Physics 239

Lecture 9 – A Closer Look at MRI

i.) Review of MRI Physics

ii.) Vertical Field – Full Analysis

iii.) Resistivity, Viscosity, Partial Ionization

iv.) Evolution of the MRI

- Vertical Field + Accretion
- 'Mixing Length Theory' of MRI
  (Mixing Length Models, Reconnection)
- Fingering and Profile Evolution
- Turbulence – a look
- Disk Dynamo – Comments

→ tbc later

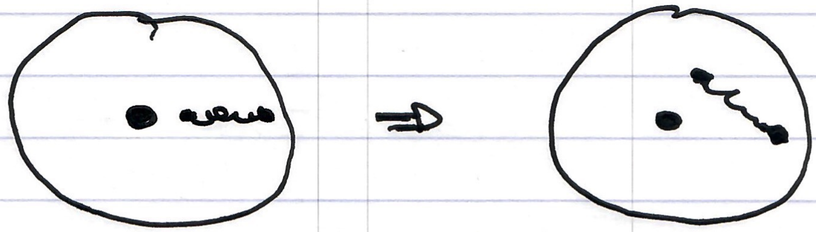
⇒ next:

Fate of the Field: Magnetic Buoyancy, Parker Instability

⇒ Hot disk coronae.

(ii) Review of MRI Physics

- recall



Follows L-B & P
B & H

vertical field:

$$k^2 v_A^2 + \frac{d\,\Omega^2}{d\ln r} < 0 \quad \Rightarrow \text{instability criterion}$$

- key points:
  - $\frac{d\,\Omega^2}{d\ln r} < 0$ not $\Phi < 0$ - its not

    → differntl rotation

    $\Phi = \frac{2\Omega}{r}\frac{d}{dr}(r^2\Omega)$
- weak field:

  $k v_A < \Omega$ ⇒ stable for strong magnetization

  $(Hk > 1)$
- strong ⇒ $\gamma_{max} \sim \Omega$.

ii.) Vertical Field - Full Analysis

$\nabla \cdot v = 0$; ignore vertical, radial $\Big\rangle$ thermal stratification

$$\tilde{v},\ \tilde{B} \sim e^{i(\underbrace{k_r r + k_z z}_{\text{variation}} - \omega t)} \qquad k^2 = k_r^2 + k_z^2$$

$$-i\omega \tilde{v}_r + i k_r \frac{\tilde{p}}{\rho} - \underbrace{2\Omega \tilde{v}_\phi}_{\text{Rotation}}$$

$\frac{\tilde{p}}{\rho} \rightarrow \tilde{P}_*$

$$+ i\frac{k_r}{4\pi\rho}(B_z \tilde{B}_z) - i\frac{k_z B_z}{4\pi\rho}\tilde{B}_r = 0$$

Magnetic tension - Alfven

$$-i\omega \tilde{v}_\phi + \tilde{v}_r \frac{\kappa^2}{2\Omega} - i\frac{k_z B_z}{4\pi\rho}\tilde{B}_\phi = 0$$

$\rightarrow$ Alfven.

$\kappa^2 \equiv$ epicyclic frequency.

$$\kappa^2 = 2\Omega \frac{\partial}{\partial r}(r^2\Omega)$$

$$-i\omega \tilde{v}_z + i k_z \frac{\tilde{P}}{\rho} + i\frac{k_z}{4\pi\rho}\cancel{\left(B_\phi \tilde{B}_\phi + B_r \tilde{B}_r\right)}$$

$$- i\cancel{k_r \frac{B_r}{4\pi\rho} \tilde{B}_z} = 0$$

trivial

and

$$-i\omega \tilde{B}_r - ik_z B_z \tilde{V}_r = 0$$

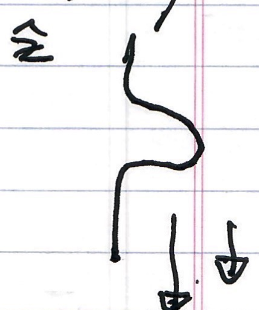
$$-i\omega \tilde{B}_\phi - \boxed{\tilde{B}_r \, r\frac{\partial\Omega}{\partial r}} - ik_z B_z \tilde{V}_\phi = 0$$

$$\underline{B}\cdot\nabla\underline{V} = \tilde{\underline{B}}\cdot\nabla\underline{V} + \underline{B}_0\cdot\nabla\tilde{\underline{V}}$$

