Physics 239

Lecture 10 – { MRI-induced transport, Fate of the Field

- Mixing Length Model for MRI, Loops, Viscosity

- What happens to the Field?
  - $\langle \tilde{B}^2 \rangle$ amplification by $r\,\partial\Omega/\partial r$
  - Buoyancy ⇒ Vertical transport of field
  - ⇒ Hot disk coronae, disk flares.

- Physics of Magnetic Buoyancy (i.e. Flux tubes rise, too).

- Parker Instability

→ Mixing Length Model for MRI
→ Estimates and Issues

Recall:

- Goal of MRI study ⇒ physics of turbulent viscosity / angular momentum transport

→ MRI is only viable game-in-town, though problemmatic for protoplanetary disks (cool → magnetic coupling → $\eta i \gamma \gtrsim \Omega_0$)

$\sim \nu_{in}$

- transport model → mixing length theory

→ Mixing scale, length ℓ (→ scale of parcel displacement)

→ $\tilde{V} \sim \tilde{V}(\ell)$

⇒ transport

showed $F_{conv} \sim \rho c_p \left[\frac{\partial\rho}{\partial T}\frac{1}{\rho} g\right]^{1/2} |\nabla T' - \nabla T|^{3/2} \ell_m^2$

$\rightarrow -\rho \chi \left(\nabla T - \nabla T_{crit}\right)$

χ is lin, ∇T dependent (nonlinear?)

see Shu, Vol. II and Lect 7 comments.

- seek analogues for disk angular momentum transport

N.B. key: $-\dfrac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi}$ $\Rightarrow$ Maxwell stress

and torque: $-r\dfrac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi\rho}$

Disk

- Need address
  - correlation of $\tilde{B}_r, \tilde{B}_\phi$
  - magnitude

Can crudely note: (cf. Eardley, Lightman '76)

$$\partial_t \underline{B} + \underline{v}\cdot\underline{\nabla}\,\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{v} + \eta\nabla^2 \underline{B}$$

$$\partial_t \tilde{B}_\phi + \frac{1}{\tau_{mix}} \tilde{B}_\phi = \tilde{B}_r (r \partial_r \Omega)$$

mixing time $\quad (\underline{V} \cdot \underline{\nabla}\, \underline{B} - \underline{B} \cdot \underline{\nabla} V)$

$$\tilde{B}_\phi \sim \tau_{mix} \tilde{B}_r (r \partial_r \Omega)$$

< 0

$\Rightarrow$

$$-\frac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi} \sim -\frac{\langle \tau_{mix} \tilde{B}_r^2 \rangle}{4\pi} r \partial_r \Omega$$

> 0 $\qquad \sim O(1)$

Now, $\tau_{mix} \to \frac{\tau \Omega}{\Omega} \sim \frac{\delta}{\Omega}$ $\qquad$ mixing time replaces length.

$\Rightarrow$

$$-\frac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi} \sim -\delta \frac{\langle \tilde{B}_r^2 \rangle}{4\pi \Omega} r \partial_r \Omega$$

$$\sim -\frac{\rho \delta}{\Omega} \underbrace{\frac{\langle \tilde{B}_r^2 \rangle}{4\pi \rho}}_{\sim v_A^2 \to c_s^2} r \partial_r \Omega$$

and recall:

surf. density

$$\Sigma \left(\partial_t + \underset{\substack{\uparrow \\ \text{accretion flow}}}{v_r \partial_r}\right) r^2 \Omega = \frac{1}{r}\left(r^2 \Sigma \underset{\substack{\uparrow \\ \text{viscosity}}}{\nu_r} \frac{\partial \Omega}{\partial r}\right)$$

so

$$\nu \sim \frac{\langle \tilde{B}_r^2 \rangle h}{\Sigma \Omega}$$

$$\sim V_A^2 / \Omega \qquad \text{if } \tilde{B}_r \sim B_0$$

more generally:

$$\boxed{\frac{\langle \tilde{B}_r^2 \rangle}{B_0^2} \frac{V_{A0}^2}{\Omega} = \nu}$$

$$\boxed{\begin{aligned} \alpha_{ss} &= \nu / c_s h \\ &= \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2} \frac{V_{A0}^2}{c_s^2} \end{aligned}}$$

