

→ Disk Relaxation

- have shown disk will relax, i.e. lower energy, if inner part of disk spins up outer part

- i.e. L-B "2 particle" argument

↳ Lowering gravitational energy requires angular momentum outflow

how? - thin disk → turbulence eddys → viscous transport

- galaxy → spiral waves → wave transport
(flex wave energy, L_0)

→ (distinction better put as $Q \gg 1$ vs

$Q \gtrsim 1$)

- trailing spiral has correct sense to spin up.

To see:

$$\underline{f_g} = \underline{\rho g}$$

$$= -\underline{\rho \nabla \phi}$$

$$\nabla^2 \phi = 4\pi G \rho$$

$$\underline{f_g} = -\frac{\underline{\nabla^2 \tilde{\phi}}}{4\pi G} \underline{\nabla \tilde{\phi}}$$

wave $\tilde{\phi}$

$$\langle \underline{f_g} \rangle = -\langle \underline{\nabla^2 \tilde{\phi}} \underline{\nabla \tilde{\phi}} \rangle$$

Mean

(2D)

$$= -\left\langle \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} \right) \left(\frac{\partial \tilde{\phi}}{\partial x} \hat{x} + \frac{\partial \tilde{\phi}}{\partial y} \hat{y} \right) \right\rangle$$

$$= -\left\langle \frac{\partial}{\partial x} \left[\frac{1}{2} \left(\frac{\partial \tilde{\phi}}{\partial x} \right)^2 \right] + \frac{\partial}{\partial y} \left[\left(\frac{\partial \tilde{\phi}}{\partial y} \right) \left(\frac{\partial \tilde{\phi}}{\partial x} \right) - \left(\frac{\partial \tilde{\phi}}{\partial x} \right) \left(\frac{\partial \tilde{\phi}}{\partial y} \right) \right] \right\rangle \hat{x}$$

$$- \left\langle \left[\frac{\partial}{\partial x} \left(\frac{\partial \tilde{\phi}}{\partial x} \frac{\partial \tilde{\phi}}{\partial y} \right) - \left(\frac{\partial \tilde{\phi}}{\partial x} \frac{\partial^2 \tilde{\phi}}{\partial x \partial y} \right) \right] + \frac{\partial}{\partial y} \frac{1}{2} \left(\frac{\partial \tilde{\phi}}{\partial y} \right)^2 \right\rangle \hat{y}$$

and re-combine to:

$$= -(\partial_x, \partial_y) \begin{bmatrix} \tilde{\phi}_x \tilde{\phi}_x, & \tilde{\phi}_x \tilde{\phi}_y \\ \tilde{\phi}_y \tilde{\phi}_x, & \tilde{\phi}_y \tilde{\phi}_y \end{bmatrix} \\ + \frac{1}{2} \begin{bmatrix} \partial_x (\tilde{\phi}_x^2 + \tilde{\phi}_y^2), & 0 \\ 0, & \partial_y (\tilde{\phi}_x^2 + \tilde{\phi}_y^2) \end{bmatrix}$$

$$= -\underline{\nabla} \cdot \underline{T} = \underline{G}$$

$$\underline{T} = \begin{pmatrix} \tilde{\phi}_x \tilde{\phi}_x, & \tilde{\phi}_x \tilde{\phi}_y \\ \tilde{\phi}_y \tilde{\phi}_x, & \tilde{\phi}_y \tilde{\phi}_y \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \tilde{\phi}_x^2 + \tilde{\phi}_y^2, & 0 \\ 0, & \tilde{\phi}_x^2 + \tilde{\phi}_y^2 \end{pmatrix}$$

and since periodic in $y \Rightarrow (y = \phi)$

$$\langle \partial_y(\cdot) \rangle = 0$$

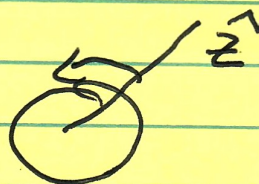
$$\langle \underline{F} \rangle = -\frac{\partial}{\partial x} \left\langle \frac{(\tilde{\phi}_x)^2}{2} \right\rangle \hat{x} - \frac{\partial}{\partial x} \langle \tilde{\phi}_x \tilde{\phi}_y \rangle \hat{y} + \frac{1}{2} \frac{\partial}{\partial x} \left\langle (\tilde{\phi}_x)^2 \right\rangle$$

