Physics 239

<u>Lecture 13</u> – Planetesimals 2

N.B.: $\boxed{\text{Goldreich \& Ward '73 must read}}$

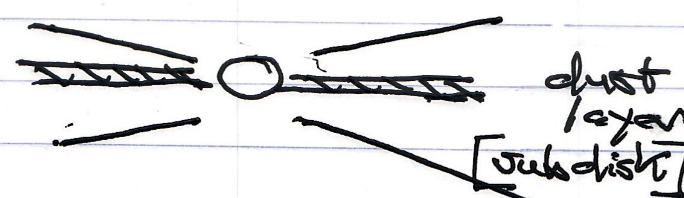
Recall:

→ Planetesimal Problem:

How 1 μ grain → 1 km rock in turbulent disk

– Processes: Sedimentation, Coagulation and Fragmentation } simultaneous

in Turbulence.



Disk → Subdisk

– Key Physics: Aerodynamic Drag

Epstein: $\ell < \ell_{mfp}$ $\quad \underline{F_D} \sim -\rho s^2 v_{th} \underline{V}$

Stokes: $\ell > \ell_{mfp}$ $\quad F_D \sim -\rho \nu s \underline{V}$

⇒

– Settling Time and Settling velocity

so $$t_{fric} \sim \frac{s\,\rho_d}{\rho\, v_{th}}$$

$$\sim 3\text{ sec}$$

small particles tightly coupled to gas.

$$V_{settle} \sim \frac{\rho_d}{\rho}\frac{s}{v_{th}}\Omega^2 Z \qquad \sim .06\text{ cm/sec}$$

$$t_{settle} \sim \frac{Z}{V_{set}} \sim \frac{\Sigma}{\rho_d s\,\Omega}\exp\left[-Z^2/2H^2\right]$$

~~[illegible]~~

$\sim 10^5$ yrs < age disk

and coagulation:

$$\begin{cases} \dfrac{dm}{dt} = \dfrac{3}{4}\dfrac{\Omega^2 f}{v_{th}} Z\, m \\[2ex] \dfrac{dZ}{dt} = V_{set} = -\dfrac{\rho_d}{\rho}\dfrac{s}{v_{th}}\Omega^2 Z \end{cases}$$

$\rightarrow$ grain grows by sweep-and-stick

$\Rightarrow$ $\boxed{\text{Grains grow to few mm} \rightarrow \text{cm on midplane in } 10^3\text{-}10^4 \text{ yrs.}}$

→ But turbulence ?

→ turbulent field → vertical

$$\frac{dz}{dt} = \tilde{V}_z(x,t) - \frac{\Omega^2 z}{\gamma}$$

so Schmoluchowski Equation:

1D vertical only

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial z}\left[\left(\frac{\Omega^2 z}{\gamma} n\right) + \frac{\partial}{\partial z} D_{z,z} n\right]$$

can expand to z, r.

$n \equiv$ dust density

$$D_{z,z} \sim \int_0^{\infty} \langle \tilde{V}_z(0)\, \tilde{V}_z(s) \rangle$$ → vertical diffusivity

$$h_d \sim \left[\left(D_{z,z}/\Omega^2\right)\gamma\right]^{1/2}$$

$$\left\{ \begin{array}{l} \frac{\partial n}{\partial t} = 0 \\ \Gamma_z = 0 \end{array} \right.$$

h_d ↓ thickness dust layer.

$$\boxed{\frac{h_d}{H} \sim \left[\alpha \Big/ \underbrace{\Omega\, t_{eno}}\right]^{1/2},}$$

thickness of disk.

