Physics 289

Part II - Galaxies and Galactic Dynamics

Lecture 14

- Vlasov Equation 1.

Material: - Binney and Tremaine, "Galactic Dynamics" (encyclopaedia) (*)

- Binney and Merrifield, "Galactic Astronomy" - qualitative, zoological

- F. Shu, Vol. 2, especially Chapters 11, 12 on spirals (*)

- Luca Ciotti, "Introduction to Stellar Dynamics"

- F. Shu, "The Physical Universe" (*)
  - UG general Astro. book with a kick. Good treatment of galaxies, spirals.

⇒ - Phys 218a Notes: Fall 2018
  Dept. site - large overlap between GD and Vlasov - Poisson plasma.

c.f. Lifshitz and Pitaevski, "Physical Kinetics"

Useful
Web sites

astro.yale.edu/vdbosch/galdyn.html (*)

astro.utu.fi/~cflynn/galdyn/teaching_galdyn.html

[courses-archive.maths.ox.ac.uk/node/44909
or
www-thphys.physics.ox.ac.uk/people/JohnMag-
orrian/cm21/
(no hyphen).

Topics:

- What? Time Scale Ordering, Models
- Vlasov Equation, Vlasov - Poisson system.

  (Vlasov = Collisionless Boltzmann = Jeans)

  origin, meaning, derivation(?)
- Jeans Eqns and Application Virial Thm.
- Orbits (basics) - c.f. 200q
- IOM, Jeans Thm., BGK solution - structure (disk, spheroidal)
- Relaxation: Phase Mixing, Landau Damping
  (n.b. collisionless) Violent Relaxation (Lynden-Bell)
- ? Jeans Instability - Variational Principle ?
- Spiral Density Waves (warning: B+T is weak here)
  - Lin-Shu Hypothesis
  - Dispersion relations, resonances
  - amplification, propagation
  - relation to accretion (Lynden Bell and Kalnajs)
  - why ubiquitous?

N.B. : → The spiral wave, which clearly propagates and transports angular momentum, allows galaxy to increase gravitational binding energy by disposing of angular momentum outward.

→ In accretion disk:

accretion ⟷ angular momentum transport

↓

B-field, MRI - specific case of viscosity → / viscous stress

In galactic disk:

concentration stars at the galactic center ("churning") ⟷ angular momentum transport ("burning") outward

↓

Spiral Wave

∴ Spiral structure discussion unifies Part I and Part II of course.

- Galaxies

⇒ for astronomical perspective, read M+B.

⇒ for physics perspective:

galaxy = stars + gas + B-field + CR + Dark Matter

$\simeq$ stars + (DM) + ..

↓

dominate mass

$\boxed{\text{N-body system, } N \gg 1\text{, classical gravitating point particles.}}$

⇒ collisionless → what is equilibrium?

⇒ hierarchical structure:

stars → galaxies → cluster → ..

⇒ Basic Issues:

① – structure ↔ eqbrm.

spheroidal, irregular, disk, [bars, bulge] triaxial,

N.B. As collisionless, CBE / VE governs:

$f = f(\underline{x}, \underline{v}, t)$

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f - \underline{\nabla}\phi\cdot\frac{\partial f}{\partial \underline{v}} = 0$$

$$\nabla^2\phi = 4\pi G\int d^3v\, f$$

$\partial f/\partial t \to 0 \;\Rightarrow$ equilibrium ↔ "BGK mode/state"

$f = f(\text{IOMs})$

not a Maxwellian!

N.B. In classic Landau Problem

$$f = \underbrace{\langle f\rangle}_{\text{Local Maxwellian Eqbrm}} + \delta f$$

set by $\dfrac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \dfrac{q}{m}\underline{E}\cdot\dfrac{\partial f}{\partial \underline{v}} = C(f)$

δf satisfies:

$$\partial_t \delta f + v\partial_x \delta f + \frac{q}{m}\tilde{E}\partial_v \delta f = -\frac{q}{m} E \partial_v \langle f \rangle$$

Max.

$$\nabla^2 \tilde{\phi} = -4\pi q \int d^3v \, \delta f$$

② Patterns, stability

→ Jeans instability - collisionless

→ "What makes those pretty spiral arms?" - ubiquitous

- gravitational instability in differentially rotating system

⇒ - "Spiral density waves"

③ Models:

- Vlasov - Poisson - derivation?