So see 2 limits:

$$\underline{B} \cdot \underline{\nabla} \to \tilde{\underline{B}} \cdot \tilde{\underline{\nabla}}, \quad \underline{B} \cdot \underline{\nabla} \to B_0 \partial_z$$

(1) $\frac{\langle \tilde{B}_r^2 \rangle}{B_0^2} \sim 1$ i.e. $\tilde{B} \cdot \tilde{\nabla} \sim B_0 \partial_z$

'weak' MHD turbulence

$$\alpha_{ss} \sim v_{A0}^2 / c_s^2 \sim 1/\beta \ll 1$$

$\Rightarrow$ weak turbulence.

this ammounts to $\langle \tilde{B}_r^2 \rangle \sim B_0^2$.

(2) Equipartition

$$\frac{\langle \tilde{B}_r^2 \rangle}{B_0^2} \sim \frac{1}{\beta}$$

[strong MHD turbulence.

$$\boxed{\alpha_{ss} \sim 1}$$

$\langle \tilde{B}_r^2 \rangle \gg B_0^2$.

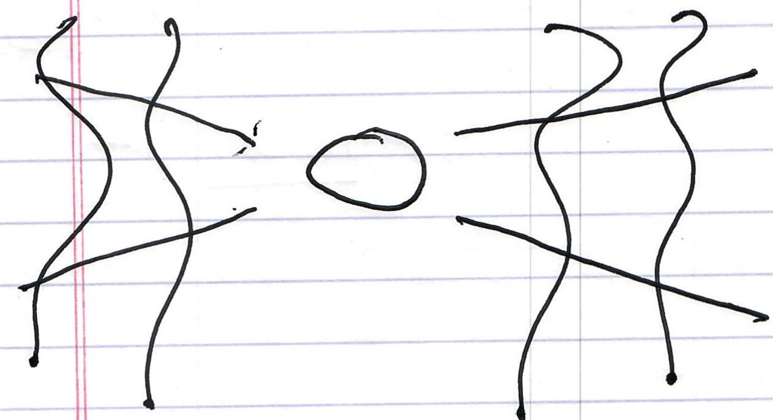
corona β_{∞}

why not $\alpha_{ss} > 1$? $\Rightarrow$ [Buoyancy, Parker instability

– or somewhere in between.

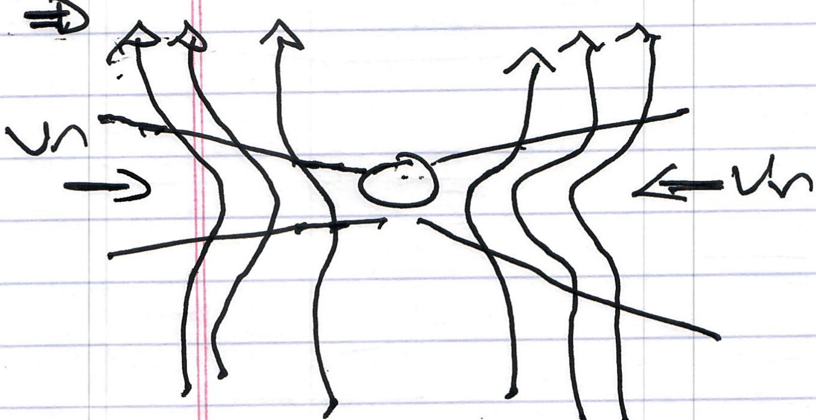
→ A bit more analysis:

- With accretion ($\langle V_r \rangle < 0$), field structure of disk is modified, due freezing-in law



cf. Shu + 2007.

⇒



accretion flow drags field in (freezing-in!)