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$$\langle \underline{f} \rangle = -\frac{1}{2} \partial_x \langle \tilde{\phi}_x^2 - \tilde{\phi}_y^2 \rangle \hat{x} - \partial_x \langle \tilde{\phi}_x \tilde{\phi}_y \rangle \hat{y}$$

For torque density,

$$\underline{\tau} = \underline{r} \times \langle \underline{f} \rangle$$



$$\tau = \tau_z = -\partial_x \langle \tilde{\phi}_x \tilde{\phi}_y \rangle$$

Thus,

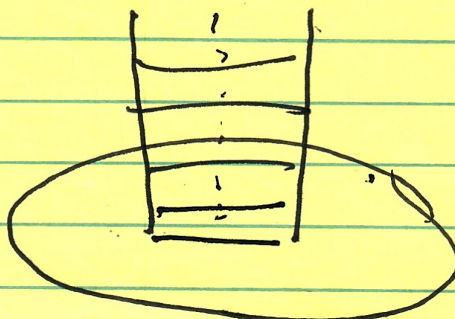
torque density

$$\frac{d}{dt} \oint \circ = \tau_z = -\partial_x \langle \tilde{\phi}_x \tilde{\phi}_y \rangle$$

↓
angular
momentum
density

cylinder.

Now, for torque
couple



Torque exerted on outside cylinder by side $\Rightarrow$

$$C = \int_{\tilde{r}}^{\infty} r dr [-\partial_r \langle \tilde{\Phi}_r \tilde{\Phi}_\phi \rangle]$$

$$C(r) = \int r \langle \tilde{\Phi}_r \tilde{\Phi}_\phi \rangle d\Omega$$

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Spiral wave gravitational couple is:

$$C(r) = \int r \langle \tilde{\Phi}_r \tilde{\Phi}_\phi \rangle d\Omega$$

$$\tilde{\Phi} = \tilde{\Phi}_0 \exp[i[g(r) + m\phi - \omega t]]$$

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$$C(r) = \int r |\Phi|^2 g'(r) M d\Omega$$

in
eikonal
approx.

Gravitational couple:

$$C(r) = \int r |\phi_0|^2 g'(r) m \, dS$$

$$C > 0 \Rightarrow g'(r) m > 0$$

$$\Rightarrow k_r(r) m > 0.$$

but trailing spiral:

$$\frac{k_r}{m} > 0 \Rightarrow \frac{k_r}{m} m^2 > 0 \quad \checkmark$$

For inner-outer gravitational
coupling which acts to spin up
the outer part of disk, need
trailing spiral!

Trailing
 → Spikes promote galactic equivalent
 of evolution.

→ Spikes vs viscosity? → range of
 wave propagation → Q

$Q \gg 1 \rightarrow$ no principal range
 → no spiral propagation

Need viscous stress (short length).

$Q \sim 1 + \rightarrow$ principal range
 → wave transport by
 spikes.

→ Dynamics of Wave Growth and Maintenance

Essence of problem:

- spirals are waves → stable ($Q=1+$) so how excited, propagate, amplify.
- How do spirals transport angular momentum, aid in relaxing galaxy

⇒ Wave Kinetics for spirals ??

Now:

①

- wave energy and angular momentum

②

- negative energy spiral waves → Physics!

[N.B.: Negative energy waves easily amplified by:

- dissipation
- coupling to positive energy waves

[Negative energy waves almost as much fun as instabilities...]

