Physics 239

Lecture 15 — Vlasov Equation 2

$\boxed{\rightarrow \text{B+T } 1^{st} \text{ ed.}}$

Recall:

- considered stellar dynamics, as model of Galaxy
- Galaxy $\cong$ N-body problem of classical gravity
- Collisionless

$\Rightarrow$ Vlasov - Poisson system is good working model

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f - \underline{\nabla} \phi \cdot \frac{\partial f}{\partial \underline{v}} = 0 \\ &\nabla^2 \phi = 4\pi G \int d^3v\, f \end{aligned}}$$

$\rightarrow$ phase space density conserved along stellar orbits

$\leftrightarrow$ some aspects of F-D statistics.....

→ f as density in incompressible flow

→ Collisionless ⇒ ~~Boltzmann H-Theorem~~

so begs question of what is

stationary (not necessarily "equilibrium")

distribution, i.e.

$$\underline{v} \cdot \underline{\nabla} f - \underline{\nabla} \phi \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$$\nabla^2 \phi = 4\pi G \int d^3v\, f$$

i.e. solution is not necessarily an "attractor".

stability?

a la' "BGK modes" in Plasma.

so

→ Base state in stellar dynamics

is solution of Vlasov - Poisson eqn

$\partial f / \partial t = 0$

$\Rightarrow$ How characterize?

— **Jeans Theorem**:

$f = f(\mathrm{IOM})$

i.e. E, $\underline{L}$, z_0, ϕ_0

all interesting?

Today:

① Do all IOM matter? — No!

$\Rightarrow$ strong Jeans Theorem:

Only isolating integrals matter for regular motion.

$f = f(\text{Isolating IOM})$

② Some examples of stationary states $\rightarrow$ mostly spheroidal galaxy models.

Later:

(3) What is "most probable state" or "most probable stationary solution"?

⇒ $\boxed{\text{Violent Relaxation}}$

⇔ an approach to relaxation in collisionless systems.....

Stationary Solutions - Basics

Strong Jeans Thm (Lynden-Bell '62)

> The distribution function of a steady state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion - or of the action integrals.

N.B.: A regular orbit in a system with n dofs is uniquely specified by the n isolating IOMs (in involution).

⇒ f ~ function giving the probability of finding a star (particle) on any one ~~[illegible]~~ of the phase space tori?

What does this mean?

→ Isolating Integrals:

~ reduce dimensionality of trajectory by one.

~ n d-o-fs, i isolating integrals

⇒ trajectory restricted to $2n-i$

dimensions in $2n$ phase space.

i.e. – spherical potential $V = V(r)$

– $2n = 6$ dofs

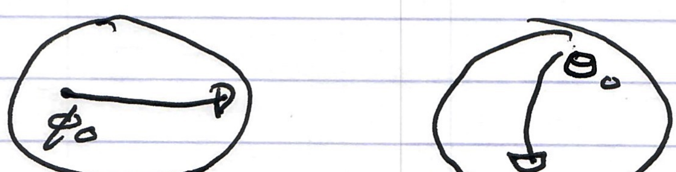
– $\underline{4}$ isolating integrals

E, L_x, L_y, L_z

so

– trajectory restricted to $2n - i = 2$ dimensions in 6 d-o-f phase space.

– Non isolating integrals are 2 initial condition angles θ_0, ϕ_0



i.e. phase on trajectory

→ keeping track of Non-isolating integrals doesn't reduce dimensionality of trajectory

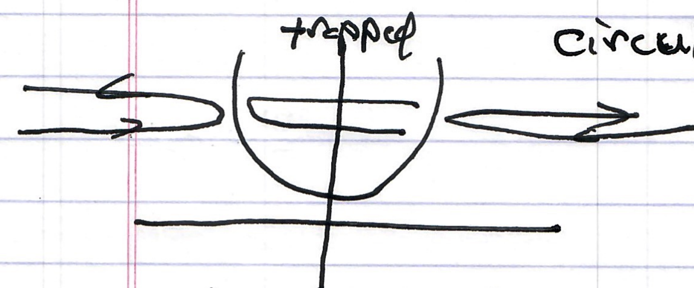
⑥ → Non-isolating integrals irrelevant in time-averaged sense.

$$\text{Expect } \underbrace{\langle f \rangle}_{\text{ensemble avg.}} \Leftrightarrow \overset{\text{time avg}}{\overline{f}} \quad \text{i.e. ergodicity}$$

→ useful.