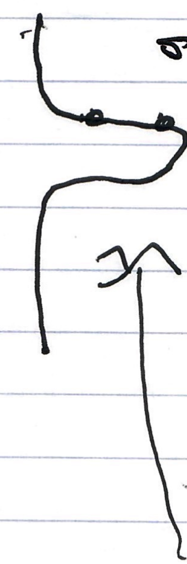
$$\tilde{B}_r\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) = \underbrace{\tilde{B}_r \, r\frac{\partial\Omega}{\partial r}}_{\text{shear}}$$

(see 4a)

key effect



$\Rightarrow$ B_r tilts B_z into differential rotation
$\Rightarrow$ field stretched



spring from tilt!

$$-i\omega \tilde{B}_z - ik_z B_z \tilde{V}_z = 0$$

N.B. :- Field Tilting (magnetic) + differential rotation is key to MRI.

- Tilting generates magnetic connection - $\boxed{\text{spring}}$

$$\frac{\partial \underline{B}}{\partial t} + \overset{①}{\underline{V}\cdot\underline{\nabla}\,\underline{B}} = \overset{②}{\underline{B}\cdot\underline{\nabla}\,\underline{V}}$$

② $$\left(\underline{B}\cdot\underline{\nabla}\,\underline{V}\right)_\phi = B_r \frac{\partial V_\phi}{\partial r} + \cancel{\frac{B_z}{r}\frac{\partial V_\phi}{\partial z}} + \cancel{\frac{B_\phi}{r} V_r} + \cancel{\frac{B_\phi}{r}\frac{\partial V_\phi}{\partial \phi}}$$

vert field

$$\simeq \tilde{B}_r \frac{\partial V_\phi}{\partial r}$$

① $$\left(\underline{V}\cdot\underline{\nabla}\,\underline{B}\right)_\phi = \cancel{V_r \frac{\partial B_\phi}{\partial r}} + \cancel{\frac{V_\phi}{r}\frac{\partial B_\phi}{\partial \phi}} + \cancel{V_z \frac{\partial B_\phi}{\partial z}} + \boxed{V_\phi \frac{B_r}{r}}$$

$$\simeq V_\phi \frac{\tilde{B}_r}{r}$$

$$\Rightarrow \frac{\partial \tilde{B}_\phi}{\partial t} = \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{V}\cdot\underline{\nabla}\,\underline{B}$$

$$= \tilde{B}_r\left(\frac{\partial V_\phi}{\partial r} - \frac{V_\phi}{r}\right) = \tilde{B}_r\, r \frac{\partial \Omega}{\partial r}$$

but you know it <u>must</u> be this.

Solid body rotation can't stretch $\tilde{B}_r$ into $\tilde{B}_\phi$. Must be differential.

so, cranking it out:

$$\boxed{\bar{\omega}^4 \frac{k^2}{k_z^2} - \bar{\omega}^2 \left(4\Omega^2 + \frac{d\Omega^2}{d\ln r}\right) - 4\Omega^2 k_z^2 V_A^2 = 0}$$

$k^2 = k_r^2 + k_z^2$

$\bar{\omega}^2 = \omega^2 - (k_z V_A)^2$

dispersion relation for axisymmetric MRI

stability: - $\bar{\omega}^2$, ω^2 real

- $\omega^2 > 0$ so $\omega^2 > 0$ to $\omega^2 < 0$ transition $\Rightarrow$ instability

so look for $\omega^2 \to 0$ as $k_z V_A \to 0$

$$\Rightarrow \boxed{\frac{\partial \Omega^2}{\partial \ln r} < 0} \rightarrow \text{as before}$$

- departs familiar hydro $\frac{\partial}{\partial r}(r^2 \Omega) < 0$

- azimuthal $\langle B \rangle$ MRI also unstable

– If vertical stratification:

$$\underset{\substack{\text{Brunt-}\\ \text{Vaisala}\\ \text{Freq.}}}{N^2} = + \underset{>0}{g_z} \underset{\substack{>0\\ \text{for stable}}}{\frac{\partial S}{\partial Z}} \qquad S \sim \ln\left(P\rho^{-\gamma}\right)$$

$$\boxed{N^2 + \frac{\partial \Omega^2}{\partial \ln r} > 0.}$$

for stability

⇒ stable stratification vertically works against MRI.

