N.B.: See course topics, posted

Physics 239

Lecture 18: Spiral Density Waves I

Refs:

F. Shu – "The Physical Universe"
especially "Spiral Structure" – Chapt. 12

→ excellent UG survey text, useful all levels
→ great for key physics, big picture

F. Shu – Vol 2
Chapt 11, 12
11 – basics
12 – wave energetics and amplification

→ beware – these two chapters are above the level of the book, though quite good!
→ notation is confusing .....

B+T – Chapt. 6

→ useful, not the strong point of B+T.
(putting it politely....)

+ postings

Spiral Density Waves: 3 classes

(1)
→ Key physics, dispersion, properties, resonances

(2)
→ Wave Action, Energy Theorems; Excitation, Amplification

(3)
→ Why Spirals? Kinetics, etc.
↓
Stellar Dynamics

A.)
Spirals – Background

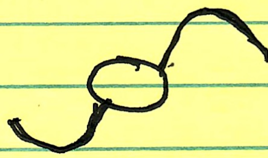
– Spirals are a density wave, not a pin-wheel. Spiral arms ↔ high density (Σ) regions

i.e.

– Galaxy is a disk, not pinwheel.
Arms are standing density wave

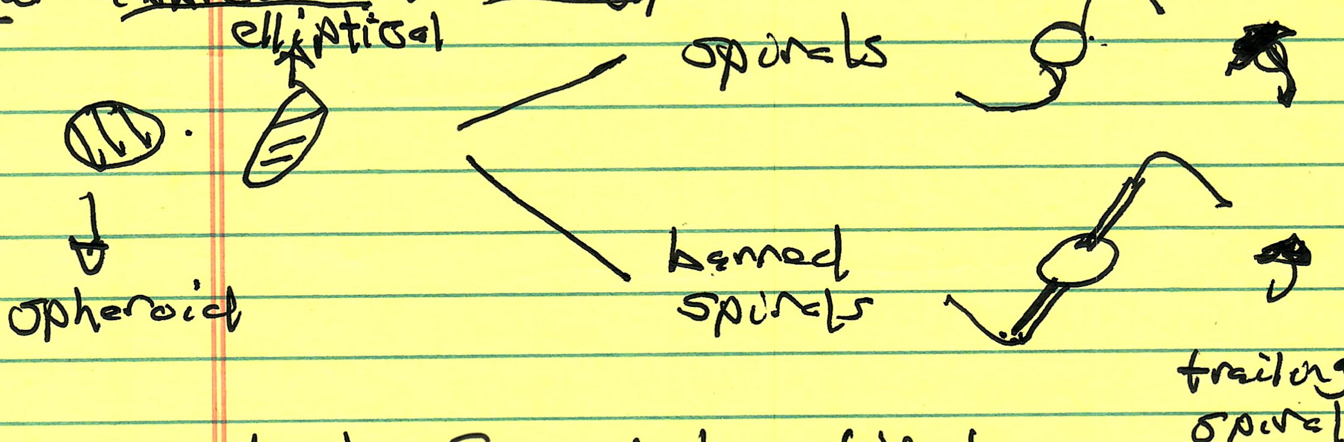
[Disk rotates differentially, albeit non-Keplerian.

–  → high density region

Not  → would wind-up too rapidly

– Most galaxies: spheroidal or disk

i.e. abbreviated zoology:



→ ellipticals formed by tidal interactions with spheroid, (c.f. "Violent relaxation")

→ disk are ubiquitous → as for stars, etc. → accretion // assembly "bottom-up"

→ As disks ubiquitous, <u>spirals</u> are ubiquitous:

- "grand design" (2 large arms → $m = 2$ mode)

- multiple arms (Milky Way → 4)

- flocculant (multiple weak arms)

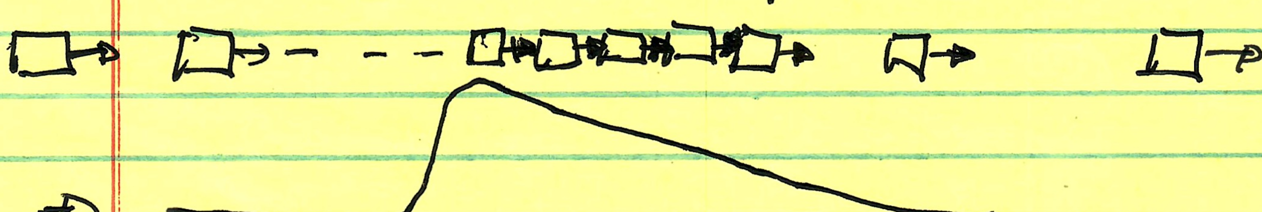
⋮ spiral turbulence ?

