Physics 239

Lecture 8 - Magnetorotational Instability (MRI)

→ Why MRI ? → Mechanism for ν

→ MRI Physics

→ Simplified MRI / Heuristics.

→ Calculation. TBC

→ why MRI?

- Need mechanism to produce turbulence to exert shear stresses.

$$T_{r\phi} = \Sigma \cdot r \, \nu \, \frac{\partial \Omega}{\partial r}$$

↳ viscosity - due fluctuations

$\nu \sim \tilde{V} \, l_{mix}$

↑ not ∇L_ϕ!

~~xxxx~~ Hydro-turbulence:

$\langle \tilde{V}_r \tilde{V}_\phi \rangle \rightarrow \nu$ turbulent Reynolds stress

MHD:

$\langle \tilde{V}_r \tilde{V}_\phi \rangle - \langle \tilde{B}_r \tilde{B}_\phi \rangle \rightarrow \nu$

MRI = Alfven wave + rotation

[Reynolds + Maxwell stress

⇒ Is there an instability which taps free energy in $\partial \Omega / \partial r$?

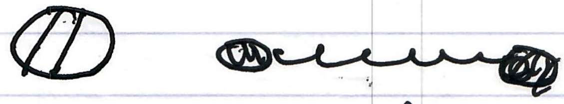
↳ Such is source/driver of turbulent transport → viscosity mechanism.

→ MRI Physics

Recall cartoon derivation from Lynden-Bell 2 particle argument.

Key point: 2 particles conserve total $\left\{ \begin{array}{l} L \\ M \end{array} \right.$

but can exchange.

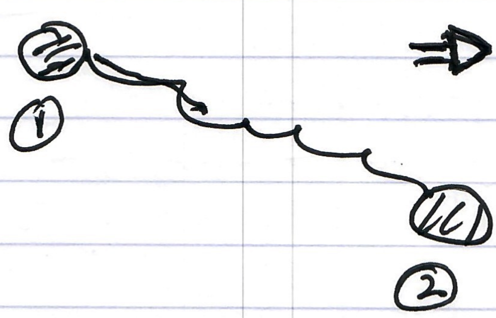


spring couples particles

Then differential rotation ⇒

\+ critical perturbation

[inner-ahead]



⇒ Ω' stretches spring

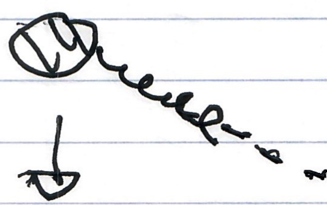
outer behind

- but the particles are "donkeys"

⇒ do the opposite of the nudge.