Contrast to:

$$N^2 + \frac{1}{r^3} \frac{\partial}{\partial r}\left(r^4 \Omega^2\right) > 0$$

in hydro.

Recall $\underset{\substack{>0\\ <0}}{g} \leftrightarrow \underset{>0}{\Omega^2}$ connection

⇒

$$\omega^2 = \frac{k_r^2 N^2 + k_z^2 \Phi}{k_r^2 + k_z^2}$$

iii) <u>Dissipation</u>

- retain ν, η

- $\gamma = -i\omega$

- $\underline{k} = k\hat{z}$

[here - collisional]

- saturation ? ν_T, η_T to render most unstable mode marginal

$$\left(\gamma + \nu k^2\right)\tilde{v}_r - 2\Omega \tilde{v}_\phi - \frac{\tilde{\rho}}{\rho^2}\frac{\partial P}{\partial r}$$

$$+ i\frac{k_z B_z}{4\pi\rho}\tilde{B}_r = 0$$

$$\left(\gamma + \nu k^2\right)\tilde{v}_\phi + \tilde{v}_r\frac{\kappa^2}{2\Omega} - i k_z B_z\frac{\tilde{B}_\phi}{4\pi\rho} = 0$$

and

$$\left(\gamma + \eta k^2\right)\tilde{B}_r - i k_z B_z \tilde{v}_r = 0$$

$$\left(\gamma + \eta k^2\right)\tilde{B}_\phi - \tilde{B}_r\frac{\partial\Omega}{\partial \ln r} - i k_z B_z \tilde{v}_\phi = 0.$$

⇒ Dispersion relation:

$$\left[\gamma^2 + k^2V_A^2 (\eta+\nu)\gamma k^2\right]^2 + \kappa^2\left[\gamma^2 + (k_z V_A)^2 + 2\eta\gamma k^2\right] - 4\Omega^2 (kV_A)^2 = 0$$

reduces for $\eta, \nu \to 0$.

$\mapsto \gamma$, with η, ν

Now, $\gamma' = \gamma_0 - \gamma$

γ': $\boxed{\text{dissipation rate}}$ $\to$ shift in γ

$\gamma_0 \mapsto \gamma$, absent η, ν

crank:

$$\gamma' \cong \eta k^2 \left[\frac{\kappa^2(1+P)\left[\gamma_0^2+(kV_A)^2\right]}{\kappa^2 + 2\left[\gamma_0^2+(kV_A)^2\right]}\right]$$

for $\gamma'/\gamma_0 < 1$ — wk. damping.

$P = \nu/\eta$ → Magnetic Prandtl # — relative importance.

For Keplerian: $P = \nu/\eta$

$\gamma_0^2 = 9/16\,\Omega^2$

$(kv_A)^2 = (15/16)\,\Omega^2$

$$\boxed{\gamma' = \underbrace{\frac{5}{8}\eta k^2}_{\text{resistive dissp}^n.}\Big(1 + \frac{3}{5}\underbrace{P}_{\nu/\eta}\Big)}$$

[routine]

~ shift additive, to #5

~ Exercise: η, ν effect on Shear Alfven wave.

Summary:

- $P \sim \left(T/10^5\right)^{4} \dfrac{10^{16}}{n}$

very sensitive to T!

- $P \ll 1 \to$ need really weak field to care abt dissp^n.

$P \gg 1 \to$ only significant effect for weak field.

→ Dissipation does little to MRI

→ Ideal MHD is good!

⇒ What of Partially Ionized $\left\{\begin{array}{l}\text{Gases}\\ \text{Fluids}\end{array}\right.$?

With neutrals → drag on ions (recall):

$$\underset{\substack{\downarrow\\ \text{force density}}}{\underline{f}} = -\gamma \rho_i \rho_n (\underline{V}_i - \underline{V}_n)$$

2-Fluid analysis ⇒ $\left\{\begin{array}{l}\text{Blaes}\\ \text{Balbus}\end{array}\right.$

$$(k_z V_A^*)^2 > -\frac{d\Omega^2}{d\ln r} \quad \text{for } \underline{\text{stability}}$$

$$V_A^{*2} = \frac{B^2}{4\pi\rho_i}\left[\frac{\omega^2 + \gamma^2\rho_0^2}{\omega^2 + \gamma^2\rho_i\rho_n}\right]$$

$\gamma \to 0 \quad [\;\;] \to 1$

$\gamma \to \infty \quad [\;\;] \to \rho_i/\rho_n \Rightarrow V_A^{*2} = B^2/4\pi\rho_n$

→ strong drag replaces ρ_i in V_A^2 with ρ_n.

i.e. neutrals constitute 'effective mass' of system, coupled by collisions.