→ Galaxy → SMBH in core

→ What is a 'density wave' ?

- canonical example: <u>traffic jam</u>

i.e. bottleneck / toll booth



ρ →

(backward shock!)

Point:

- ⊕ stationary density bump
- but - cars move

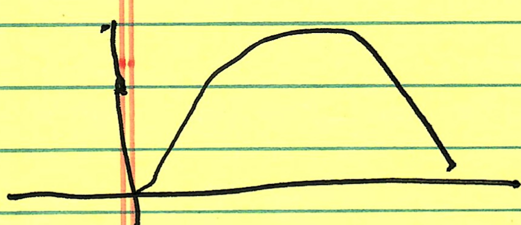
obviously, $\underline{c_{pattern} \neq V_{car}}$.

A little mathematics (1D traffic flow)

c.f. - Lighthill and Whitham papers
- Whitham "Linear and Nonlinear Waves"

Consider: 1D
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho V(\rho)\right) = 0$$
(continuity)

$V(\rho) \equiv$ flow(-rate) of density



empirical

(see Whitham)

<u>Given</u>

$Q(\rho) = \rho V(\rho)$ so

$$\frac{\partial \rho}{\partial t} + \frac{dQ}{d\rho}\,\partial_x \rho = 0$$

$$\frac{dQ}{d\rho} = V(\rho) + \rho\frac{dV}{d\rho} \equiv c(\rho)$$

→ pattern speed.

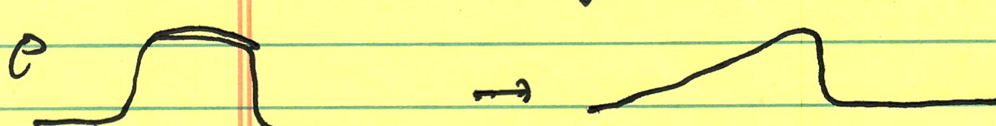
then $$\frac{\partial \rho}{\partial t} + c(\rho)\,\partial_x \rho = 0$$

Note: for $\frac{dV}{d\rho} < 0$, $c(\rho) < V(\rho)$

i.e. [realizes idea of difference between (particle) flow speed and pattern speed.

Also:

- usual shock (gas-dynamics) → forward



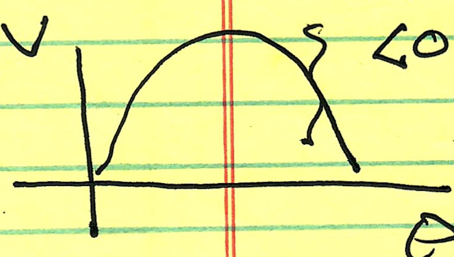
steepening ⇒ over-taking, as speed increases with density.

$c(\rho)$ s/t $\partial c(\rho)/\partial\rho > 0$ $\left(P = P_0(\rho/\rho_0)^\gamma\right)$

Now, for traffic flow with $\partial V/\partial\rho < 0$;

$$\frac{\partial c}{\partial\rho} = \frac{\partial V}{\partial\rho} + \frac{\partial V}{\partial\rho} + \rho\frac{\partial V}{\partial\rho^2}$$

$$= 2\frac{\partial V}{\partial\rho} + \rho\frac{\partial^2 V}{\partial\rho^2} < 0$$



<0 $\qquad$ <0

⇒ speed decreases with density

⇒ backward shock → a la' bottleneck.

$V(\rho) \longrightarrow$

here: $\qquad \longleftarrow c(\rho)$ $\qquad$ opposite.

→ Traffic jam is excellent example of ⊕ stationary density wave.

More on spirals:

→ Galaxy = Stars + Gas
n.b. not Keplerian; ⊕ flat $V(r)$ rotation curve
- density wave → shocks gas
→ star formation
} see spiral arms bright in blue....

→ What are Spirals?