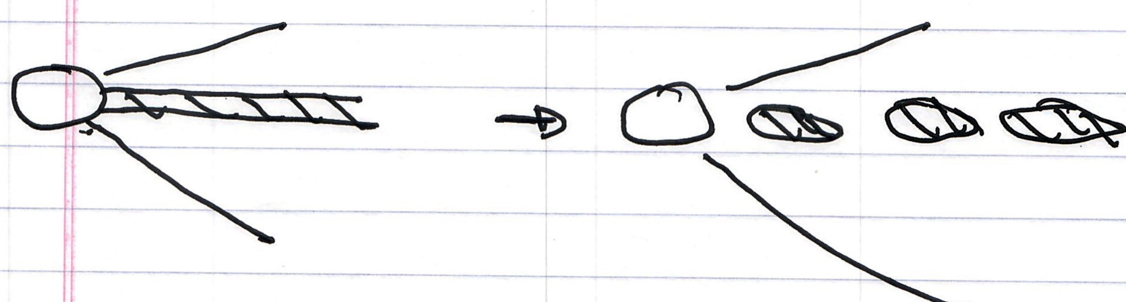
Need $\Omega t_{fric} > \alpha$ for $h_d \ll H$

$\Rightarrow$ For realistic α need substantial particle growth (so $\Omega t_{fric} > \alpha$) ($t_{fric} \sim a^3$ - Epstein) prior settling..

Now: What Else ?

N.B.: Big picture: subdisk layer goes Jeans/ gravitationally unstable

$\Rightarrow$ mass clumps $\rightarrow$ planetesimals



$$\omega^2 = \Phi - 2\pi G \Sigma |k_r| + k_r^2 c_s^2$$

$\rightarrow$ sedimentation driven

$\Rightarrow$ break up to rings, etc.

→ Radial Drift

- Have developed the theory for vertical sedimentation, now consider radial drift.

- Observations: → Need take care re: speeds gas, speeds grains, rocks.

→ Disk supports radial pressure gradient
why now? → modest difference between gas flow and Keplerian V_ϕ

$$\frac{V_\phi^2}{r} = \frac{GM}{r^2} + \frac{1}{\rho_g}\frac{dP_g}{dr}$$

if $P = P_0 r^{-n}$

$$\Rightarrow \quad V_\phi^2 = \underbrace{V_K^2}_{\text{Keplerian velocity}} - n \underbrace{\frac{P_g}{\rho_g}}_{c_s^2}$$

$$V_\phi^2 = V_K^2 - n c_s^2$$

sub Keplerian $\boxed{V_\phi = V_K(1-\eta)^{1/2}}$ $\boxed{\eta = \frac{n c_s^2}{V_K^2}}$

Then:

→ small particles dust tightly coupled to gas ($\Omega t_{fric} \ll 1$)

↓

→ will be swept along with gas

↓

→ speed will be sub-keplerian, so

↓

→ won't be able to balance gravity

⇒ $\boxed{\text{spiral inward}}$ → inward radial drift

→ large particles

→ naturally ~ Keplerian

but

→ gas sub-Keplerian so

↓

⇒  large particle feels headwind of gas

→ drag → loss angular momentum.

∴ Large particles "donkeys" ⇒

spiral inward → $\boxed{\text{inward radial drift}}$

→ grains, planetesimals tend to spiral / drift inward to star.

→ will diffuse radially as well

⇒ 2D Fokker-Planck Problem.
(r, z)

Here → radial drift

→ Calculation – Radial Drift

Consider equations for particulate motion on $z=0$ plane

planetesimal

$$\frac{dV_r}{dt} = \underbrace{\frac{V_\phi^2}{r}}_{\text{centrifugal}} - \overbrace{\Omega_K^2 r}^{\text{gravity}} - \frac{1}{t_f}\overbrace{(V_r - V_{r\,gas})}^{\text{accretion}}$$

t_f → frictional rate (Epstein, Stokes)

centrifugal ⊖

→ decay to gas speed (via coupling)

$$\frac{d}{dt}(rV_\phi) = -\frac{r}{t_f}(V_\phi - V_{\phi\,gas})$$

To simplify: estimate $\frac{d}{dt}(rV_\phi)$

Now, $\frac{d}{dt}(rV_\phi) \cong V_r \frac{d}{dr}(rV_K)$ $\quad$ $V_\phi \sim V_K$

$$\simeq \frac{V_r V_K}{2}$$

so

$$\frac{V_r V_K}{2} = -\frac{r}{t_f}(V_\phi - V_{\phi g})$$

simplify to:

$$\boxed{V_\phi - V_{\phi g} \cong -\frac{1}{2}\, t_f \frac{V_r V_K}{r}}$$

Now, $$\frac{dU_r}{dt} = \frac{V_\phi^2}{r} - \Omega_K^2 r - \frac{1}{t_f}(U_r - V_g)$$

$$= \left(V_{\phi g} - \tfrac{1}{2} t_f \frac{V_r V_K}{r}\right)^2 \Big/ r - \Omega_K^2 r - \frac{1}{t_f}(V_r - V_g)$$

to first order:

$$= \frac{V_{\phi g}^2}{r} - V_{\phi g}\, t_f \frac{V_r V_K}{r^2} - \Omega_K^2 r - \frac{1}{t_f}(V_r - V_g)$$

$\partial P/\partial r$

but $$V_{\phi g}^2 = V_K^2 (1 - \eta)$$

$$\frac{dV_r}{dt} = \frac{V_K^2}{r}(\cancel{1} - \eta) - \cancel{\Omega_K^2 r} - V_{\phi g}\, t_f \frac{V_r V_K}{r^2} - \frac{1}{t_f}(V_r - V_g)$$

but $$-\frac{1}{2} t_f V_r V_K = V_\phi - V_{\phi g}$$

$$\frac{dV_r}{dt} = -\eta \frac{V_K^2}{r} + 2\frac{V_K}{r}(V_\phi - V_{\phi g}) - \frac{1}{t_f}(V_r - V_{rg})$$

Now V_r will achieve terminal velocity on t_f time scale, so RHS balance.

$$0 = -\eta \frac{V_K^2}{r} - \frac{1}{t_f}(V_r - V_{rg}) + \frac{2V_K}{r}\left(-\frac{1}{2} t_f V_r V_K\right)$$

$$\boxed{V_r = \frac{\frac{r}{V_K t_f} V_{rg} - \eta V_K}{\frac{V_K}{r} t_f + \frac{r}{V_K t_f}}}$$

[Particle radial drift

$$t_f \Omega_K = \tilde{T}_f$$ (de-dimensionalize)

$$\boxed{V_r = \left(\frac{V_{rg}}{\tilde{T}_f} - \eta V_K\right) \Big/ \left(\tilde{T}_f + 1/\tilde{T}_f\right)}$$

radial drift

10a

$$\boxed{\text{- particle } \cancel{\text{goes}} \text{ drifts inward with (accreting) gas, and also due pressure.}}$$

- consider $\tau_f \ll 1$ — small particles

$$\boxed{v_r \simeq v_{rg} - \eta \tau_f v_K}$$

~ particles drift inward with gas, and experience drift linear on stopping time

- consider $\tau_f \gg 1$ — large particles

$$\boxed{v_r = -\frac{\eta v_K}{\tau_{fric}}}$$

$v_r \sim 1/\tau_{fric}$

- $\boxed{v_{r\,peak} \simeq -\frac{1}{2}\eta v_K}$ → @ peak speed.

→ Why care:

$$\boxed{t_{drift} \sim r/V_{max_r} \sim \underline{10^3 yr.}}$$ fast!

⇒ 2 points:

① - planetesimal formation must be rapid.

otherwise, grains drift to star and are evaporated.

② - radial flow, re-distribution of solids likely. Local enhancements or depletions of solids relative to Σ_g will occur.

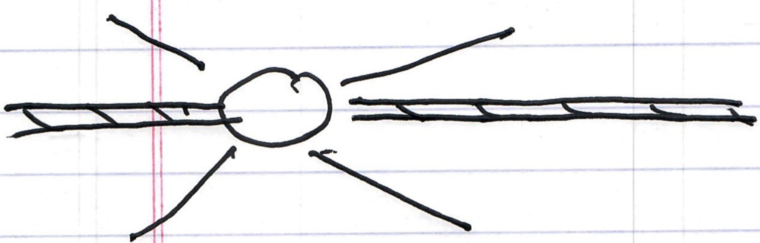
→ Much more to be said about radial diffusion, co-agulation etc.