© - Eikonal theory and wave kinetics for spirals $\rightarrow$ wave Action evolution

For wave energy, angular momentum:

- consider adiabatic switch-on of external perturbation, from $t = -\infty$

$$\frac{dE_w}{dt} = \int_0^\infty r dr \int_0^{2\pi} d\phi \tilde{v} \frac{\partial \tilde{\phi}_{ext}}{\partial t}$$

$$\frac{dE}{dt} = \frac{\partial \tilde{\phi}_{ext}}{\partial t}$$

$$E_{wave} = \int_{-\infty}^t dt' \int_0^\infty r dr \int_0^{2\pi} d\phi \tilde{v} \frac{\partial \tilde{\phi}}{\partial t}$$

likewise

$$r \frac{\partial}{\partial \phi}$$

$$L_{wave} = \int_{-\infty}^t dt' \int_0^\infty r dr \int_0^{2\pi} d\phi \left(-\tilde{v} \frac{\partial \tilde{\phi}_{ext}}{\partial \phi} \right)$$

Now, need relate external perturbation to self-consistent field.

in plasma:

(recall basic EM)

$$\nabla^2 \phi = -4\pi \left(\underbrace{\tilde{\rho}_{ind}}_{\substack{\text{p-polarization} \\ \text{charge}}} + \tilde{\rho}_{ext} \right)$$

or

$$\underbrace{\epsilon(k, \omega)}_{\substack{\downarrow \\ \text{dielectric}}} \phi_{k, \omega} = \tilde{\rho}_{k, \omega}^{ext} \frac{4\pi}{k^2}$$

$$\underbrace{\epsilon}_{\substack{\downarrow \\ \text{dielectric} \\ \text{d.e. dynamic} \\ \text{collective response}}} = 1 + 4\pi \underbrace{\chi(k, \omega)}_{\substack{\downarrow \\ \text{susceptibility} \\ \text{relates } \rho, \phi}}$$

For spirals (and self-gravitating matter, in general),
do the same!

$$\nabla^2 \phi = 4\pi G \nabla \cdot \sigma(z)$$

$$\nabla^2 \phi - 4\pi G \nabla_0 \frac{\tilde{\nabla}_0}{\nabla_0} \delta(\phi) = 4\pi G \nabla_0 \frac{\tilde{\nabla}_{\text{ext}}}{\nabla_0} \delta(\phi)$$

$$\int_{-G}^{+G} dz \Rightarrow$$

$$-2|k| \tilde{\phi}_k - 4\pi G \nabla_0 \left(\frac{\tilde{\nabla}_k}{\nabla_0} \right) = 4\pi G \nabla_0 \frac{\tilde{\nabla}_{\text{ext}}}{\nabla_0}$$

$\downarrow$
2D

Now,

$$\frac{\tilde{\nabla}_k}{\nabla_0} = \chi_{\text{grav}}(k, \omega) \tilde{\phi}_{k, \omega}$$

$\downarrow$
gravitational susceptibility

$$\left[|k| + 2\pi G \nabla_0 \hat{\chi}(k, \omega) \right] \tilde{\phi}_k = -2\pi G \nabla_0 \frac{\tilde{\nabla}_{\text{ext}}}{\nabla_0}$$

$$D(k, \omega) = 1 + \frac{2\pi G \nabla_0 \hat{\chi}(k, \omega)}{|k|}$$

$$\epsilon_{\text{eff}}^{\text{grav}}(k, \omega)$$

$$\hat{\chi} = \delta(\tilde{\nabla}_k) / \delta \phi$$

$\leadsto$ effective dielectric (gravitational screening)

For $D(k, \omega)$ relate $\tilde{V}/V_0$ to $\tilde{\phi}$
have, in tight winding:

$$-i\bar{\omega} \tilde{V}_r - 2\Omega \tilde{V}_\theta = -ikr(\epsilon_0^2 \tilde{V}/V_0 + \tilde{\phi})$$

$$-i\bar{\omega} \tilde{V}_\theta + \frac{\tilde{V}_r}{r} \frac{d}{dr}(r^2 \Omega) = 0$$

$$-i\bar{\omega} \tilde{\phi}_{kr} + ikrV_0 \tilde{V}_r = 0$$

$$\frac{\tilde{V}}{V_0} = kr^2 \tilde{\phi} / (\bar{\omega}^2 - k^2 - kr^2 \epsilon_0^2)$$

$\Rightarrow$

$$\chi(k, \omega) = kr^2 / (\bar{\omega}^2 - k^2 - kr^2 \epsilon_0^2)$$

$$D(k, \omega) = 1 + \frac{2\pi\sigma V_0 / kr}{\bar{\omega}^2 - k^2 - kr^2 \epsilon_0^2}$$

check: $D(k, \omega) = 0$

$$\bar{\omega}^2 = k^2 + k^2 c_s^2 - 2\pi k n_0 / k$$

Recover Lin-Shu ✓

Exercise: Extend to kinetics, using known reduction factor.