⇒ bending develops

$$\boxed{\langle B_r \rangle \neq 0 \text{ develops}}$$

Can relate B_r^+ ~ radial field at surface - to $B_z(r,t)$:

$$B_r^T = \int_0^\infty k_{r0}\,(r/r_{cyl})\,B_z(r,t)\,r\,dr/r_{cyl}^2$$

- For turbulent viscosity, seek Maxwell stress by mixing length approach

i.e. $\langle \tilde{B}_r \tilde{B}_\phi \rangle / 4\pi \rightarrow f[B^{T2}]$

then link $\tilde{B}_r \lesssim B_r^T \rightarrow \langle B_z \rangle$

$\tilde{B}_\phi$?

- Model: $\rightarrow$ δr ~ radial Mixing length

$$\frac{\partial \tilde{B}_\phi}{\partial t} + \underline{\tilde{V}} \cdot \nabla \tilde{B}_\phi - \underline{\tilde{B}} \cdot \nabla \tilde{V}_\phi \sim B_r\, r \frac{\partial \Omega}{\partial r}$$

$$\sim \tilde{B}_r\, r\,\partial\Omega/\partial r$$

$$\tilde{V}_r \frac{\partial}{\partial r} \tilde{B}_\phi \qquad\qquad \frac{\partial}{\partial r} \sim \frac{1}{\delta r}$$

so
$$\boxed{\tilde{V}_r \tilde{B}_\phi \sim \tilde{B}_r\, r\,\Omega'\, \delta r}$$

but $\tilde{V}_r \sim \Omega\,\delta r$

fastest time scale $\rightarrow$ same as before

$$\tilde{B}_\phi\, \Omega\, \cancel{\delta r} \sim \tilde{B}_r\, r\Omega'\, \cancel{\delta r}$$

Mixing length cancels!

$$\boxed{\tilde{B}_\phi \sim \tilde{B}_r \frac{r\Omega'}{\Omega}}$$

$\Rightarrow$

$$\langle \tilde{B}_r \tilde{B}_\phi \rangle \sim \langle \tilde{B}_r^2 \rangle \frac{r\Omega'}{\Omega}$$

$\langle\ \rangle$ as avg.

$$\boxed{\langle \tilde{B}_r \tilde{B}_\phi \rangle \geq (B^+)^2 \frac{r\Omega'}{\Omega} h}$$