- $m \neq 0$ counterpart of Toomre instability (Jeans in differential rotation)

i.e. for $|k_r| \gg k_\theta \sim m/r$
(tight winding limit →

$$\Omega = \Omega(r)$$

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G|k|\Sigma_0 + k^2 c_s^2$$

$k = k_r \ldots$ $\quad \Phi$

Lin-Shu dispersion relation

- (magnetized) Jeans instabilities / $\boxed{\text{Waves.}}$
  ⟺ Theory is essentially wave dynamics.
- can calculate via:

(*) → gas dynamics (cf. Toomre)

(easy)

→ stellar dynamics (i.e. solve Vlasov equation)

(more difficult).

Stellar Dynamics

N.B. Resembles magnetized plasma dispersion relation (Bernstein).

$$J_n^2 = J_n^2(k_\perp v_\perp/\Omega)$$

i.e. $$\epsilon = 1 + \frac{\omega_p^2}{k^2}\int d^3v \sum_n \frac{\left(\frac{n\Omega}{v_\perp}\frac{\partial\langle f\rangle}{\partial v_\perp} + k_\parallel \frac{\partial\langle f\rangle}{\partial v_\parallel}\right) J_n^2}{\omega - k_\parallel v_\parallel - n\Omega}$$

$\Omega = \Omega_{\text{cycl.}}$

and re-group

→ **Why** Spirals?

- Spirals enable galaxies to increase gravitational binding energy, i.e. relax, lower energy.

⇒ Accretion! (c.f. Shu in "Physical Universe")

how?

- Recall L-B. "2 particle" argument:

$\boxed{\text{N.B.: "2 particles" of Rayleigh → Lynden-Bell → Balbus is core theme of this course}}$

i.e. - interchange of 2 particles conserving sum of angular momentum and mass

$\Rightarrow$

- $\Delta E < 0$ $\rightarrow$ relaxation if mass gain inner, angular momentum gain outer.

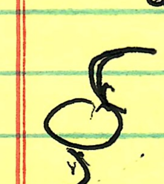
- End state: Accreted mass + 1 particle at ∞.

Here $\rightarrow$ Spiral wave (n.b. broken azimuthal symmetry) carries angular momentum

$\Leftrightarrow$ exchanges angular momentum between 2 particles.

$\rightarrow$ Alternative is viscous torque, produced by MRI

$\rightarrow$ Only trailing spirals carry angular momentum outward.



$\uparrow$ $\leftrightarrow$ sign $(k_r k_\theta)$

→ History:

- Pre-history : Bertil Lindblad
(50's)

- Hey day : 60's → Lin, Shu
early 70's. A. Toomre
Lynden-Bell, Goldreich
Kalnajs

Density Wave
Stellar Dynamics

- 70's : Goldreich, Tremaine
(late) (excitation)

quiescent, but many open issues.

Recently:

Spirals are "back" ⟷ protoplanetary disks
dust → massive .

B) Spiral Waves in Gas Dynamics

a/a' Toomre:

$$\sigma = \sigma_k e^{i(\alpha(r) + m\theta - \omega t)}$$

α ↓ phase

$\alpha' = k_r$

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\underline{v}\sigma) = 0$$

$$\sigma_0 \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla P - \sigma \nabla \phi$$

$$\nabla^2 \phi = 4\pi G \sigma \delta(z) \qquad \text{thin disk}$$

$\Rightarrow$

$$-i(\omega - m\Omega)\tilde{\sigma} + \sigma_0 (i k_r \tilde{v}_{\underline{k}} + i k_\theta \tilde{v}_{\theta_{\underline{k}}}) = 0$$

$$-i(\omega - m\Omega)\tilde{v}_{r_{\underline{k}}} - 2\Omega \tilde{v}_{\theta_{\underline{k}}} = -i k_r (\tilde{\phi}_{\underline{k}} + \tilde{h}_{\underline{k}})$$

$\tilde{h}$ ↓ enthalpy $\sim dP/\rho$

$$-i(\omega - m\Omega)\tilde{v}_{\theta_{\underline{k}}} + \frac{\tilde{v}_{r_{\underline{k}}}}{r} \frac{\partial}{\partial r}(r^2\Omega) = -i k_\theta (\tilde{\phi}_{\underline{k}} + \tilde{h}_{\underline{k}})$$