Observation: Dispersion? $\rightarrow$ What happens?

n.b. need for $\int_{-\infty}^t dt$ to converge.

$$\epsilon(k, \omega) = \epsilon_r(k, \omega) + i \epsilon_{\text{IM}}(k, \omega)$$

or $D(k, \omega)$ ϵ_{small}

$$\epsilon(k, \omega) = 0 \Rightarrow$$

$$\underbrace{\epsilon_r(k, \omega)}_{i\gamma_n} + (\omega - \omega_n) \underbrace{\frac{d\epsilon_r}{d\omega}}_{\omega_n} = -i \epsilon_{\text{IM}}(k, \omega_n)$$

00

$$G_r = 0$$

$$\gamma_H = - \frac{E_{IM}}{\partial E / \partial \omega|_{\omega_r}}$$

$$\partial E / \partial \omega > 0 \rightarrow E_{IM} > 0 \rightarrow \text{damping} \quad (\text{usual})$$

$$\partial E / \partial \omega < 0 \rightarrow E_{IM} < 0 \rightarrow \text{growth}$$

$$\partial E / \partial \omega < 0 \rightarrow E_{IM} > 0 \rightarrow \text{growth!}$$

$$E_{IM} < 0 \rightarrow \text{damping}$$

$\xrightarrow{\text{Flip}}$
 $\downarrow$
 negative energy wave.

$\Rightarrow$ suggests that $\partial D / \partial \omega$
 is crucial to spiral wave
 dynamics.

Sign $\rightarrow$ $\oplus$ energy wave.

00 back to wave energy....

$$\frac{dE_w}{dt} = \int_0^{2\pi} d\phi \int_0^{\infty} r dr \quad \vec{T} \cdot \frac{\partial \vec{\phi}_{\text{ext}}}{\partial t}$$

but $D(k, \omega) \hat{\phi}_{k, \omega} = \hat{\phi}_{k, \omega}^{\text{ext}}$

Near or at collective resonance (i.e. wave) :

$$D(k, \omega) = D(k, \omega_k) + (\omega - \omega_k) \left. \frac{\partial D}{\partial \omega} \right|_{\omega_k} + i \frac{\gamma_k}{2} D(k, \omega_k) \quad \text{for now}$$

so

$$(\omega - \omega_k) \left. \frac{\partial D}{\partial \omega} \right|_{\omega_k} \hat{\phi}_{k, \omega} = \hat{\phi}_{k, \omega}^{\text{ext}}$$

δ
 $i\gamma_k$

$$\frac{\partial \hat{\phi}^{\text{ext}}}{\partial t} = -i\omega_k \hat{\phi}^{\text{ext}} = \gamma_k \left. \frac{\omega \partial D}{\partial \omega} \right|_{\omega_k} \hat{\phi}_{k, \omega}$$

Now, From Poisson eqn. :

$$\hat{\phi}_{k, \omega} = -\frac{|k|}{2\pi G} \hat{\phi}_{k, \omega} \exp[i(k \cdot x - \omega t) e^{\gamma_k t}]$$

$$\frac{d\varepsilon_w}{dt} = \frac{v_e}{2} \left(\nabla^2 \frac{\partial \phi}{\partial t} \right)$$

$$\begin{aligned} \varepsilon_w &= \int_{-\infty}^t dt e^{2\gamma_k t} \left(\frac{-|k|}{4\pi G} \right) \omega_k \frac{2\gamma_k}{2} \frac{\partial D}{\partial \omega} \bigg|_{\omega_k} |\tilde{\phi}_{\gamma\omega}|^2 \\ &= \int_{-\infty}^t dt' (2\gamma_k) e^{2\gamma_k t} \left(\frac{-|k|}{8\pi G} \right) \omega_k \frac{\partial D}{\partial \omega} \bigg|_{\omega_k} |\tilde{\phi}_{\gamma\omega}|^2 \end{aligned}$$

$\Rightarrow$

$$\boxed{\varepsilon_w = \left(\omega_k \frac{\partial D}{\partial \omega} \bigg|_{\omega_k} \right) \left(\frac{-1}{8\pi G} |k|^2 |\tilde{\phi}_{\gamma\omega}|^2 \right)}$$

↓
wave energy density

Obvious counterpart of:

$$\boxed{\varepsilon_{w_k} = \omega_k \frac{\partial \varepsilon}{\partial \omega} \bigg|_{\omega_k} \frac{|E_k|^2}{8\pi t}}$$

i.e. field energy + media energy.