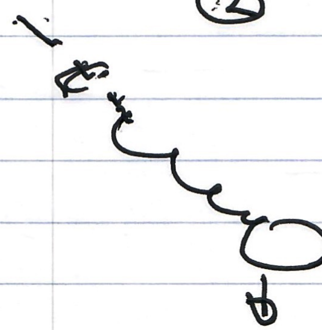
so

①

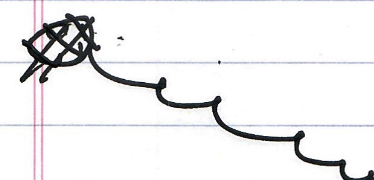


loses angular momentum
so
drops to inner orbit, with higher Ω

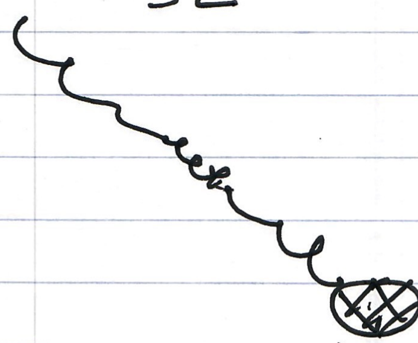
②



gains angular momentum
so
moves to outer orbit, with lower Ω

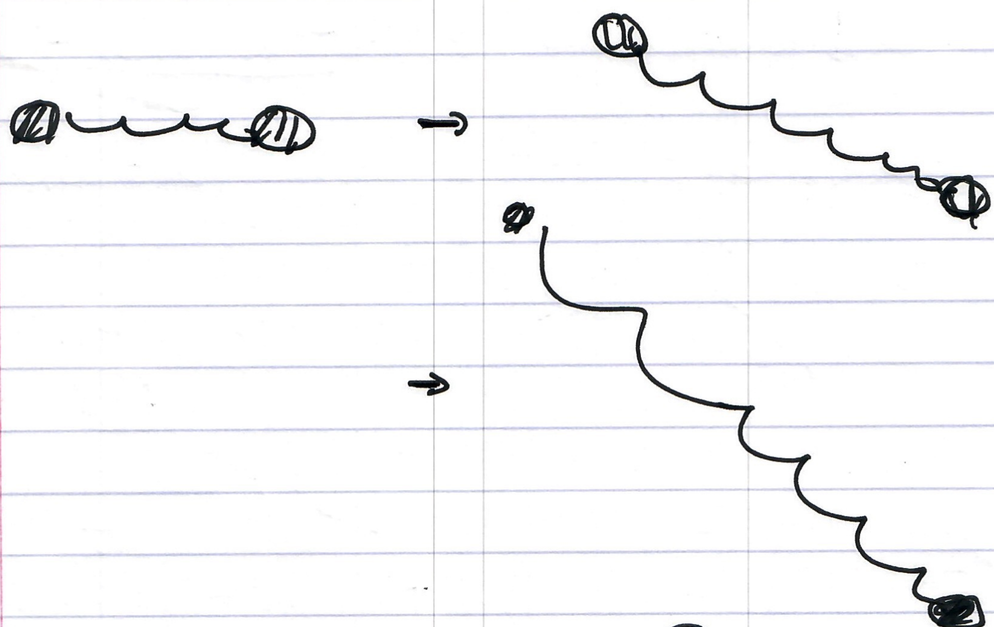


⇒ moves further ahead



⇒ drops further behind

⇒ stretching the spring results in further stretching of spring i.e.



⇒ perturbation self-reinforces.

⇒ instability

Note:

- spring "facilitates" instability via interaction, but is **not** the free energy source.

- Ω' is what drives instability.

$\downarrow$

differential rotation - gravity.

$\Omega' < 0$ matters.

- Observe contrast between:

$\Omega' < 0$ and $\Phi = \frac{2\Omega}{r}\frac{\partial}{\partial r}(r^2\Omega) > 0$

- L-B & P arguments apply ⇒ angular momentum transported outward.

⇒ Noting: "Spring loop" ⟺ Magnetic Tension.
+ attachment ⟺ Freezing-in Law.

⇒ MRI mechanism is simple, robust linear instability mechanism leading to turbulence and angular momentum transport in Keplerian disks

which are ionized ⟺ hot

⇒ Caveat Emptor: Proto-planetary disks.

Velikov '59
* Chandrasekhar '61
Balbus, Hawley '91 et. seq.

→ MRI : Simplified Model

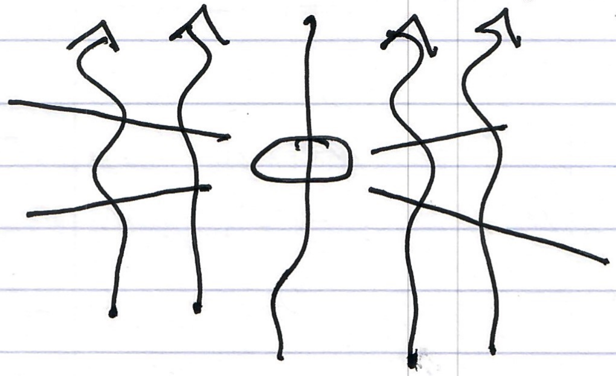
Consider ideal MHD :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad , \quad \underline{\nabla} \cdot \underline{v} = 0$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} = -\frac{1}{\rho} \nabla \left(P + \frac{B^2}{8\pi} \right) - \underbrace{\underline{\nabla} \Phi}_{\text{gravity}} + \frac{1}{4\pi\rho} \underline{B} \cdot \underline{\nabla} \underline{B}$$

$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{B} = \underline{B} \cdot \underline{\nabla} \underline{v}$$

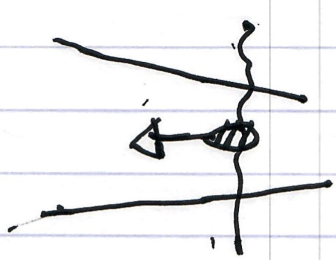
and disk, threaded by vertical field



2 cases:

- vertical field $\underline{B} = B_0 \hat{z}$
- azimuthal $\underline{B} = B_\phi(r) \hat{\phi}$

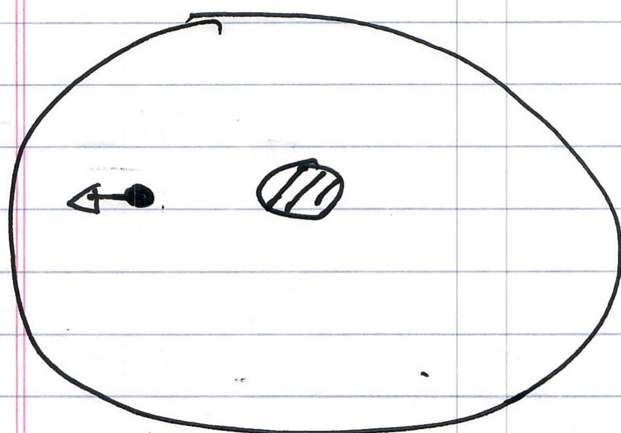
Consider:



perturb fluid element attached to B_0.