→ in reality;

$$\gamma^2 \rho_i \rho_n \gg k^2$$

$$\boxed{V_A^{*2} \to \frac{B^2}{4\pi\rho_n}\left[1 + \frac{k^2}{\gamma^2 \rho_i^2}\right]}$$

so for $\boxed{\rho_i \gamma \gg k \sim \Omega,}$

MRI as usual with $B^2/4\pi\rho_n$.

⇒ * $\boxed{\text{ion-neutral collision rate must exceed rotation rate (dynamical time scale) for magnetic coupling and MRI}}$

Can relate to:

$$\left.\frac{n_i}{n_n}\right|_{\text{min needed}} = 10^{-13}\left(1+\frac{m_n}{m_i}\right)\left(\frac{M}{M_\odot}\right)^{1/2} R_{AU}^{-3/2}\left(\frac{n_n}{10^{15}}\right)^{-1}$$

→ small.

→ important exception → protoplanetary disks.

→ accretion.

→ Some Questions:

- why the apparent discontinuous change in stability criteria with B?

Point: - B enters as $(\underline{k}\cdot\underline{V_A})$

- B will have effect for $k \sim \Omega / V_A$

- so long as dissipation not relevant can alway choose high/small wavelength.

$\hookrightarrow$ scale?

– Key: when does dissipation kick in?

– Angular Momentum Transport?

$$\Pi_{r,\phi} = r\left[\langle \tilde{v}_r \tilde{v}_\phi \rangle - \frac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi\rho}\right]$$

sign, alignment?

Recall:

$$\partial_t \tilde{B}_\phi = \tilde{B}_r \, r \frac{\partial \Omega}{\partial r} + B_z \partial_z \tilde{V}_\phi$$

response to tilt generates correlation

$\Rightarrow$

$$\tilde{B}_\phi \sim \int \tilde{B}_r \, r \frac{\partial \Omega}{\partial r} \sim \tau_{eff} \tilde{B}_r \, r \frac{\partial \Omega}{\partial r}$$

$$-\frac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi\rho} \rightarrow \underbrace{-\langle \tilde{B}_r^2 \rangle \tau_{eff} \, r \underbrace{\frac{\partial \Omega}{\partial r}}_{<0}}_{>0}$$

and magnetic stress ⇒ outward flux,

(iii) Estimating the Viscosity
— A Mixing Length Theory for
MRI

— Why doing all this? ⇒ MRI →
turbulence → momentum transport

i.e. get ν!

→ Caveat Emptor!

— How? — numerical simulation (OO)
→ assumptions? — boundary conditions
— $\boxed{\text{resolution}}$
↳ ν_{eff}
→ interpretation?
— detailed theory? — tough
— $\boxed{\text{Reduced model} \rightarrow \text{mixing length theory} \quad \text{(guide)} \rightarrow \text{intuition}}$

boundary conditions:
— radial → inner
— vertical
— field

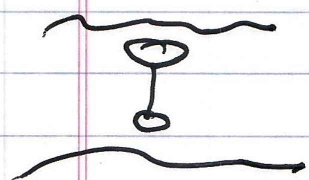
see Shu, 2007 on

Aside: Mixing Length Theory for Convection

- classic example of mixing length models ⇒ Convection.

- Recall convection:

Cf.: 2820, 218b.



$$\rho g = \frac{dP}{dz}$$

$$\rho', P' < 0$$

$$P\rho^{-\gamma} \sim \text{const.}$$

blob buoyant if:

$$\frac{1}{\gamma}\frac{1}{P_1}\frac{dP_1}{dz} < \frac{1}{\rho_1}\frac{d\rho_1}{dz} \quad \Rightarrow \quad \underbrace{\frac{dS}{dz} < 0}_{\text{super-adiabatic}}$$

⇒ Schwarzchild condition for convective instability (condition on $\frac{1}{T}\frac{dT}{dz}$)

→ based on idea of a <u>slug</u> or parcel.