→ In general, between 1, 2n-1 isolating integrals

→ Why "isolating" ? Consider H-O.



E "isolates" different classes of particles.

Revisiting Strong Jeans:

Equivalent: If almost all orbits regular $\rightarrow$ $\boxed{\text{non-resonant frequencies}}$

$(\underline{m} \cdot \underline{\omega} \neq 0)$, then at most 3 (indep.)

isolating integrals J_1, J_2, J_3

i.e. $\boxed{\text{3 actions}}$

N.B.: Recall:

$\rightarrow$ orbits $(\underline{J}, \underline{\theta})$; $H = H(\underline{J}, \underline{\theta})$

$$\frac{d\underline{J}}{dt} = -\frac{\partial H}{\partial \underline{\theta}}$$

$$\frac{d\underline{\theta}}{dt} = \underline{\omega}(\underline{J}) = \frac{\partial H}{\partial \underline{J}}$$

$\rightarrow$ Integrability $\Leftrightarrow$ separability of Hamilton-Jacobi equation

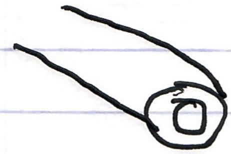
$\rightarrow$ separation constants - IoM.

6D $\rightarrow$ 3 constants $\rightarrow$ action $\underline{J}$

+3 phases IoM.

i.e. Integrability $\Rightarrow$ $\frac{dJ}{dt} = 0$

nested tori



$$\frac{d\theta}{dt} = \omega(J)$$ phase non-iso

$$\theta = \omega(J)t + \theta_0$$

And $f \sim$ probability of finding a star on any particular phase space torus

Re strong Jeans:

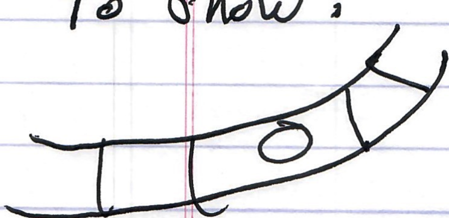
Lemma: When regular orbit is non-resonant, average time T that orbit spends in any region D of torus is $\sim V(D) \sim \int_D d^3\theta$

volume of D

i.e. Ergodic Thm (time avg ~ phase space average)

To show: f_D s/t $f_D(\theta) = 1$, $\theta \in D$

$f_D = 0$ otherwise



$$f_0(\underline{\theta}) = \sum_{\underline{n}=-\infty}^{+\infty} F_{\underline{n}} \, e^{i\underline{n}\cdot\underline{\theta}}$$

$$\int_{\text{torus}} d^3\theta \; f_0(\theta) = \int_0 d^3\theta \, f = V(\theta)$$

$$V(\theta) = \int_{\text{torus}} d^3\theta \; f_n(\underline{\theta}) = (2\pi)^3 \, F_0$$

i.e. kills all but $\underline{n} = 0$.

Now, time of dwell in θ:

$$T_T(\theta) = \frac{1}{T}\int_0^T dt \, f_0(\theta(t))$$

$$= \frac{1}{T}\int_0^T dt \sum_{\underline{n}} F_{\underline{n}} \, e^{i\underline{n}\theta}$$

$$= \frac{1}{T}\int_0^T dt \sum_{\underline{n}} F_{\underline{n}} e^{i\underline{n}\cdot\underline{\Omega} t} \, e^{i\underline{n}\cdot\underline{\theta}(0)}$$