Define $\bar{\omega} = \omega - m\Omega$

$\therefore$

$$\bar{\omega}\tilde{v}_r - 2i\Omega\tilde{v}_\theta = k_r(\tilde{\phi} + \tilde{h})$$

$$\frac{i}{r}\frac{\partial}{\partial r}(r^2\Omega)\,\tilde{v}_r + \bar{\omega}\tilde{v}_\theta = k_\theta(\tilde{\phi} + \tilde{h})$$

$$K^2 = \Phi = \frac{2\Omega}{r}\partial_r(r^2\Omega)$$

$\Rightarrow$

$$\tilde{v}_r = \left[\bar{\omega}^2 - K^2\right]^{-1}\left\{(\bar{\omega}k_r - 2ik_\theta\Omega)(\tilde{\phi} + \tilde{h})\right\}$$

$$\tilde{v}_\theta = \left[\bar{\omega}^2 - K^2\right]^{-1}\left\{\left(\bar{\omega}k_\theta - k_r\frac{i}{r}\frac{\partial}{\partial r}(r^2\Omega)\right)(\tilde{h} + \tilde{\phi})\right\}$$

Note: Resonance at $(\tilde{v}_r, \tilde{v}_\theta \uparrow)$

$$\bar{\omega}^2 = K^2 \Rightarrow \boxed{\omega = m\Omega \pm K}$$

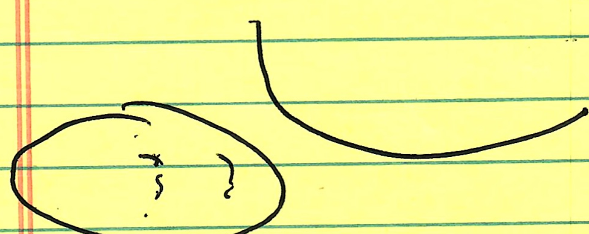
$\boxed{\text{Lindblad resonance}}$ $\quad r = r_{LR}$.

At Lindblad resonance, locally Doppler shifted frequency matches ± epicyclic frequency.

⇒ spiral pattern pushes continually on star.

⇒ Response to external forcing or torque strong at Lindblad resonance

⇒ LBR ↔ wave emission/absorption sites



Now, in tight winding limit → $k_r \gg k_\theta$

$$\tilde{\sigma}/\sigma_0 = k_r \tilde{v}_r / \bar{\omega}$$

$$\tilde{v}_r = (\bar{\omega}^2 - \kappa^2)^{-1} (\bar{\omega} k_r) (\tilde{\phi} + \tilde{h})$$

$$\tilde{v}_\theta = (\bar{\omega}^2 - \kappa^2)^{-1} \left(-i \frac{k_r}{r} \partial_r (r^2 \Omega) \right) (\tilde{\phi} + \tilde{h})$$

As before:

$$\nabla^2 \Phi = 4\pi G \sigma_0 \frac{\tilde{\sigma}}{\sigma_0} \delta(z)$$

$$\left(\frac{\partial^2}{\partial z^2} - k_r^2\right) \tilde{\Phi} = 4\pi G \sigma_0 \delta(z) \frac{\tilde{\sigma}}{\sigma_0} \qquad k_r \gg k_\theta$$

$$-2|k_r| \tilde{\Phi}_u = 4\pi G \sigma_0 \frac{\tilde{\sigma}_u}{\sigma_0}$$

$$\tilde{\Phi}_u = -\frac{2\pi G}{|k_r|} \sigma_0 \left(\tilde{\sigma}_u / \sigma_0\right)$$

and,

$$\tilde{\Phi}_u + \tilde{h}_u = \left(-\frac{2\pi G \sigma_0}{|k|} + c_s^2\right) \tilde{\sigma}_u / \sigma_0$$

$$\tilde{\sigma}_u / \sigma_0 = \frac{k_r \tilde{v}_r}{\bar{\omega}} = \frac{k_r}{\bar{\omega}} \left(\frac{\bar{\omega} k_r (\tilde{\Phi}_u + \tilde{h}_u)}{\bar{\omega}^2 - \kappa^2}\right)$$

so $\frac{\tilde{\sigma}_u}{\cancel{\Sigma_0}} = \frac{k_r^2}{\overline{\omega}^2 - \kappa^2}\left(-\frac{2\pi G \Sigma_0}{|k_r|} + c_s^2\right)\frac{\tilde{\sigma}_u}{\cancel{\Sigma_0}}$

$\kappa^2 \Rightarrow$ low k
stability
$k_r^2 c_s^2 \Rightarrow$ high k

so $\Rightarrow$

$$\boxed{\overline{\omega}^2 = \kappa^2 - 2\pi G \Sigma_0 |k_r| + k_r^2 c_s^2}$$

$\overline{\omega} = \omega - m\Omega$

Lin - Shu dispersion relation!