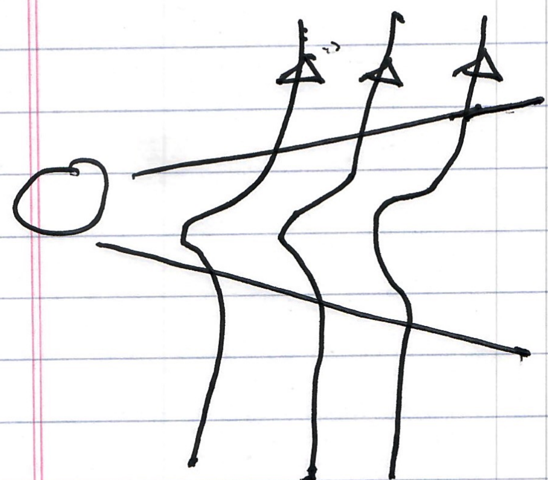
$\Rightarrow$

$$\boxed{\nu \sim B_r^{+2}\, h / \Sigma\Omega}$$

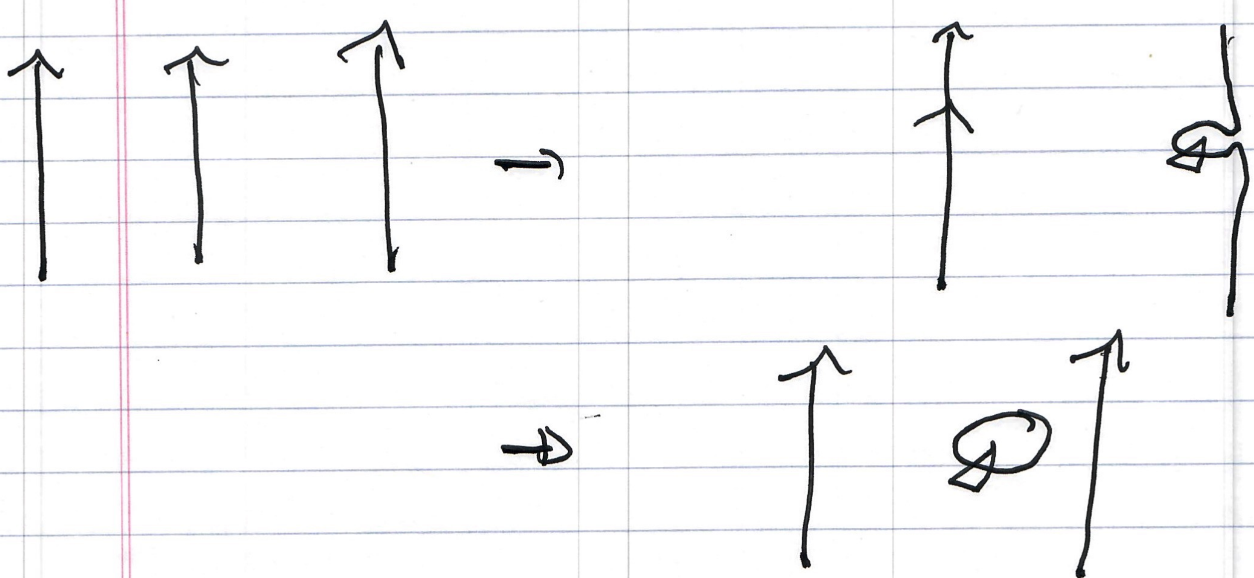
$B_r^+ \equiv B_0^+ \{B_z\}$

$$\sim V_A^{+2}/\Omega.$$

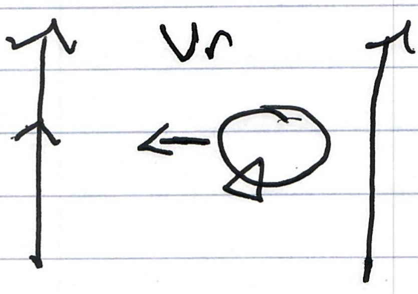
→ Loops [redacted]
Flux Tubes
} magnetic structures, with coupled fluid



Field lines distorted, $\tilde{B}_r$,



+ accretion flow



so $\alpha_{SS} \sim V_A^{+2} / c_s h \Omega$

$$\boxed{\alpha_{SS} \sim 1/\beta^{+}}$$

$$\boxed{\frac{1}{\beta} \leq \alpha_{SS} \leq 1}$$

N.B.: - Key: $\langle \tilde{B}_r^2 \rangle \rightarrow B^{+2}$
$\sim \langle B_r \rangle^2$

- likely $\langle \tilde{B}_r^2 \rangle > \langle B_r \rangle^2$.

open question → simulations.
how much. (variety of opinion)

→ can view calculation as lower bound on viscosity.
Bounds on convective flux –
L. N. Howard, Malkus ... → pursue.

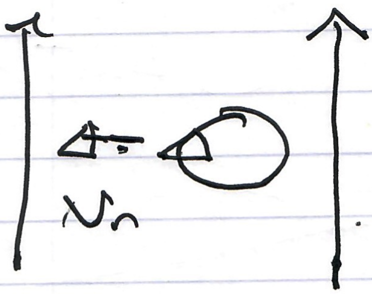
- key question → $\langle \tilde{B}_r^2 \rangle$?

Diff'nt notation → B_ϕ

→ But: Mixing Length Theory ⇒ Parcels, slugs, ...

What is the parcel here ?