i.e. oscillates

$$= F_0 + \frac{1}{T}\sum_{\underline{n}\neq 0} e^{i\underline{n}\cdot\theta(0)} \; \frac{F_n e^{i(\underline{n}\cdot\underline{\Omega})T} - 1}{i\underline{n}\cdot\underline{\Omega}}$$

if $\underline{n}\cdot\underline{\Omega} \neq 0$ (non-resonant!) $\Rightarrow$

3 isolating integrals

$$\Rightarrow \boxed{\lim_{T\to\infty} T_T(D) = F_0 = V(D)/(2\pi)^3}$$

$\underline{n}\cdot\underline{\omega} \neq 0 \Rightarrow$ Volume (D) ~ Dwell Time (D)

phase space

Ergodic Thm ✓ i.e. in practical terms

time avg = ensemble average.

N.B. - if resonance: $n_1\Theta_1 + n_2\Theta_2 + n_3\Theta_3 = 0$

some n_1, n_2, n_3

$\Rightarrow$ additional isolating integral!

Star confined to phase space island.

- if multiple overlapping resonances

$\Rightarrow$ chaos!

$$\boxed{\begin{array}{l}\text{Vlasov equation becomes } \underline{\text{Fokker-Planck}}.\\ \sim \underline{\text{Flattening}} \text{ (Plateau - 1D)}\end{array}}$$

Use: → calculating expectations, ⇒

So, strong Jeans: ⇒ for any observable Q:

$$\langle Q \rangle = \int d^3x \int d^3v \; Q \, f$$

$f \to f(\underline{x}, \underline{v}, t)$

$Q(\underline{x}, \underline{v})$
smooth, time indep

in stationary state,

$$f(\underline{x}, \underline{v}, t) \to \bar{f}(\underline{x}, \underline{v})$$

$$\to \bar{f}(\underline{\theta}, \underline{J})$$

$$\langle Q \rangle = \bar{Q} = \int d^3\theta \int d^3J \; Q(\underline{\theta}, \underline{J}) \, \bar{f}(\underline{\theta}, \underline{J})$$

Now, $\bar{f} = \lim\limits_{T \to \infty} \dfrac{1}{T} \displaystyle\int_0^T dt \, f(\underline{x}, \underline{v}, t)$

but, by Ergodic thm (the Lemma) ⇒

$$\boxed{\bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, f(\underline{x}, \underline{v}, t)}$$

time avg = ensemble avg.

$\therefore$ $\bar{f}\, d^3\underline{J}\, d^3\underline{\theta} \Rightarrow$ probability in $d^3\underline{J}\, d^3\underline{\theta}$ volume

$$\sim \frac{d^3\underline{\theta}}{(2\pi)^3} \text{ probability actions in } d^3\underline{J}$$

$$\sim \frac{d^3\underline{\theta}}{(2\pi)^3} \left[d^3\underline{J}\, f_J(\underline{J}) \right]$$

$$\sim \frac{d^3\underline{\theta}}{(2\pi)^3}\quad d^3\underline{J} \,\langle f(\underline{J}) \rangle$$

$$\downarrow$$

$$F_0$$

$$\boxed{\langle \underline{Q} \rangle = \int_V d^3\theta \int d^3 J \quad Q\,(\underline{\theta}, \underline{J})\; f_J(\underline{J})}$$

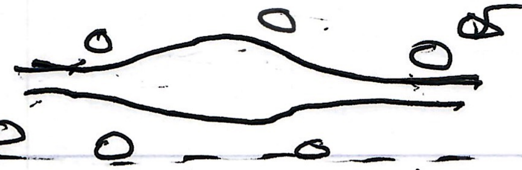
as, as intuitive.

i.e. $\bar{f}(\underline{\theta}, \underline{J}) = \frac{1}{T}\int_0^T dt\; f(\underline{x}, \underline{v}, t)$

$$= f_J(\underline{J}) \Rightarrow \text{prob. on torus at } \underline{J}.$$

Regard $f_J(\underline{J}) \Rightarrow \langle f(\underline{J}) \rangle$.