- Jeans Equations - moments V-P. Closure?

- Gas dynamics (!?).

N.B. Vlasov equation introduces kinetic phenomena — Landau damping, etc.

and much more ....

→ Time Scales / Vlasov Eqn.

Recall: dilute gas: (Boltzmann Eqn works)

$$d < \bar{r} \sim n^{-1/3} < l_{mfp} < L$$

d: range force; $\bar{r}$: interparticle spacing; l_{mfp}: $1/n\sigma$ mean free path; L: system.

small parameter: $d^3 n \ll 1$

Plasma:

$$\bar{r} < \lambda_D < l_{mfp} < L$$

λ_D: Debye length (screening)

small parameter: $1/n\lambda_D^3 \ll 1$

Given small parameter, can convert:

Liouville Eqn. for $f_N(x_1, p_1, \ldots x_N, p_N)$

to $\left.\begin{matrix} BE \\ CBE \end{matrix}\right\}$ for $f(x, p)$

i.e. Liouville

$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^{N} \underline{v}_i \cdot \frac{\partial f^N}{\partial \underline{x}_i} - \frac{\partial f^N}{\partial \underline{p}_i} \cdot \sum_{j<i} \frac{\partial V_{ij}}{\partial x_i} = 0$$

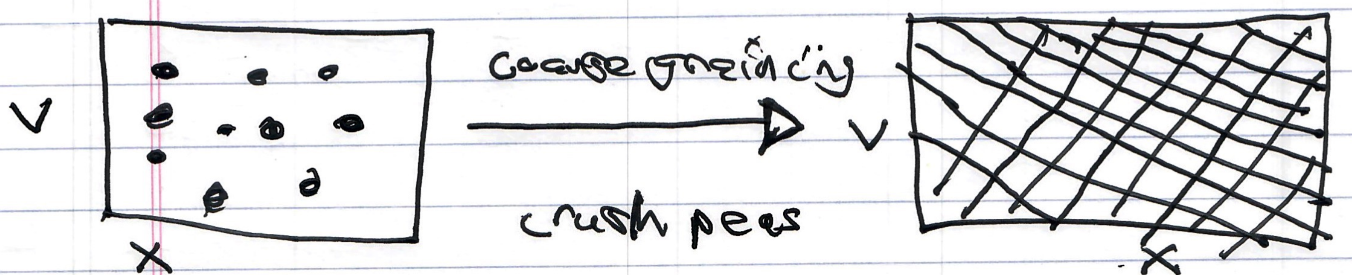
$$\begin{cases} \dot{x}_i = \underline{v}_i \\ \underline{\dot{p}}_i = - \partial_j \sum_{j<i} V_{ij} \end{cases}$$

$\rightarrow$

$$\begin{cases} \frac{\partial f(x, v, t)}{\partial t} + \underline{v} \cdot \nabla f - \underline{\nabla} \phi \cdot \frac{\partial f}{\partial \underline{v}} = C(f) \\ BE, \ CBE \end{cases}$$

– Procedure is "BBGKY Hierarchy"

c.f. see B+T, L+P, 210b (2020) notes.



peas
↕
stars here

pea soup
(after T.H. Dupree)
→ phase space fluid

Vlasov Equation is Continuity equation for the (fluid of the pea soup) in phase space.

i.e. $\partial f + \nabla\cdot(f\underline{V}) = 0$

$$\frac{\partial f}{\partial t} + \underline{V}\cdot\underline{\nabla} f + f\,\underline{\nabla}\cdot\underline{V} = 0$$

Here $\underline{V} = \begin{pmatrix} \dot{\underline{x}} \\ \dot{\underline{p}} \end{pmatrix} \rightarrow \begin{pmatrix} \dot{\underline{q}} \\ \dot{\underline{p}} \end{pmatrix}$ [phase space flow