⇒ Mixing length theory seeks to model heat flux in terms of a single length scale or <u>mixing length</u>

N.B.: 'slug' concept more natural for heat

⇒ Aim is to estimate $\left\{ \begin{array}{l} \text{heat flux} \\ \text{thermal diffusivity } (\chi) \end{array} \right.$

useful in stellar structure calculations

↳ transport coefficient (γ)

Seek convective heat flux:

rising parcel

$$\underline{F}_{conv} = \rho \, (\Delta h) \, \underline{V} \Leftrightarrow \rho c_p \langle \tilde{v} \tilde{T} \rangle$$

Δh → excess enthalpy (heat)

↳ parcel velocity

and parcel exists for length scale

l_m → mixing length. ($\sim l_{mfp}$)

based on analogy with kinetic theory → questionable

$$F_{conv} \sim \rho \left(\Delta h \, V_z \right)$$

$\Delta h \sim c_p \, \Delta T$ → excess temperature of rising blob, relative surroundings

↳ heat of slug.

$$\Delta T \sim T' - T$$

T': slug, T: surrounding.

so $\quad \Delta T \sim l_m \underbrace{(\nabla T' - \nabla T)}_{\text{difference in gradient}} \cdot \hat{n} \qquad \hat{n} = -\frac{\mathbf{g}}{|\mathbf{g}|}$

For v:

$$\mathbf{v}\cdot\nabla\mathbf{v} \sim g\rho \longrightarrow f_d \sim \rho A_s v^2$$

$$\rho \frac{v^2}{l_m} \sim \Delta\rho\, |g|$$

$l_m \rightarrow$ dynamic drag

$$\Delta\rho \sim (\partial\rho/\partial T)\, \overset{l_m}{\Delta T} \quad \text{(buoyancy)}$$

so

$$v \sim \left[\frac{|g| \left(\frac{\partial\rho}{\partial T}\right) \Delta T\, l_m}{\rho}\right]^{1/2}$$

so

$$\boxed{F_{conv} \simeq \rho c_p \left[\frac{\partial\rho}{\partial T}\frac{1}{\rho} g\right]^{1/2} |\nabla T' - \nabla T|^{3/2}\, l_m^{2}}$$

basic scaling of convective flux

Typically: $\quad l_m = \alpha H \rightarrow$ fraction of pressure scale height.

$$H^{-1} = 1/p \frac{\partial p}{\partial z}$$

and can insert F_{conv} into stellar heat transport model. Furthur assumptions/estimates allow statements re: profiles / stratification, cf Shu.

Point : - single characteristic length scale

- Link velocity, Temp perturbation } l_m

⇒ estimate flux.

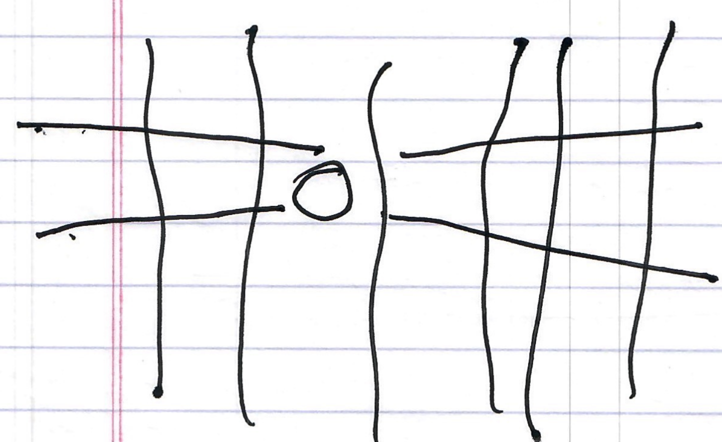
⇒ transport model.

- NB: Can have bistable/two-scale mixing length models see Balmforth et. al. JFM '98 ⇔ multiple states transport.

Variant: Model η_T, ν_T and calculate conditions for marginality, near marginality.

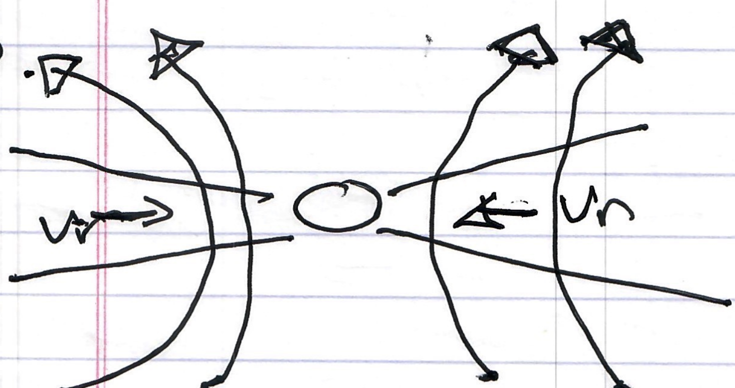
Now, to Disk:

- With accretion ($\langle V_r \rangle < 0$), field structure of disk is modified, due freezing-in law



cf Shu + 2007

$\Rightarrow$



accretion flow drags field $\Rightarrow$ bending.