(2) ↦ Globular cluster



→ Parker

→ **Spherical Systems** (symmetry)

→ spheroidal galaxies

easiest

$$f = f(E, |L|)$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G \int d^3v \, f\left(\frac{v^2}{2} + \Phi, \; |\underline{r} \times \underline{v}|\right)}$$

Eqn. for stationary state of Vlasov-Poisson system.

Now, to clarify:

$\Psi = -\Phi + \Phi_0$ relative pot.

$\mathcal{E} = -E + \Phi_0$

energy $= \psi - \frac{v^2}{2}$

b.c. $\psi \to \Phi_0$ at ∞.

then Poisson $\Rightarrow$ $\boxed{\nabla^2 \psi = -4\pi G \rho}$

Now, if $f = f(\mathcal{E})$

$$= f\left(\psi - v^2/2\right)$$ for isotropic dispersion

$\Rightarrow$

$$\boxed{\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) &= -16\pi^2 G \int_0^{\sqrt{2\psi}} f\left(\psi - v^2/2\right) v^2 d^3 v \\ &= -16\pi^2 G \int_0^{\psi} f(\mathcal{E}) \sqrt{2\Big(\psi - \underbrace{v^2/2}_{\mathcal{E}}\Big)}\, d\mathcal{E} \end{aligned}}$$

Coupled velocity, spatial structure.

- ϕ_0 s/t $f(\mathcal{E}) \to 0$ for $\mathcal{E} < 0$

i.e. untrapped/unbound gone

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = -16\pi^2 G \int_0^{\psi} f(\mathcal{E}) \sqrt{2(\psi - \mathcal{E})}\, d\mathcal{E}}$$

- nonlinear equation for ψ

i.e. velocity $\to$ position $\to$ $\begin{cases} \text{Plummer} \\ \text{King} \\ \text{Isothermal sphere.} \end{cases}$

f

nb after stars

or, equivalently:

→ Linear equation for $f(\mathcal{E})$ given Ψ

position → velocity

(Eddington)

⇒ ∃ a zoology of time-independent Vlasov solutions. See B & T

- sample a few here

- Asymmetry escalates complexity rapidly

(N.B. Plasma → BGK industry is 1D)

B_0 strong

- Stability? → Jeans?

- More Useful: Most Probable Solution?

→ but how measure probability?

→ Violent Relaxation.

Polytropes → stellar structure.

Some Examples

(after stars)

→ Polytropes / Plummer

$$f(\mathcal{E}) = F\mathcal{E}^{n-3/2} \qquad \mathcal{E} > 0$$

$$= 0 \qquad \mathcal{E} < 0$$

no untrapped.

$$\rho = 4\pi \int_0^\infty f\left(\Psi - \frac{v^2}{2}\right) v^2 dv$$

$$= 4\pi F \int_0^{\sqrt{2\Psi}} (\Psi - v^2/2)^{n-3/2} v^2 dv$$

Now, $v^2 = 2\Psi \cos^2\theta$

$$\rho = c_n \Psi^n$$

$$c_n = 2^{7/2} F\pi \left[\int_0^{\pi/2} \sin^{2n-2}\theta \, d\theta - \int_0^{\pi/2} \sin^{2n}\theta \, d\theta \right]$$

$$= (2\pi)^{3/2} (n - 3/2)! \, F / n!$$

⇒ Nonlinear Poisson Equation:

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) = -4\pi G c_n \psi^n}$$

and further:

$$s = r/b$$

$$\psi = \psi/\psi_0$$

$$\Psi_0 = \psi(0)$$

$$b \equiv \left(4\pi G \Psi_0^{n-1} c_n\right)^{-1/2}$$

⇒
$$\boxed{\begin{aligned}\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi}{ds}\right) &= -\psi^n & \psi > 0\\ &= 0 & \psi \leq 0\end{aligned}}$$

Lane-Emden Equation

b.c. $\left.\frac{d\psi}{ds}\right|_0 = 0$

i.e. no gravitational force center.

$\psi(0) = 1$, defn.