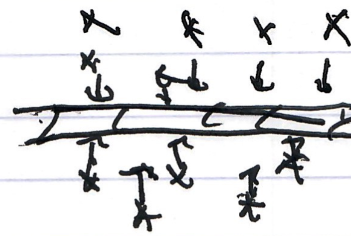
↓

extend Fokker-Planck analysis.

N.B. 2D (r, z) analysis ⇒ Fokker-Planck w/ cross coupling?

→ but Gravity !? (Goldreich-Ward)





fueled

- sedimentation produces thin, radially varying subdisk.

- sheet accumulates sufficient Σ → gravitationally unstable

- sheet breaks up into rings, etc.

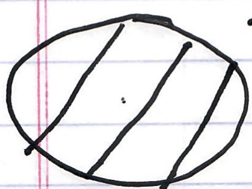
⇒ $\boxed{\text{large planetesimals}}$, etc.

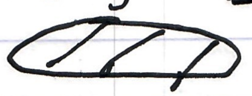
Key physics: self-gravitation ~~[illegible]~~ instability of sheet.

→ Gravitational Instability of Differentially Rotating Sheet (Toomre)

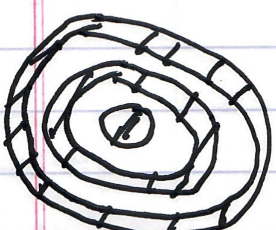
Consider thin sheet: (limiting case)

Σ, c_s^2, Φ ⇒ given





⇒ break up into rings, by finite k_r gravitational instability ⇒ ?



Jeans instability finite k_r.

⇒ later, $k_\theta \neq 0$ ⇒ spiral wave

Then, in gas dynamics:

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\underline{v}\,\sigma) = 0$$

σ = 2D density

grav potential

$$\sigma_0\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla} p - \sigma\nabla\phi$$

(2D) pressure gravitational force

$$\nabla^2\phi = 4\pi G\,\sigma\,\delta(z) \qquad \text{(sheet)}$$

Now consider axisymmetric (i.e. rings, a la' Rayleigh) only

$$\tilde{\sigma} = \sigma_0\, e^{ik_r r}\, e^{-i\omega t}$$

$\Rightarrow$

$$-i\omega\,\tilde{\sigma}_k + \sigma_0\left(ik_r\,\tilde{v}_{r_k}\right) = 0$$

$$-i\omega\,\tilde{v}_{r_k} - 2\Omega\,\tilde{v}_{\theta_k} = -ik_r\left(\tilde{\phi}_k + \tilde{P}_k\right)$$

$$-i\omega\,\tilde{v}_{\theta_k} + \frac{\tilde{v}_{r_k}}{r}\frac{\partial}{\partial r}\left(r^2\Omega\right) = 0$$

$\Rightarrow$

$$\tilde{v}_{r_k} = \left(\omega^2 - \Phi\right)^{-1}\omega k_r\left(\tilde{\phi}_k + \tilde{P}_k\right)$$

epicyclic frequency

$$\Phi = \kappa^2 = \frac{2\Omega}{r}\frac{\partial}{\partial r}\left(r^2\Omega\right) \rightarrow \text{Rayleigh discriminant}$$

(B)

$$-i\cancel{\omega}\frac{\tilde{\sigma}_u}{\sigma_0} + \frac{i k_r(\cancel{\omega} k_r)(\tilde{\Phi}_k + \tilde{P}_u)}{\omega^2 - \Phi} = 0$$

$$\tilde{P}_k = c_s^2\,\tilde{\sigma}_u/\sigma_0$$

$$-i\cancel{\omega}\frac{\tilde{\sigma}_u}{\sigma_0} + i k_r^2\cancel{\omega}\,\frac{(\tilde{\Phi}_k + c_s^2\,\tilde{\sigma}_u/\sigma_0)}{\omega^2 - \Phi} = 0$$

$$\left[-1 + \frac{k_r^2 c_s^2}{\omega^2 - \Phi}\right]\frac{\tilde{\sigma}_u}{\sigma_0} = -\frac{k_r^2\tilde{\Phi}_u}{\omega^2 - \Phi}$$