As
$$\frac{\partial D}{\partial \omega} = \frac{-2\pi G \sigma_0 / k_1}{(\omega^2 - k^2 - k^2 G^2)^2}$$

$$\epsilon_w = \frac{\omega_r \bar{\omega}_r}{(\omega^2 - k^2 - k^2 G^2)^2} \int_0^{\frac{2\pi}{\lambda}} \left(\frac{k_1 A_1}{2} \right)^2$$

→ wave energy density

Obvious: - $\bar{\omega}_r < 0$ for $r < r_0$

$\epsilon_w < 0$ → spiral is negative energy inside co-rotation

- $\omega_r > 0$ for $r > r_0$

$\epsilon_w > 0$

junction procedure →

$$\boxed{L_z = \frac{m \bar{\omega} \sigma_0}{(\omega^2 - k^2 - k_{\text{ce}}^2)^2} \left(k^2 \frac{|A_0|^2}{2} \right)}$$

Wave angular momentum density

Note:

$$- \Sigma_{\text{wave}} = (\omega/m) L_{\text{wave}}$$

$$\underline{\sigma} \quad \Sigma = \omega N$$

$$\rightarrow P = k_0 N$$

semi-classical

$$\underline{P} = \underline{k} N$$

$$L_z = m N$$

$$\text{and } \Sigma_w / L_w = \omega / m$$

$$\left(\begin{array}{c} \Sigma_w \\ L_w \end{array} \right) > 0 \text{ for } \frac{\omega}{m} > \Omega \rightarrow r > r_0$$

$$\left(\begin{array}{c} \Sigma_w \\ L_w \end{array} \right) < 0 \text{ for } \frac{\omega}{m} < \Omega \rightarrow r < r_0$$

$$N = \frac{\bar{\omega}_k}{(\bar{\omega}^2 - k^2 - k^2 c_s^2)^2} \propto \left(k^2 \frac{|\Phi_k|^2}{2} \right)$$

Wave Action Density $\rightarrow N = \mathcal{E}/\omega$
(Adiabatic invariant)

What does this mean?

— Wave Energy relative to certain frame $\Rightarrow$ not Galilean invariant

— medium active $\Rightarrow$ supports energy, possibly accessible
(differential rotation)

ie Plasma $\epsilon = 1 - \frac{\omega_p^2}{\omega^2} +$
 $\rightarrow \oplus$ energy, waves

Beam-Plasma $\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kv_b)^2}$ Beam Velocity
 $\rightarrow$ can be $\ominus$ energy.

- Positive energy wave excited by adding energy to wave
- Negative energy wave excited by extracting energy
- destabilized by damping.

Thus any damping/dissipation of a spiral wave would dissipate

→ a $\oplus$ amount of $\left(\frac{\mathcal{E}}{\mathcal{L}_0}\right)$ for $r > r_0$

→ a $\ominus$ amount of $\left(\frac{\mathcal{E}}{\mathcal{L}_0}\right)$ for $r < r_0$

but:

→ $\oplus$ wave $\mathcal{L}_0$ dissipated → positive wave angular momentum transferred to medium → medium angular momentum increases.

$\rightarrow$ $\ominus$ wave $\overset{\text{dissipated}}{S_0^1} \rightarrow$ negative wave angular
 momentum transferred to medium $\rightarrow$
 medium angular momentum decrease.

$\Rightarrow$ Spirals can couple angular
 momentum from smaller radius
 to larger radius.

$\Rightarrow$ good for accretion / lowering energy.

But: Need deal with Q barrier.

TBC.