→ given tight winding,

Toomre + $\omega \to \omega - m\Omega$ recovers
Lin - Shu, no surprise

→ issue of excitation open, → must
address how spirals excited, maintained.

→ For stellar dynamics (kinetics):

$\overline{\omega}^2 = \kappa^2 - 2\pi G |k| \Sigma_0 + k^2 c_s^2$

$= \kappa^2$ + collective
(→ single particle

→ Then, straightforward to see that kinetics enter via pressure (only!)

i.e.

$$\bar{\omega}^2 = \kappa^2 - 2\pi \Sigma_0 G |k_r| + k_r^2 \left[\frac{\delta P_k}{\delta(\tilde{\Gamma}/\Gamma_0)_k} \right]$$

or gas dynamics:

$$\delta P / \delta(\tilde{\Gamma}/\Gamma_0)_k = c_s^2 \quad \checkmark$$

in Vlasov theory:

$$\delta P = \left(\int d^3v \, \frac{\delta \hat{f}}{\delta \phi} \, v^2 \right) \hat{\phi}$$

- i.e. compute pressure perturbation by moment of linear Vlasov response i.e. $\hat{f}_{\underline{k}} = \dfrac{\nabla\hat{\phi}_{\underline{k}} \cdot \partial\langle f\rangle/\partial v}{-i(\omega - \underline{k}\cdot\underline{v})}$, formally

but Poisson equation $\Rightarrow$

$$\hat{\phi}_{\underline{k}} = \frac{-2\pi G}{|k_r|} \Sigma_0 \left(\tilde{\Sigma}/\Sigma_0\right)_k$$

$\Rightarrow$

$$\delta p = \int d^3v \left(\frac{\delta f}{\delta\phi} v^2\right)\left(\frac{-2\pi G \Sigma_0}{|k_r|}\left(\tilde{\Sigma}/\Sigma_0\right)_{\underline{k}}\right)$$

$\therefore$

$$\bar{\omega}^2 - \kappa^2 = -2\pi G |k_r| \Sigma_0 \left[1 + \int d^3v \left(v^2 \frac{\delta f}{\delta\phi_k}\right)\right]$$

So, in stellar dynamics:

variance v

$$\boxed{\bar{\omega}^2 = \kappa^2 - 2\pi G \Sigma_0 |k_r| \, \mathcal{F}\left(\bar{\nu}, k_r v_{Ith}/\kappa\right)}$$

$\bar{\omega}/\kappa \equiv \nu$ (kinetic) "Reduction Factor."

where: $$\overline{\epsilon} = 1 + \int d^3v\, v^2\, \delta f_n / \delta \phi_n$$

$$= 1 + \int d^3v\, v^2 R(\bar{\omega}, k_r, v_\perp)$$

response fct.

actually:

$$\overline{\epsilon} = \frac{2}{x}(1-\nu^2)e^{-x} \sum_{n=1}^{\infty} \frac{n^2 I_n(x)}{n^2-\nu^2}$$

$$\nu = \bar{\omega}/\kappa$$

$$x = k_r \langle \Delta v \rangle / \kappa$$

compare Bernstein

Exercise:

- Work thru all this.
- Calculate $\overline{\epsilon}$ for stellar dynamics. (see B+T appendix)

  Warm-up by deriving Bernstein dispersion relation.
- Discuss differences in high k_r behavior, gas and kinetics.

(c) Patterns

Now, proceeding in gas dynamics equations : seek!

- patterns (i.e. $k(\omega)$, as $\omega \leftrightarrow$ forcing)
- resonances / barriers

  $\downarrow$

  response to forcing.