Then: $\underline{\nabla} \cdot \underline{V} = \frac{\partial}{\partial \underline{x}} \dot{x} + \frac{\partial}{\partial \underline{p}} \dot{\underline{p}}$

$$= \frac{\partial}{\partial \underline{x}} \underline{p} + \frac{\partial}{\partial \underline{p}} \cdot (-\underline{\nabla}\phi)$$

Liouville's Thm $= 0$

$\Rightarrow$

Phase space flow incompressible. No sources or sinks

$$\frac{\partial}{\partial t} \rho + \underline{V} \cdot \underline{\nabla} \rho + \rho \cancel{\underline{\nabla} \cdot \underline{V}} = 0$$

$$f(\underline{x}, \underline{v}, t) = \rho$$

density of phase space fluid

$$\underline{V} \cdot \underline{\nabla} = \underline{v} \cdot \frac{\partial}{\partial \underline{x}} - \underline{\nabla}\phi \cdot \frac{\partial}{\partial \underline{v}}$$

so

$$\boxed{\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f - \underline{\nabla}\phi \cdot \frac{\partial f}{\partial \underline{v}} = 0 \qquad \text{Vlasov Eqn.}}$$

→ describes phenomena on t s/t

$t < t_{coll} \sim (v/l_{mfp})^{-1}$.

→ ϕ determined self-consistently

$$\nabla^2\phi = 4\pi G \int f \, d^3v$$

Vlasov - Poisson system.

But: For classical gravity, "small parameter" is far from clear

i.e.

- no "d" in gravity → long range, scale free force

- no "λ_D" in gravity → only 1 sign of "charge", no screening.

* - no rigorous truncation of BBGKY hierarchy for gravity.

See: Pgs. 497-500 B+T
dodges the key issue.

- Can <u>physically</u> argue for truncation
  by demonstrating/showing 'weak correlation'
  so $\quad f(1,2) \to f(1)\,f(2)$

What can be said about basic time scales?

$\tau_* \sim r_0/\bar{v} \;\to$ time of collisional interaction

r_0 ↓ typical range of (strong) collision – info lost, entropy created

$\tau_0 \sim l_{mfp}/\bar{v} \;\to$ time between collisions

$\Theta_0 \sim L/\bar{v} \;\to$ macro-relaxation time

Estimating the scales

① $r_0 \sim b$ of $\quad \dfrac{GM}{b} \sim V^2 \qquad$ distance closest approach

$r_0 \sim GM/V^2$

so $\boxed{\tau_* \sim Gm/V^2}$ → time of collision

③ $\tau_0 \simeq \frac{l_{mfp}}{V}$, $l_{mfp} \sim 1/n\sigma$

For star cluster: scale R

$n = \frac{N}{R^3}$ → # stars → Volume

$\sigma \simeq \pi b^2$

$$l_{mfp} = \frac{R^3}{\pi N b^2} = \frac{V^4}{\pi (Gm)^2 N}$$

$$\boxed{\tau_0 = V^3 R^3 / G^2 m^2 N}$$

For $\tau_* < \tau_0$

$$\frac{Gm}{V^3} \ll \frac{V^3 R^3}{G^2 m^2 N}$$

collision interaction time ≪ time between collisions → diluteness

(heuristic)

so

$$\frac{G^3 m^3}{R^3} \ll \frac{v^6}{N}$$

$\Rightarrow$

$$\frac{Gm}{R} \ll \frac{v^2}{N^{1/3}}$$

$$\underbrace{\frac{GmN}{R\,v^2}}_{\uparrow\ \text{!}} \ll N^{2/3}$$

but "virialization" $\Rightarrow$ approximate bulk energy balance for cluster

$\Downarrow$

$$\frac{GmN}{R} \sim \overline{v^2}$$

then

$$\left.\begin{aligned} \tau_* &\cong \tau_G N^{-3/2} \\ \tau_0 &\cong \tau_G N^{1/2} \end{aligned}\right\} \quad \tau_G \sim (R^3/Gm)^{1/2}$$

time scale for virialization $N \gg 1$.