Now, sheet at $z=0$.

$$\nabla^2\phi = 4\pi G \sum \frac{\tilde{\sigma}}{\sigma_0}\,\delta(z) \quad \Rightarrow \text{Poisson Eqn.}$$

$$\nabla^2 = \frac{\partial^2}{\partial z^2} - k_r^2$$

$$\int_{0_-}^{0_+} \frac{\partial^2 \tilde{\phi}_k}{\partial z^2} = \int_{0_-}^{0_+} 4\pi G \Sigma \frac{\tilde{\sigma}_k}{\sigma_0} \delta(z)$$

$$\left.\frac{\partial \tilde{\phi}}{\partial z}\right|_{0_-}^{0_+} = 4\pi G \Sigma \frac{\tilde{\sigma}_k}{\sigma_0}$$

$$-2|k_r| \tilde{\phi}_k = 4\pi G \Sigma \frac{\tilde{\sigma}_k}{\sigma_0}$$

$$\phi_k = -\frac{2\pi G}{|k_r|} \Sigma \frac{\tilde{\sigma}_k}{\sigma_0}$$

So

$$\left[-1 + \frac{k_r^2 c_s^2}{\omega^2 - \Phi}\right] \cancel{\frac{\tilde{\sigma}_k}{\sigma_0}} = + \frac{2\pi G |k_r| \Sigma}{\omega^2 - \Phi} \cancel{\frac{\tilde{\sigma}_k}{\sigma_0}}$$

$$\boxed{\omega^2 = \Phi - 2\pi |k_r| G \Sigma + k_r^2 c_s^2}$$

⇒ $\boxed{\begin{array}{l}\text{Jeans/Toomre Criterion for axisymmetric} \\ \text{modes in a self-gravitating sheet:} \\ \omega^2 = \Phi - 2\pi|k_r|G\Sigma_0 + k_r^2 c_s^2\end{array}}$

- important!

- $\Phi = \frac{2\Omega}{r}\partial_r(r^2\Omega) = \kappa^2$

  κ ↳ epicyclic frequency

  Σ_0 = surface density

  c_s^2 = sound speed.

N.B.: Exercises:

(a)

- Show for uniformly rotating homogeneous system, $k_z = k_\theta = 0$:

$$\omega^2 = 4\Omega^2 - 4\pi G\rho_0 + k^2 c_s^2$$

→ HW problem in Shu.

(b) How generalize to kinetic/stellar
* dynamic treatment ?

i.e. Vlasov equation ?

- Observe:

→ differential rotation stabilizes at low kr

→ acoustics stabilizes at high kr.

→ Unstable range ?

$$\nu^2 \equiv \omega^2 / \kappa^2$$

$$|k_r| = \frac{\pi G \Sigma_0}{c_s^2} \left[1 \pm \left(1 - \frac{4\kappa^2}{c_s^2} \left(\frac{c_s^2}{2\pi G \Sigma_0} \right)^2 (1-\nu^2) \right)^{1/2} \right]$$

For marginal: $\omega = \nu = 0$

$$\frac{|k|}{k_T} = \frac{2}{Q^2}\left[1 \pm \left(1-Q^2\right)^{1/2}\right]$$

For instability, k real so:

$$\boxed{Q < 1 \Rightarrow \left(\kappa c_s / \pi G \Sigma_0\right) < 1}$$

Toomre criterion! → need know.

→ need $\Sigma > \Sigma_{crit}$ s/t $Q = 1$.

→ $\kappa \quad c_s$ vs Σ_0

$\downarrow \quad \downarrow$ collapse

[redacted] short

long

→ For Keplerian (or $\sim$ Keplerian) profile rotation, main effect is c_s vs. Σ_0.

