Conservative advection:

N.B. $\left.\begin{array}{l}\text{chaos}\\ \text{phase mixing}\end{array}\right\}$ fundamentally similar processes

of stretching + partition scale.

∴ chaos → exponential divergence

phase mixing → algebraic divergence.

Now, to see phase mixing analytically:

- coarse graining as tiny ~~but finite~~ diffusion

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{q}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} \cong \underset{\substack{\downarrow\\ \nu \to 0.}}{\nu}\frac{\partial^2 f}{\partial v^2}$$

$q/m \to 1$ for gravity

write as

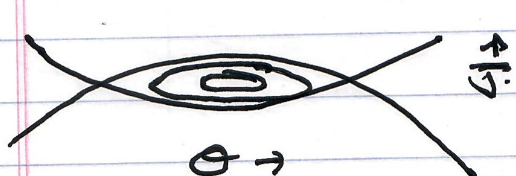
$$\partial_t f + \{f, H\} = \nu\frac{\partial^2 f}{\partial v^2}$$

and, in action-angle variables:

$$\frac{\partial f}{\partial t} + \omega(J)\frac{\partial f}{\partial \theta} = \nu \frac{\partial^2 f}{\partial J^2}$$

ν const

i.e. differential rotation in phase space for trapped particles (in wave)



$$J = \oint p(\epsilon, x)\, dx$$

$$\epsilon = H = \frac{p^2}{2m} + q\phi$$

$$p = \left[2m(H - q\phi)\right]^{1/2}$$

Now, as $\nu \to 0$,

$$f(\theta, J, t) = f(\theta - \omega(J)t, J, 0)$$

all particles rotate on track

so

$$f(\alpha,J,t) = F_0(J) + \sum_n F_n(J)\, e^{in(\theta-\omega t)}$$

$\downarrow$ fine structure

$\to$ streams

$F_0 = \langle f(t=0)\rangle_\theta$ initial coarse-grained

then

$$\left(\frac{\partial}{\partial t} + \omega(J)\frac{\partial}{\partial \theta} - \nu\frac{\partial^2}{\partial J^2}\right)\left[F_0(J) + \sum_n F_n(J)\, e^{in(\theta-\omega t)}\right] = 0$$

$n=0$

$$\partial_t F_0 = \nu\frac{\partial^2 F_0}{\partial J^2}$$

$\to$ mean evolves slowly, diffusively

and $n \neq 0$

$$\sum_n \left[\cancel{-in\omega F_n} + \frac{\partial}{\partial t}F_n + \cancel{in\omega F_n} + \nu n^2\left(\frac{\partial \omega}{\partial J}\right)^2 t^2 F_n\right] e^{in(\theta-\omega t)} = 0$$

$\uparrow$ n large. $\to$ fine structure

$\omega = \Omega$

so $\quad \partial_t F_n = -\nu n^2 \left(\frac{\partial \Omega}{\partial J}\right)^2 t^2 F_n$

$$F_n = \exp\left[-\frac{\nu n^2}{3}\left(\frac{\partial \Omega}{\partial J}\right)^2 t^3\right] F_n(0)$$

decay rate due "phase mixing"

!

= differential rotation + coarse graining

$$1/\tau_{decay} \simeq \left(\frac{\nu n^2}{3}\left(\frac{\partial \Omega}{\partial J}\right)^2\right)^{1/3} \sim \nu^{1/3}\,\omega'^{2/3}$$

$\nu^{1/3}$!

'shear' on phase space rotation

so $t \to \infty$

$F \to F_0$, $\quad \rightarrow$ all fine structure damped.

$\rightarrow$ F_0 relaxes diffusively

demonstrates decay of coarse grained distribution + physics of decay.

② Entropy, Distribution, etc.

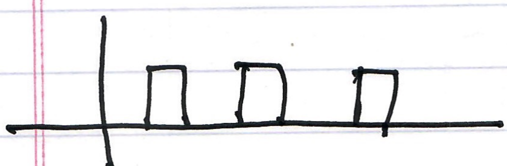
Consider:

→ system stirred in phase space, starting from i.c.

→ $\frac{df}{dt} = 0$, some finite set of η's.

For simplicity f's $= \eta \to \eta_0 \to$ all levels phase space density the same.

i.e.



and $\underline{\nabla} \cdot \underline{V}_{\Gamma} = 0$ → phase volume of each element conserved.