Note : $\rightarrow \tau_{*_u} \ll \tau_0$

$\rightarrow$ approximate "diluteness"

Now, 3rd scale:

$$\Theta_0 = R / \bar{V}$$

but

$$\frac{\tau_0}{\Theta_0} \simeq \frac{\tau_G N^{1/2} \bar{V}}{R}$$

but $\bar{V} \sim \left(\frac{GmN}{R}\right)^{1/2}$

$$\frac{\tau_0}{\Theta_0} \simeq \tau_G \frac{N^{1/2}}{R} \left(\frac{Gm}{R}\right)^{1/2} N^{1/2}$$

$$\simeq \cancel{\left(\frac{R^3}{Gm}\right)^{1/2}} \frac{N}{\cancel{R}} \cancel{\left(\frac{Gm}{R}\right)^{1/2}} \sim N$$

i.e. Time between collisions greatly exceeds streaming time thru macroscopic scale

So ① Can argue for 'diluteness' effectively

② No rigorous BBGKY truncation

③ Macro-dynamics are collisionless

$\Rightarrow$

$$\boxed{\text{Vlasov Equation / Vlasov - Poisson system is good working model of Galaxy, or Stellar Dynamics (N pt. system)}}$$

Comments on Vlasov Equation:

– VE $\rightarrow$ phase space density conserved along particle orbits

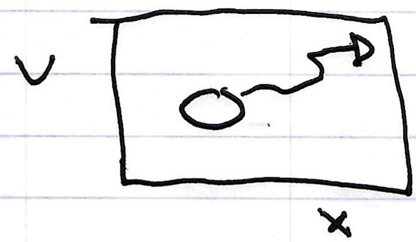
$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f - \nabla \phi \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$$\underline{v} = \frac{d\underline{x}}{dt} \quad , \quad -\nabla \phi = \frac{d\underline{v}}{dt}$$

$$\frac{\partial f}{\partial t} + \frac{d\underline{x}}{dt} \cdot \underline{\nabla} f + \frac{d\underline{v}}{dt} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

so

$$\frac{d}{dt} f(\underline{x}, \underline{v}, t) = 0$$



conservation of f for Hamiltonian EOMs

- characteristic equations of Vlasov equation (PDE) are EOMs (Hamiltonian) for particles (stars)

- VE conserves entropy density:

$$s = \int d^3v\,(-f \ln f) \quad , \quad S = \int d^3x\, s$$

$$\frac{ds}{dt} = \int d^3v \left[-f \frac{1}{f} \frac{df}{dt} - \ln f \frac{df}{dt} \right]$$

$$= 0.$$

i.e. Entropy production / H Thm is due to $C(f)$ in BE.

- Enter wave-particle resonance.
  (mode-star)

i.e if solution to:

$$\underline{v} \cdot \nabla f - \nabla \underline{\phi} \cdot \frac{\partial}{\partial \underline{v}} = 0$$

$$\nabla^2 \phi = \int 4\pi G f \, d^3 v$$

$\Rightarrow f_0$. (take indep. position, convenience)

so $f = f_0 + \delta f$

Then, for stability

$$\frac{\partial}{\partial t} \delta f + \underline{v} \cdot \underline{\nabla} \, \delta f = \underline{\nabla} \tilde{\phi} \cdot \frac{\partial f_0}{\partial \underline{v}}$$

$$\nabla^2 \tilde{\phi} = 4\pi G \int d^3 v \, \tilde{f}$$

$$f = f_{k,\omega} \, e^{i(\underline{k} \cdot \underline{x} - \omega t)}$$

$$-i(\omega - \underline{k}\cdot\underline{v}) f_{\underset{\omega}{k}} = i\underline{k}\,\tilde{\phi}_{\underset{\omega}{k}} \cdot \frac{\partial f_0}{\partial \underline{v}}$$

$$f_{\underset{\omega}{k}} = -\tilde{\phi}_{\underline{k},\omega} \frac{\underline{k}\cdot\partial f_0/\partial\underline{v}}{\omega - \underline{k}\cdot\underline{v}}$$

what happens at $\omega = \underline{k}\cdot\underline{v}$?