$k = k_r$

so

$$\overline{\omega}^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + k^2 c_s^2$$

$$|k|^2 - \frac{2\pi G \Sigma_0}{c_s^2} |k| + \frac{\kappa^2}{c_s^2}\left(1 - \frac{\overline{\omega}^2}{\kappa^2}\right) = 0$$

$$\nu^2 = \overline{\omega}^2 / \kappa^2$$

$$\nu^2 = 1 \rightarrow \omega = m\Omega \pm \kappa$$

(Lindblad resonances)

so

$$|k_r| = \frac{\pi G \Sigma_0}{c_s^2}\left[1 \pm \left(1 - \frac{4\kappa^2}{c_s^2}\left(\frac{c_s^2}{2\pi G \Sigma_0}\right)^2 (1 - \nu^2)\right)^{1/2}\right]$$

so $Q^2 = \kappa c_s / \pi G \Sigma_0$

→ return of Toomre

$$k_r = \frac{\pi G \Sigma_0}{c_s^2}\left[1 \pm \left(1 - Q^2(1-\nu^2)\right)^{1/2}\right]$$

$k_T = \kappa_0^2 / 2\pi G \Sigma_0$ — Toomre k.

so

$$\boxed{|k|/k_T = \frac{2}{Q^2}\left[1 \pm \left(1 - Q^2(1-\nu^2)\right)^{1/2}\right]}$$

→ $k(\omega)$ form of Lin-Shu dispersion relation

→ + → short wave
more sensitive to high k details (sound)

− → long wave. (more robust)

Related structure:

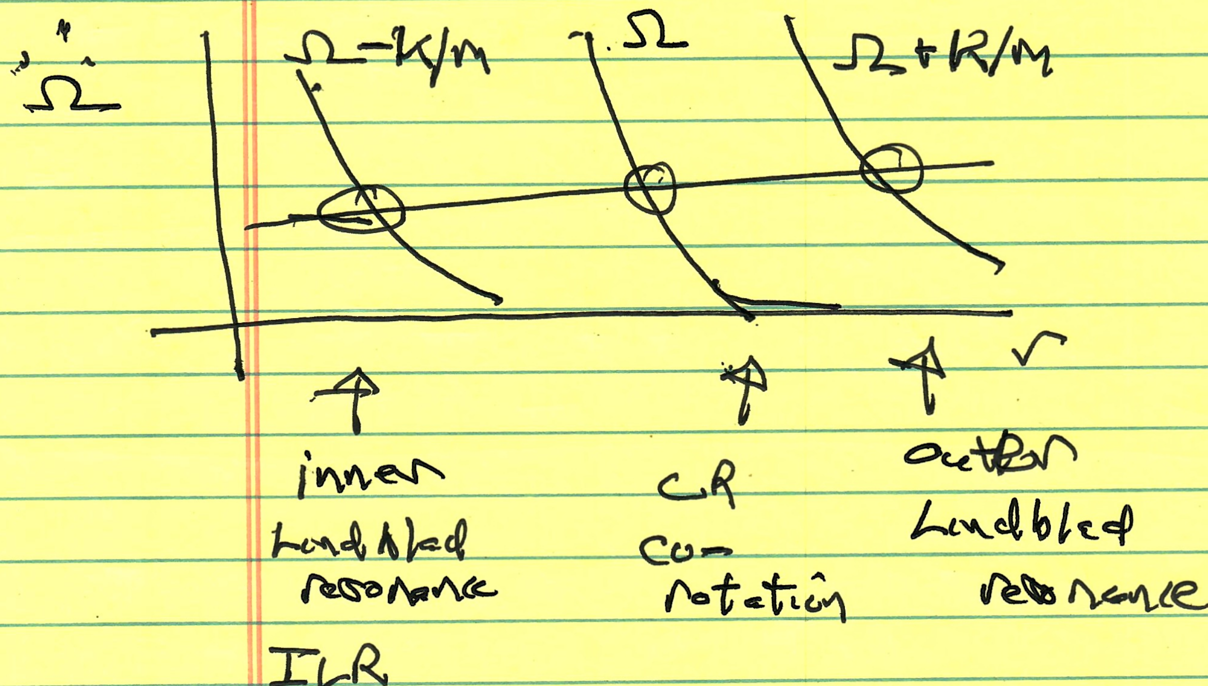
- resonances $\qquad \omega = m\Omega \pm \kappa \qquad V_0 \sim \text{const.}$

$\nu = \pm 1$

$$= m\Omega_0 \frac{r_0}{r} \pm \kappa$$

Lindblad

$$r_{res} = m\Omega_0 r_0 / (\omega \mp \kappa)$$



Lindblad resonances: $k_L = 0$

(turning pt.).