$v(\eta)$ conserved

↓

volume of element with $f = \eta$

then, introduce:

$$\boxed{\rho(\underline{x}, \underline{v}, \eta) \equiv \text{probability of finding } f = \eta \text{ at } \underline{x}, \underline{v} \text{ (within small neighborhood)}}$$

so

$$\int d\eta \, \rho(\underline{x}, \underline{v}, \eta) = 1$$

and

$$\bar{f}(x, v) = \int d\eta \, \eta \, \rho(\underline{x}, \underline{v}, \eta)$$

$\downarrow$

local coarse-grained distribution

and

$$\nabla^2 \bar{\phi} = 4\pi G \int d^3v \, \bar{f}$$

$$= 4\pi G \int d^3v \int d\eta \, \eta \, \rho(\underline{x}, \underline{v}, \eta)$$

Then argue that relevent entropy is:

$$S = -\int d\eta \, d^3x \, d^3v \, \rho \ln \rho$$ (i.e usual)

subject to constraints of constant:

① total phase volume occupied by level η (i.e. element

$$\gamma(\eta) = \int d^3x \, d^3v \, \rho(\underline{x}, \underline{v}, \eta)$$

→ effectively total mass of phase fluid with $f = \eta$

→ $\eta = 0, \eta_0$ only → equivalent to total mass ($\eta = \eta_0$)

② Energy:

$$E = \frac{1}{2}\int d^3x \, d^3v \, v^2 \bar{f} + \frac{1}{2}\int d^3x \, d^3v \, \bar{f}\bar{\phi}$$

(3) angular momentum

(4) linear momentum

Now, most probable state from:

$$\delta\Big[\,S - \overset{1/T}{\overset{\downarrow}{\beta}} E - \int d\eta\, \overset{\text{"chemical potential" of } \eta}{\overset{\downarrow}{\alpha(\eta)}}\, \gamma(\eta)\Big] = 0$$

$$\gamma(\eta) = \int \rho(\underline{x}, \underline{v}, \eta)\, d^3x\, d^3v$$

and usual crank:

$$\boxed{\begin{aligned} \rho(\underline{x}, \underline{v}, \eta) &= c \exp\left[-\alpha(\eta) - \beta\eta\left(\frac{v^2}{2} + \bar{\phi}\right)\right] \\ c &= \int d\eta \exp\left[-\alpha(\eta) - \beta\eta\left(\frac{v^2}{2} + \bar{\phi}\right)\right] \end{aligned}}$$

– Gibbs distribution

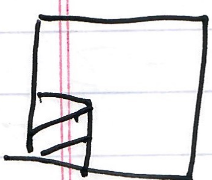
Further simplify:

- take $\eta = 0$

  or

  $\eta = \eta_0$

  i.e. single level phase density



$\eta = \eta_0$
in some part of
phase space, initially

so $\int d\eta = \sum_{\substack{\eta=0 \\ \eta=\eta_0}}$

and

$\bar{f} = \rho \eta_0$ $\left\{ \begin{array}{l} \text{linear proportionality} \\ \rho = \bar{f}/\eta_0 \end{array} \right.$

$\Rightarrow$

$$\boxed{\bar{f} = \frac{\eta_0 \exp[-\beta\eta_0(\epsilon - \mu)]}{1 + \exp[\beta\eta_0(\epsilon - \mu)]}}$$

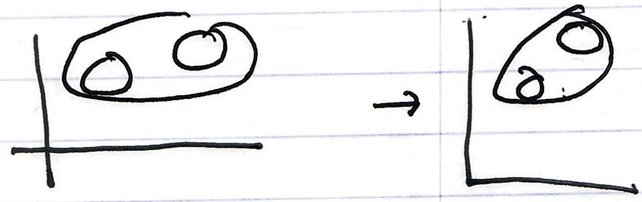
where $\epsilon = \frac{v^2}{2} + \bar{\phi}$, $\mu = -\alpha/\beta m_0$
(chem pot)

then to determine $\bar{\phi}$:

$$\boxed{\begin{aligned}\nabla^2 \bar{\phi} &= 4\pi G \int d^3v \; \bar{f} \\ &= 4\pi G \int d^3v \; \frac{m_0 \exp\left[-\beta m_0 \left(\frac{v^2}{2} + \bar{\phi} - \mu\right)\right]}{1 + \exp\left[-\beta m_0 \left(\frac{v^2}{2} + \bar{\phi} - \mu\right)\right]}\end{aligned}}$$

- Recovers ⊗ Fermi-Dirac distribution and statistics
- ϕ is same as for F-D gas of gravitationally interacting particles.
- "exclusion" ⟺ phase space density conservation (& partition!).

i.e.



can't pile blobs of phase fluid on top of each other.

f = const on trajectories, though can deform.

Partition cell ~~~~defines scale.

- in non-degenerate limit: (many, most? applications)

$\bar{f} \ll \eta_0$ i.e. small occupied phase volume.

$$\boxed{\bar{f} = c\, e^{-\beta \eta_0 G}}$$ → Maxwell - Boltzmann

N.B. Recovers L-B state of Maxwell - Boltzmann distribution without collisions.

⇒ Underpinning of Galaxy models presuming M-B distribution, Isothermal Sphere, etc.

⇒ Basic Theory of Collisionless Relaxation in (Vlasov) Stellar Dynamics.

Lots more to say .....

Also application to 2D Fluids.