$\sigma_0 = \Sigma_0$

Define: $\frac{\cancel{4}\kappa^2(c_s^2)^2}{\cancel{4}\pi^2 c_s^2 G^2 \Sigma_0^2} = \frac{\kappa^2 c_s^2}{(\pi G \Sigma_0)^2}$

$\equiv Q^2 \rightarrow$ Toomre stability factor

(N.B. You will see this!)

$$\boxed{Q^2 = \frac{\kappa^2 c_s^2}{\pi^2 G^2 \Sigma_0^2}}$$

so

$$|k_r| = \frac{\pi G \Sigma_0}{c_s^2}\left[1 \pm \left(1 - Q^2(1-\nu^2)\right)^{1/2}\right]$$

if $\underbrace{k_T}_{\text{Toomre}} = \kappa^2 / 2\pi G \Sigma_0$

$$\frac{|k|}{k_T} = \frac{2}{Q^2}\left[1 \pm \left(1 - Q^2(1-\nu^2)\right)^{1/2}\right]$$

~ cooling, as usual, is of paramount importance to Jeans condensation.

→ Punchline:

- Disk can break up, by gravitational condensation, onto rings with $\lambda \cong \Delta r \sim 2\pi / k_m$

$$k_m = k_T \frac{2}{Q^2}\left[1 \pm (1-Q^2)^{1/2}\right]$$

$Q \sim 1 \Rightarrow$ scale $k_m \sim 2 k_T$

$\Rightarrow$ inter-planetary spacing ?

see G & W !

N.B. → Swindle for dust subdisk
$\Rightarrow$ Granular flow (i.e. sand)

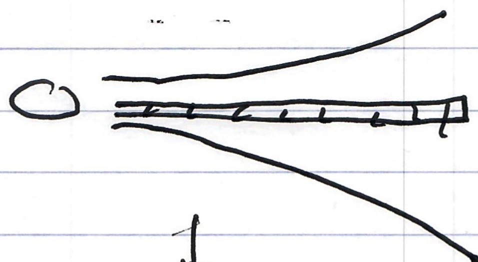
$\Rightarrow c_s^2 \rightarrow c_{s\,\text{granular}}^2$, granular temp.

see Tremaine, Goldreich, Haff ....

⇒ Back to Planets

–

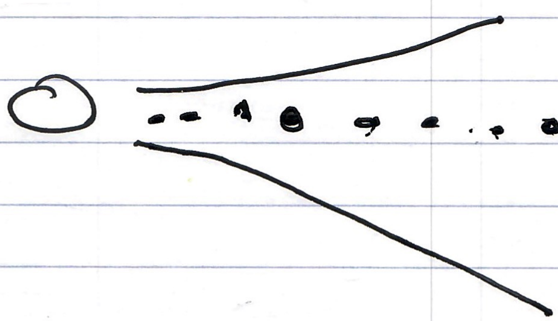
↓ sedimentation



↓

Toomre

inner region
Σ can rise
due radial
drift.



Q criterion,
some form

⇒ clumps,
planetesimals

punch line: $\boxed{\ell \sim 5-10\ \text{km}}$

$M_p \sim 10^{18}\,g$.

rapid ($\sim$ 1 yr)! formation

Does
- Work ?

 - Thin layer of many small grains optimal

 - avoids "drift into star.
 - also "2 stream" clumping.

- Catch ?

 - KH ⇒ sharp vertical gradient + $\left\{ \begin{array}{l} v_{gas} \\ v_{\phi d} \end{array} \right.$
 unstable
 ⇒ KH thickens layer $\Downarrow v_{K}$

 → generates turbulence

 ⇒ barrier to sheet formation ?!
 Σ_0 large enough ?
 - Planetesimal collisions ⇒ maintain dust.
 - but radial drift ⇒
 [enhance gravitational collapse
 in inner region. → 2D problem

⇒ $\boxed{\text{continuing story}}$ ...

N.B. streaming instability — clumping.

So

→ Planetesimal story is evolving, still, but 1 km planetesimals can form by many means.

→ Next is:

Planetesimals → Terrestrial & Gas Giant Planets

→ TBC.

→ Next: Galaxy Dynamics.