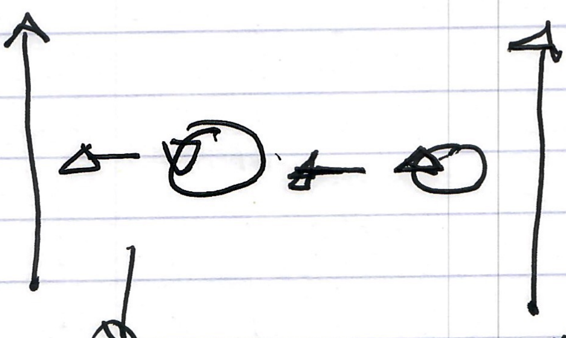
→ The issue:



Cannot reconnect
→ incorrect polarity!

→ no irreversible mixing

Need 3D flow → twist:



reconnection possible → opposite field polarity

∴

- loops/parcels random walk and mix due 3D turbulence

- 1D random walk/mixing requires 3D turbulence.

- direction of walk set by accretion flow.

⇒ Much nonlinear physics remains in MRI to be understood

especially re: planet forming disks

This brings us to:

⇒ Buoyancy / Parker Instability / Corona

Now:

- MRI/ $\tilde{B}_r$ + $r\Omega'$ ⇒ stretch and amplify B_ϕ

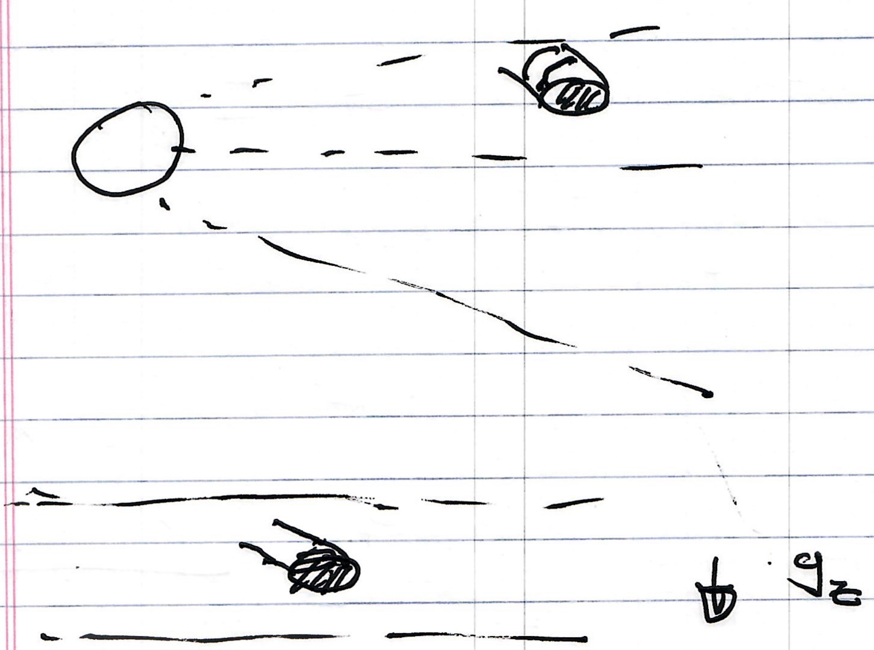
⇒

- accreting disk as sea of loops or flux tubes.

⇒ What happens to them ?

$\Rightarrow$ They rise, to top of disk

why ?



$$p + \frac{B^2}{8\pi} \sim \text{const} \Rightarrow \text{pressure balance.}$$

so

$$\delta p + \frac{\delta B^2}{8\pi} \sim 0$$

$\Rightarrow$ high field regions $\Rightarrow$ low pressure regions but $P = P_0 (\rho/\rho_0)^\gamma$

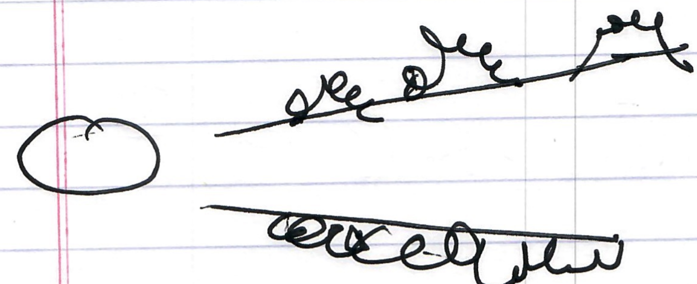
so $\delta P < 0 \Rightarrow \delta \rho < 0.$

$\Rightarrow$ Low density! $\Rightarrow$ Buoyant

i.e. flux loop/tube lighter than surrounding

∴

- tube rises to top of disk!