$$-k^2 \tilde{\phi}_{k,\omega} = 4\pi G \int d^3v\, \tilde{f}_{\underset{\omega}{k}}$$

$\Rightarrow$ dispersion relation:

$$d(\underline{k},\omega) = 1 - \int d^3v \frac{\underline{k}\cdot\partial f_0/\partial\underline{v}}{\omega - \underline{k}\cdot\underline{v}} \left(\frac{4\pi G}{k^2}\right)$$

$\downarrow$

$\epsilon(k,\omega)$ for plasma

$$\boxed{d(\underline{k},\omega) = 1 - \frac{4\pi G}{k^2}\int d^3v \frac{\underline{k}\cdot\partial f_0/\partial\underline{v}}{\omega - \underline{k}\cdot\underline{v}}}$$

$d(\underline{k},\omega) = 0 \Rightarrow$ modes, stability

New features:

- velocity space dependence

- resonance: $\omega = \underline{k} \cdot \underline{v}$

  wave - particle

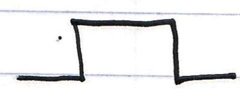
  ⇒ 'Landau damping' — exchange of energy wave - particle

- $$\frac{1}{\omega - \underline{k}\cdot\underline{v}} = \frac{P}{\omega - k\cdot\underline{v}} - i\pi\delta(\omega - \underline{k}\cdot\underline{v})$$

  introduces dIM. ⇒ Landau Damping

- Exercise:

ⓐ - 1D



- $f_0 = \dfrac{\bar{f}}{2\Delta V}$ for $V_0 - \Delta V$ [redacted] $< V < V_0 + \Delta V$

- Jeans stability ?

(b) → 1D

- $f_0 = f_{max}(v/v_{Th})$

- stability for $\frac{\omega}{k} \gtrsim v_{Th}$?

More on Landau Damping later...

→ Using the Vlasov Eqn.:

Vlasov Equation

(2) Moment Equations
closed via measurement
Jeans Eqn. - macroscopic
Virial Eqn. — gross descr.

(1) Equilibrium Theory:
$$\nabla^2\phi = 4\pi G\int d^3v\, f(\mathrm{IOM}(\phi))$$

(1) Jeans' Thm: Solution of Vlasov Equation is Function of Integrals of Motion (IOM).

$I = I(\underline{x}(t), \underline{v}(t)) \equiv$ IOM for (all) trajectories $\underline{x}(t), \underline{v}(t)$.

IOM

$$\left.\frac{dI}{dt}\right|_{\text{traj.}} = 0 \qquad \frac{\partial I}{\partial t} + \frac{d\underline{x}}{dt}\cdot\frac{\partial I}{\partial \underline{x}} + \frac{d\underline{v}}{dt}\cdot\frac{\partial I}{\partial \underline{v}} = 0$$

and $$\frac{df}{dt} = 0$$

$\Rightarrow$ $f = f(I)$ solves Vlasov Eqn.

Key Point:

For equilibrium, need solve:

$$\nabla^2\phi = 4\pi G\int d^3v\, f$$

As $f = f(\text{IOM}) = f(\text{IOM}(\phi))$, understanding IOM constrains parametrization of f and nature of equilibrium.

→ Jeans thm. not particularly useful.

⇒

→ Strong Jeans Thm. (Lynden-Bell, '62)

The distribution function f of a steady-state system in which almost all orbits are regular can be written as a function of the independent isolating integrals of motion, or of the action-integrals.

N.B.: A regular orbit in a system with n dofs is uniquely specified by the n isolating IOM (in involution).

⇒ f ~ function giving the probability of finding a star on each of the phase space tori.

What does this mean? TBC.

ie. if $V = V(r)$ - spherical potential

Isolating integrals: $E, \underline{L} \rightarrow E, L_x, L_y, L_z$

More generally:

→ Isolating Integrals -

~ reduce dimensionality of trajectory by one

~ n d-o-fs, i isolating integrals

⇒ trajectory restricted to $2n-i$ dims

in $2n$ phase space.

→ Spherical Potential: $n=3$
$i=4$

→ trajectory specified by 2 coords.

→ In general, between 1, $2n-1$ isolating integrals.