– Principal Range:

$$r_{ILR} < r < r_{OLR}$$

→ Long wave propagates between 2 Lindblad resonances

→ waves excited/absorbed at Lindblad resonances

- tidal force
- scale $l \gg k_r$ ⇒ spatially broad forcing acts at resonance (location)

– n.b. spirals happen in Saturn's rings. Moons are "forcing".

– Barriers

$$\frac{|k_r|}{k_T} = \frac{2}{Q^2}\left\{ 1 \pm \left[1 - Q^2(1-\nu^2)\right]^{1/2} \right\}$$

so if $Q(r)$ s/t $\frac{1}{Q^2} \leq (1-r^2)$

$k_{Im} \neq 0 \rightarrow$ evanescent

→ defines "Q barrier"

→ waves damp

Now, $1/Q^2 < 1-r^2$

$r^2 < 1 - 1/Q^2$

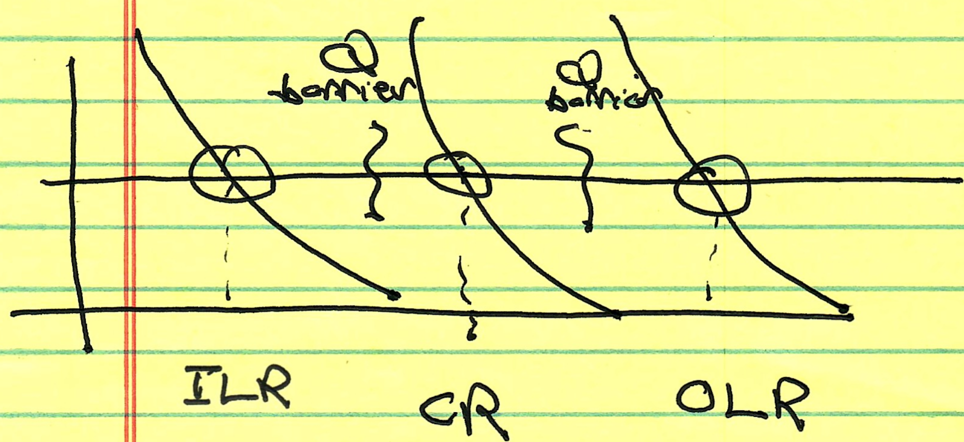
$Q^2 = 1 + \epsilon$ (~ marginal)

so $r^2 < 1 - 1/(1+\epsilon) \sim \cancel{1} - \cancel{1} + \epsilon$

$r^2 < \epsilon$

so for $Q^2 \sim 1 + \epsilon \Rightarrow$ barriers at co-rotation, where $r \rightarrow 0$.

- So now have Principal Range:



Q barrier surrounds co-rotation

$\omega = m\Omega \pm \kappa \rightarrow$ LR

$1/Q^2 = 1 - \nu^2 \rightarrow$ Q barrier region

∴ can see 2 useful limits:

$Q \gg 1 \rightarrow$ very stable (Σ_0 low)

$1/Q^2 < 1 - \nu^2$

$r_Q \rightarrow r_{LBR}$

so Q barrier at Lindblad resonances $\nu = \pm 1$

$\rightarrow$ no principal range!

i.e. 'Wisp' galaxies — disks with low σ and $Q \gg 1$ do not have spirals! No principal range

$Q \gtrsim 1 \Rightarrow$ narrow band at co-rotation

- long waves everywhere, except ring at $\omega \sim m\Omega$ (co-rotation)

- evanescent wave can tunnel/interact thru Q barrier

- feedback mechanisms

- long/short waves can refract into one another

Next: Seek understand dynamics of spiral wave propagation on Principal Range, with barriers

∴ come to:

- spiral wave $\begin{cases} \text{Action} \\ \text{Energy} \\ \text{Momentum} \end{cases}$

- Wave amplification mechanisms $\begin{cases} \text{WASER} \\ \text{SWING} \end{cases}$

- kinetics ⇒ at resonances!

- excitation/absorption.