$\Rightarrow$ magnetized, $\underbrace{\text{hot}}_{\text{coronal heating problem.}}$ corona, etc. sun.

$\Rightarrow$ Buoyancy processes are natural fate of the field.

This brings us to $\begin{cases} \text{magnetic buoyancy} \\ \text{Parker instability} \end{cases}$

⇒ ① Vertical flux of magnetic field // intensity

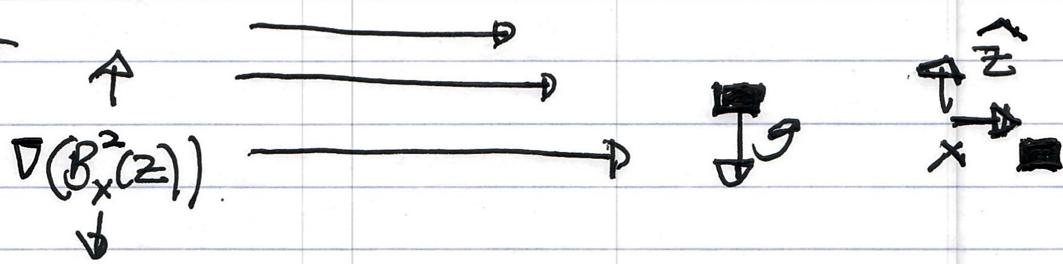
→ saturation mechanism for $\langle \tilde{B}^2 \rangle$

② change of disk structure.

→ Basic Physics of Magnetic Buoyancy

→ Consider

Slab



continuum.

$g > 0$

$$\rho_0 \left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} \right] = -g \tilde{\rho} \hat{z} - \underline{\nabla} p^* + \frac{\underline{B} \cdot \underline{\nabla}\, \underline{B}}{4\pi}$$

$$\frac{\partial \tilde{\underline{v}}}{\partial t} = -g \frac{\tilde{\rho}}{\rho_0} \hat{z} - \frac{\nabla p^*}{\rho_0} + \frac{B_{x0} \cancel{\partial_x} \tilde{\underline{B}}}{4\pi}$$

flute → $k_x = 0$.

Then, $\underline{\nabla}\cdot\underline{\tilde{V}} = 0$ $\Rightarrow$ buoyancy is slow relative to <u>magnetosonic wave</u>

$$0 \cong -\frac{\nabla^2 \tilde{p}^*}{\rho_0} - \cancel{g\,\partial_z \frac{\tilde{\rho}}{\rho_0}}$$

$$\omega^2,\ N^2 \ll \omega_{MS}^2$$

↳ magnetosonic wave

$$\tilde{p}^* \cong 0 \Rightarrow \frac{\tilde{\rho}}{\rho_0} + \frac{\tilde{T}}{T_0} + \frac{\tilde{p}_M}{p_0}$$

↓

buoyancy slow, maintains perturbed pressure balance.

$$\frac{\tilde{\rho}}{\rho_0} = -\frac{\tilde{T}}{T_0} - \frac{\tilde{p}_M}{p_0}$$

↳ magnetic pressure perturbation driven cell

↓

Temperature perturbation, as usual.