⇒ 

then, on plane disk:



consider as mechanics problem, i.e.

particle motion in r, ϕ in gravitational potential and acted on by magnetic forces (tension)

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \underbrace{\Phi}_{\text{grav. potential energy}} - \underbrace{U_B}_{\text{potential energy stored in field}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$\therefore$

$$\ddot{r} - r\dot{\phi}^2 = -\frac{d\Phi}{dr} - \underbrace{\frac{d}{dr}U_B}_{\text{generalized force.}}$$

radial force.

$$\boxed{\ddot{r} - r\dot{\phi}^2 = -\frac{d\Phi}{dr} + f_r}$$

and

$$\frac{d}{dt}\left(r^2\dot{\phi}\right) = -\frac{\partial U_B}{\partial \phi}$$

$$r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} = -\frac{\partial U_B}{\partial \phi}$$

$$\boxed{\begin{aligned} r\ddot{\phi} + 2\dot{r}\dot{\phi} &= -\frac{1}{r}\frac{\partial U_B}{\partial \phi} \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= f_\phi \end{aligned}}$$

Then, slab model $\leftrightarrow$ "shearing sheet"

$$\left.\begin{aligned} r &= r_0 + x \\ \phi &= \Omega t + \frac{y}{r_0} \end{aligned}\right] \qquad y = r_0(\phi - \Omega t)$$

then

$$\Rightarrow \quad \ddot{x} - (r_0 + x)\left(\Omega + \frac{\dot{y}}{r_0}\right)^3 = -\frac{GM}{(r_0+x)^2} + f_r$$

↳ expand

$$\ddot{x} - \cancel{r_0 \Omega^2} - \cancel{r_0}\, 2\Omega \frac{\dot{y}}{\cancel{r_0}} + h.o.t.$$

$$= -\cancel{\frac{GM}{r_0^2}} + \frac{GM}{r_0^2}\left(\frac{2x}{r_0}\right) + f_x$$

↳ repulsive!

backward h.o.

$$\Rightarrow \quad \ddot{x} - 2\Omega \dot{y} = \frac{GM}{r_0^2}\frac{2x}{r_0} + f_x$$

$$= r_0 \Omega^2 \frac{2x}{r_0} + f_x$$

$$= -x \frac{d\Omega^3}{d\ln r} + f_x$$

↳ <0.

$$\boxed{\ddot{x} - 2\Omega \dot{y} = -x \left.\frac{d\Omega^3}{d\ln r}\right|_{r_0} + f_x}$$

$$(r_0+x)\left(\frac{\ddot{y}}{r_0}\right) + 2\dot{x}\left(\Omega + \frac{\dot{y}}{r_0}\right) = f_y$$

$$\boxed{\ddot{y} + 2\dot{x}\Omega = f_y}$$

Finally:

$$\boxed{\begin{aligned} \ddot{x} - 2\Omega\dot{y} &= -x\,\frac{d\Omega^2}{d\ln r} + f_x \\ \ddot{y} + 2\Omega\dot{x} &= f_y \end{aligned}}$$

Corrolis

What are f_x, f_y ?

- forces result from "plucked" magnetic fields ~~↔~~ i.e. magnetic tension!

- Recall Alfven wave:

$$\frac{\partial \tilde{v}}{\partial t} = \cancel{-\nabla P^*} + \frac{B_0 \partial_z}{4\pi\rho_0}\tilde{B}$$

$$\frac{\partial}{\partial t}\tilde{\underline{B}} = B_0 \partial_z \tilde{\underline{V}}$$

$$\frac{\partial^2 \tilde{\underline{V}}}{\partial t} = \frac{(B_0 \partial_z)^2}{4\pi\rho_0} \tilde{\underline{V}}$$

$$\underline{V} = \partial_t \underline{\xi}$$

↳ fluid element displacement

$$\frac{\partial^2}{\partial t^2} \partial_t \underline{\xi} = V_A^2 \partial_z^2 \partial_t \underline{\xi}$$

$$\underline{\xi} = \hat{\underline{\xi}} \, e^{i(kz - \omega t)}$$

$$\boxed{\frac{\partial^2}{\partial t^2} \underline{\xi} = -k^2 V_A^2 \underline{\xi}}$$