$\langle B_r \rangle \neq 0$ develops.

- $$B_r^+ = \int_0^{\infty} K_0\left(\frac{r}{r_{cyl}}\right) B_z(\;, t) \frac{dr}{r_{cyl}^2}$$

radial field surface

- For turbulent viscosity, seek Maxwell stress, via mixing length approach

i.e. $\langle \tilde{B}_r \tilde{B}_\phi \rangle / \; \rightarrow \; \mathcal{F}\left[B^{+2}\right]$

then link $\tilde{B}_r \leq B_r^+$

$\tilde{B}_\phi$?

- Mixing Model Estimate $\rightarrow$ $\delta r \sim$ radial mixing length

$$\tilde{\underline{V}} \cdot \underline{\nabla} B_\phi \sim \tilde{B}_r \, r \frac{\partial \Omega}{\partial r} \qquad \text{(induction)}$$

$$\tilde{V}_r \frac{\partial}{\partial r} \tilde{B}_\phi \sim \tilde{B}_r \, r \partial \Omega / \partial r$$

$$\frac{\partial \tilde{V}_r}{\partial r} \sim \frac{\tilde{V}_r}{\delta r}$$

$$\tilde{B}_\phi \frac{\partial \tilde{V}_r}{\partial r} \sim \tilde{B}_r \, r \frac{\partial \Omega}{\partial r}$$

$$\boxed{\tilde{V}_r \tilde{B}_\phi \sim \tilde{B}_r \, r \, \Omega' \, \delta r}$$

but $\tilde{v}_r \sim \Omega\,\delta r$

$$\tilde{B}_\phi\,\Omega\,\cancel{\delta r} \sim \tilde{B}_r\, r\Omega'\,\cancel{\delta r}$$

$$\boxed{\tilde{B}_\phi \sim \tilde{B}_r \frac{r\Omega'}{\Omega}}$$

ML cancels!
intuitively plausible

$$\Rightarrow \langle \tilde{B}_r \tilde{B}_\phi \rangle \sim \langle \tilde{B}_r^2 \rangle \frac{r\Omega'}{\Omega}$$

$\langle\ \rangle$ ar avg. ϕ

$$\boxed{\langle \tilde{B}_r \tilde{B}_\phi \rangle = (B^+)^2 \frac{r\Omega'}{\Omega}\, h}$$

and recall;

$$\Sigma\,(\partial_t + V_r\partial_r)(r^2\Omega) = \frac{1}{r}\partial_r\left(r^3 \Sigma\, \nu\, r \frac{\partial\Omega}{\partial r}\right)$$

$$\Rightarrow \boxed{\nu \sim B_r^{+2} h / \Sigma\Omega}$$

$B_r^+ = B_r^+\{\beta_z\}$

so

$$\nu \sim B_z^2 \Big/ \underbrace{\frac{\Sigma}{h}}_{\rho}\,\Omega$$

$$\sim v_A^2/\Omega \quad \rightarrow \text{Alfven speed } \cancel{\text{sets}} \text{ viscosity}$$

so.