How compute $\tilde{p}_M$?

$$\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V}$$ (ideal, simplicity).

$$\partial_t \tilde{\underline{B}} + \tilde{V}_z\,\partial_z B_x(z) = \cancel{B_x \partial_x \tilde{\underline{V}}} - B_x \underline{\nabla}\cdot\tilde{\underline{V}}$$ (flute)

Why the $\underline{\nabla}\cdot\underline{\tilde{V}}$?

$$\cancel{\frac{\partial \tilde{\rho}}{\partial t}} + \tilde{V}_z \frac{\partial \rho_0}{\partial z} = -\rho_0 \underline{\nabla}\cdot\underline{\tilde{V}}$$

(anelastic)

$\omega \ll \omega_s, \omega_{MS}$ → neglect acoustic coupling !

kc_s

but retain weak compression

$$\underline{\nabla}\cdot\underline{\tilde{V}} = -\frac{1}{\rho_0}\frac{d\rho_0}{\partial z}\tilde{V}_z$$

$\underline{B}_0 \Rightarrow$

$$\partial_t \tilde{P}_M + \tilde{V}_z \partial_z \frac{B_x^2}{2} = +\frac{\partial \rho_0}{\partial z}\tilde{V}_z$$

$$\therefore \quad \boxed{\partial_t \tilde{P}_M = -B_0^2 \tilde{V}_z \frac{d}{dz} \ln \langle B \rangle / \rho_0}$$

→ gradient drive → B field

→ comes from compression

and can retain resistivity, etc.

For temperature perturbation:

$$\frac{\partial \tilde{S}}{\partial t} + \tilde{V}_z \frac{\partial}{\partial z} S_0 = 0$$

$$S = \ln\left[\frac{P}{P_0}\left(\rho/\rho_0\right)^{-\gamma}\right]$$

$$= \ln\left[T\left(\rho/\rho_0\right)^{-(\gamma-1)}\right]$$

and recall: $\displaystyle \frac{\tilde{\rho}}{\rho_0} = -\frac{\tilde{T}}{T_0} - \frac{\tilde{P}_M}{P_0}$

then as usual for convection,

$$\boxed{\partial_t\left[\tilde{T} + \frac{1}{c_p \rho_0}\tilde{P}_m\right] = -\tilde{V}_z \frac{T_0}{\gamma}\frac{d}{dz}\ln\left[P_0 \rho_0^{-\gamma}\right]}$$

and, as usual, form equation for component vorticity along B_0.

$$\underline{V} = \nabla\phi \times \hat{x}$$

$$\Rightarrow \boxed{\partial_t \nabla^2 \tilde{\phi} = g \partial_y \left(\frac{\tilde{T}}{T} + \frac{\tilde{p}_m}{p_0} \right)}$$

and $\tilde{T}/T_0$, $\tilde{p}_m/p_0$ equations $\Rightarrow$

$$\underbrace{\gamma^2}_{\text{growth rate}} = \frac{-g k_y^2}{k_z^2 + k_y^2} \left[\frac{\pm}{\underbrace{\gamma}_{\text{spec heat ratio}} p_0} B_0^2 \frac{d}{dz} \ln\left(\frac{\langle B \rangle}{\rho} \right) + \frac{\cancel{p_0}}{T_0 \gamma} \frac{d}{dz} \left[\ln \left(p \rho^{-\gamma} \right) \right] \right]$$

$$= \underbrace{\gamma^2_{conv.}}_{\text{usual}} \frac{-g k_y^2}{k_z^2 + k_y^2} \underbrace{\left[\frac{\pm}{\gamma p_0} B_0^2 \frac{d}{dz} \ln \left(\frac{\langle B \rangle}{\rho} \right) \right]}_{\text{magnetic buoyancy drive}}$$

this all gives a generalized Schwarzschild criterion!

need $\frac{d}{dz} \ln \frac{\langle B \rangle}{\rho} < 0$

---

for contribution to instability

i.e. buoyancy instability criterion:

$$\boxed{\frac{d}{dz}\ln\left(P\rho^{-\gamma}\right) + \frac{B_0^2}{P_0}\frac{d\ln}{dz}\left(\langle B\rangle/\rho_0\right) < 0}$$