→ Lane-Emden familiar from stellar structure

i.e. hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr}$$

$$P = K\rho^{\gamma}$$

↳ Polytropic gas

so

$$K\gamma\, \rho^{\gamma-2} \frac{d\rho}{dr} = \frac{d\Phi}{dr}$$

$\Phi = 0$, edge

$$\rho^{\gamma-1} = \frac{\gamma-1}{K\gamma}\, \Psi$$

so

$$\boxed{\rho = \left(\frac{\gamma-1}{K\gamma}\right)^{1/\gamma-1} \psi^{1/\gamma-1}}$$

$$\boxed{\rho = c_n \psi^n}$$

$n \Leftrightarrow 1/\gamma-1$

clear correspondence

i.e. density of stellar dynamics/ Vlasov polytrope sphere same as gas sphere with $\gamma_{eff} = 1 + 1/n$

Solutions ?

→ in general, unknown.

→ $n = 1$ → Helmholtz ✓

→ $n = 5$ → Schuster

$$\Psi = 1/\left(1 + \frac{\xi^3}{3}\right)^{1/2}$$ → solves L-E with $n = 5$

so $$\rho = c_5 \Psi^5$$
$$= c_5 \Psi_0^5 / \left(1 + \xi^2/3\right)^{5/2}$$ → Plummer Model

density finite everywhere.

M finite ¡ ⟺ $M = \sqrt{3}\, \Psi_0 b / G$

→ Plummer Model fits globular clusters moderately well

→ generally falls off too fast $(\rho \sim r^{-5})$ for many

→ M finite, $\rho \neq 0$ everywhere

Now, $n < 5$ — $\rho \to 0$, finite radius (?!)

$n = 5$ — ρ finite; M finite — Plummer

$n > 5$ — ρ decays slowly; M ∞

⇒ Isothermal Sphere

$n \to \infty$, $\gamma = 1$ → isothermal sphere

$$f(\mathcal{E}) = \rho_1 e^{\mathcal{E}/\sigma^2} \Big/ (2\pi\sigma^2)^{3/2}$$

$$= \frac{\rho_1 e^{(\Psi - v^2/2)/\sigma^2}}{(2\pi\sigma^2)^{3/2}}$$

Choice of velocity dispersion

$$\rho = \rho_1 e^{\Psi/\sigma^2}$$

so

$$\boxed{\frac{d}{dr}\left(r^2 \frac{d \ln \rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho}$$ (*)

but for $\gamma = 1$ polytrope,

$$P = \frac{k_B T}{m} \rho$$

hydrostatics

$$\frac{dP}{dr} = \frac{k_B T}{m} \frac{d\rho}{dr} = -\rho \frac{G M(r)}{r^2}$$

so

$$\boxed{\frac{d}{dr}\left(r^2 \frac{d \ln \rho}{dr}\right) = -\frac{G m}{k_B T} 4\pi r^2 \rho}$$ (**)

(*) same as (**)

so, for $n \to \infty$, $f(\mathcal{E}) = \exp[\mathcal{E}/\sigma^2]$ is collisionless system with same structure as collisional one.

→ Gaussian with constant dispersion ⇒ isothermal sphere structure.

To solve,

$$\rho \sim r^{\alpha}$$

$$r^2 r^{\alpha} \sim \frac{d}{dr}\left(r^2 r^{\alpha} r^{\alpha-1}\right)$$

$$r^{\alpha+2} \sim \frac{d}{dr}(r)$$

⇒ $\alpha = -2$ is solution

$$\boxed{\rho(r) = \sigma^2 / 2\pi G r^2}$$

singular density
→ resolve by core
isothermal sphere

- Industry of stationary state models (BGK solutions) → ∞.

- Rapid rise in complexity for disks, etc.
- See B&T for more, Chapt 4

Next:

- Jeans Equations (Moments of Vlasov)

  $\rightsquigarrow$ closure ?

  $\downarrow$

  data / empiricism

- Virial Theorem
- Violent Relaxation