$$\alpha_{ss} = \nu / c_s h$$

$$= v_A^2 / c_s \underline{h \Omega} \sim v_A^2 / c_s^2$$

$$\sim 1/\beta$$

$$\boxed{\alpha_{ss} \sim 1/\beta} \rightarrow \text{small;}$$

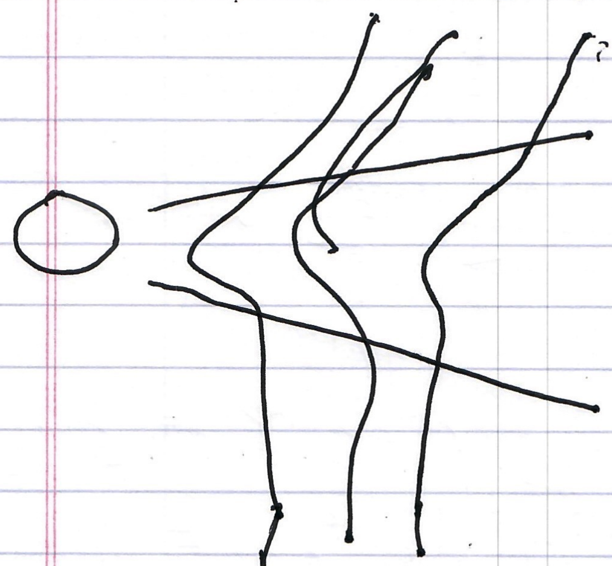
N.B.: - Likely can have $\langle \tilde{B}_r^2 \rangle > \langle B \rangle^2$

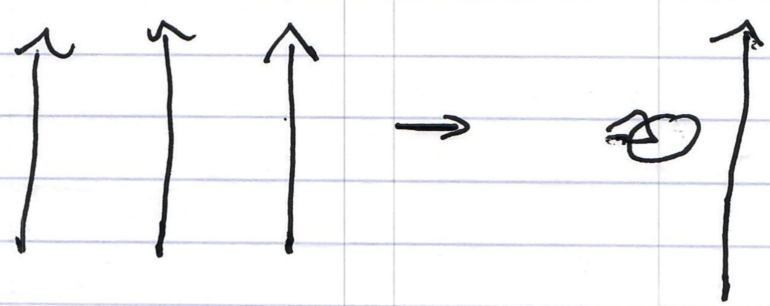
- $\alpha \rightarrow 1$? $\rightarrow$ some simulations.

open question

→ What are the Parcels / slugs ?

→ Loops ! → of magnetic field, with coupled flux





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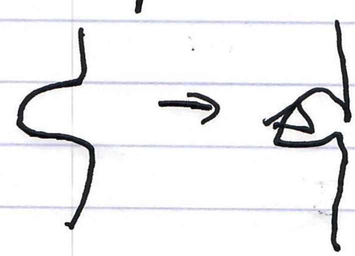
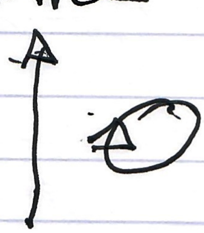


pinch

now, unless twist,



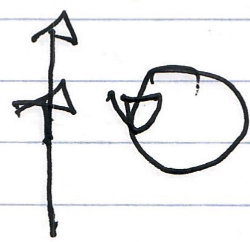
cannot reconnect

same polarity

$\Rightarrow \quad \delta r^2 \sim \text{const} \not\propto \Delta t$

not a walk $\rightarrow$ trapped.

$\rightarrow$ if 3D, twist:



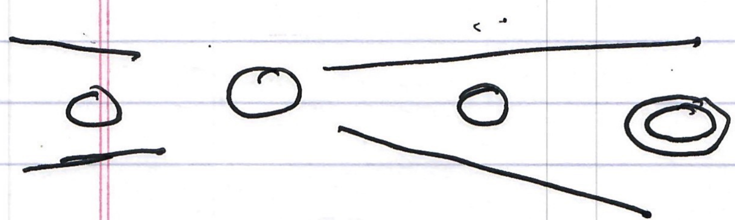
can reconnect.

∴ - parcels are magnetic loops, random walking due 3D turbulence.

- direction set by accretion flow. ?

$\Rightarrow$ Much physics in nonlinear dynamics of MRI remains!

→ Looking ahead: buoyancy



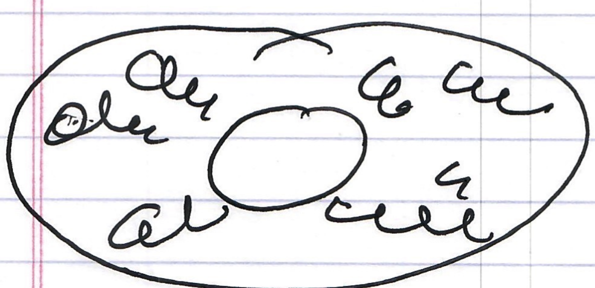
flux tube, loop.

$p + B^2/8\pi \sim$ const

high B regions → low p → low ρ

$$p = p_0(\rho/\rho_0)^\gamma$$

Low density ⇒ buoyancy → loops rise to top of disk



→ disk corona, ala' sun.