⇝ gives critical gradient of magnetic field relative to density gradient for instability (for adiabatic thermal)

$$\frac{1}{\langle B\rangle}\frac{dB}{dz} < \frac{1}{\rho_0}\frac{d\rho_0}{dz}$$

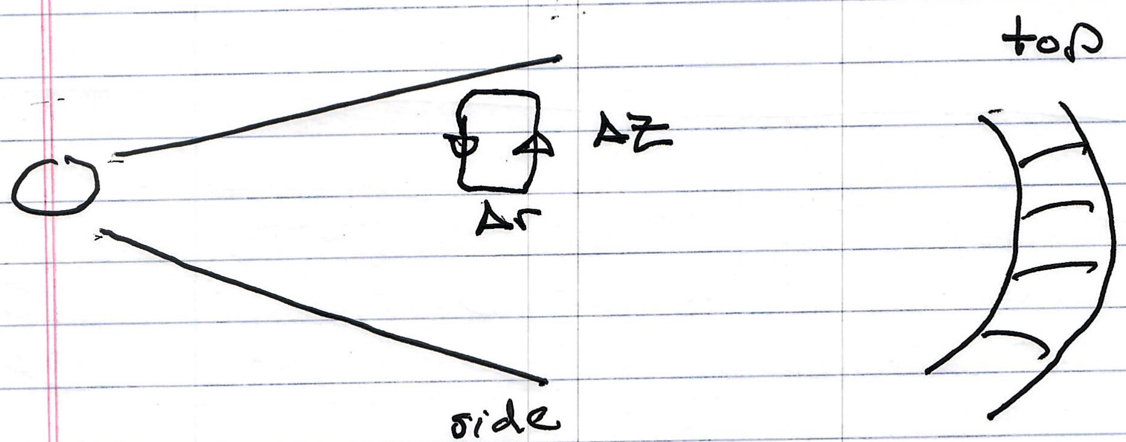
and of course $\frac{dP}{dz} = -g\rho$ gives density profile. $(H = c_s/\Omega)$

⇒ just as hot air, strong magnetic fields rise ..., and are lost from MRI process.

⇒ like thermal plumes, flux tubes/loops will rise to disk surface.

→ But this is a disk, not a slab.

As before, for buoyancy in disk:



→ consider axisymmetric rolls → else shearing limits

→ vertical motion releases buoyancy potential energy

but radial motion costs energy, i.e.

$$\omega^2 = \frac{k_z^2}{k^2}\Phi$$, as radial stratification

stable.

$\therefore$ straightforward to show:

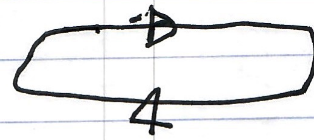
$$\omega^2 = \frac{k_r^2 N^2_{\substack{mag\\buoy}} + k_z^2 \Phi}{k_r^2 + k_z^2}$$

$$\left[\begin{array}{l} N^2_{\substack{mag\\buoy}} = +g\left[\dfrac{B_0^2}{P_0}\dfrac{d}{dz}\ln\left(\dfrac{\langle B\rangle}{\rho}\right) + \dfrac{1}{\gamma}\dfrac{d}{dz}\ln\left(P\rho^{-\gamma}\right)\right] \\ \\ \Phi = \dfrac{2\Omega}{r}\dfrac{d}{dr}\left(r^2\Omega\right) \end{array}\right.$$

or

$$\gamma^2 = \frac{\overset{\text{gain}}{\overbrace{(\Delta z^2)\left(-N^2_{\substack{mag\\buoyancy}}\right)}} - \overset{\text{energy penalty}}{\overbrace{(\Delta r)^2\,\Phi}}}{(\Delta r)^2 + (\Delta z)^2}$$

→ favors thin, tall cells (elevators, cf. Taylor–Proudman)

i.e.  not 

→ angular momentum transport inward

i.e. $\Gamma^{+}_{L_\theta} \sim -D\,\partial_r(r^2\Omega)$

$D \sim \langle \tilde{v}^2 \rangle / \gamma$.

⇒ Magnetic buoyancy results in rise of magnetic loops to top (bottom) of disk. → magnetized corona.