force per unit mass

$$\underline{\xi} = (x, y)$$

so

$$\boxed{\begin{aligned} f_x &= -k^2 V_A^2 x \\ f_y &= -k^2 V_A^2 y. \end{aligned}}$$

$$\boxed{\begin{aligned} &\Rightarrow \ddot{x} - 2\Omega\dot{y} = -x\left.\frac{d\Omega^2}{d\ln r}\right|_{r_0} - (kV_A)^2 x \\ &\ddot{y} + 2\Omega\dot{x} = -(kV_A)^2 y \end{aligned}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{-i\omega t}$$

$$\boxed{\begin{aligned} &-\omega^2 x_0 + 2i\omega\Omega y_0 = -x_0\left.\frac{d\Omega^2}{d\ln r}\right|_{r_0} - (kV_A)^2 x_0 \\ &-\omega^2 y_0 - 2i\omega\Omega x_0 = -(kV_A)^2 y_0 \end{aligned}}$$

n.b. → absent rotation:

$$-\omega^2 x_0 = -(kV_A)^2 x_0$$

$$-\omega^2 y_0 = -(kV_A)^2 y_0$$

$\frac{d}{d\ln r}\Omega^2 < 0$ is driving gradient

$\Rightarrow$ shear Alfven wave!
(stable)

→ absent B-field:

$$\begin{vmatrix} \left(-\omega^2 + \frac{d\Omega^2}{d\ln r}\right) x_0 , & 2i\omega\Omega \\ -2i\Omega\omega , & -\omega^2 \end{vmatrix} = 0$$

$$(-\omega^2)\left(-\omega^2 + \frac{d\Omega^2}{d\ln r}\right) - 4\omega^2\Omega^2 = 0$$

$$-\omega^2 + \frac{d\Omega^2}{d\ln r} + 4\Omega^2 = 0$$

$$\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln r}$$

$$= 4\Omega^2 + \frac{d\Omega^2}{d\ln r} \qquad \Omega^2 = \frac{GM}{r^3}$$

$$= \underline{4}\Omega^2 - \underline{3}\Omega^2 \; !$$

$$> 0$$

→ stable radial oscillation

B-field is facilitator of instability → allows coupling

⇒ so determinant of full system:

$$(\omega^2)^2 - \omega^2\left[\frac{d\Omega^2}{d\ln r} + 4\Omega^2 + 2(kV_A)^2\right] + (kV_A)^2\left[(kV_A)^2 + \frac{d\,\text{[redacted]}\,\Omega^2}{d\ln r}\right] = 0$$

$$X^2 + BX + C = 0 \qquad X = \omega^2$$

$$\omega^2 = -\frac{B}{2} \pm \frac{1}{2}\left(B^2 - 4C\right)^{1/2}$$

$\omega^2 > 0$ → stable → oscillating solution → wave

$\omega^2 < 0$ → unstable → growing perturbations

B [redacted] < 0

Need $\boxed{C < 0}$ → so $\underbrace{-\frac{B}{2}}_{>0} \pm \underbrace{\frac{1}{2}\left(B^2 - 4C\right)^{1/2}}_{>0}$

⇒ $\boxed{\text{root negative.}}$

For tokamak:

$$(kv_A)^2 \sim (v_A/a)^2$$

$$d\Omega^2/d\ln r \sim d(V_\phi/R)^2/d\ln R$$

$V_A \gg V_\phi$.

Field as facilitator
not driver

$\Rightarrow$ Field must be "weak"

For Keplerian:

$$(k_z v_A)^2 - 3\Omega^2 < 0$$

$\Rightarrow$ unstable for $k < k_{crit}$

$$k_{crit} V_A = \sqrt{3}\,\Omega$$

so smallest scale of instability is

$$\lambda_{\substack{min \\ z}} = 2\pi/k_{crit}$$

$$= 2\pi/\sqrt{3}\,\Omega/v_A$$

$$= \left(\frac{2\pi}{\sqrt{3}}\right)\frac{V_A}{\Omega}$$

If $\lambda_{min_z} > 2h \Rightarrow$ mode does not "fit" on disk. — analysis

so $\left(\frac{2\pi}{\sqrt{3}}\right)\frac{V_A}{\Omega} < 2h = 2\frac{C_s}{\Omega}$

$\Rightarrow \left(\frac{C_s}{V_A}\right)^2 > \# \sim \frac{2\pi^2}{3}$

Need $\boxed{\beta > \frac{2\pi^2}{3}}$

$\beta = \frac{C_s^2}{V_A^2}$

$\sim \frac{\text{Plasma pressure}}{\text{Magnetic Pressure}}$

high β system!

N.B. — $\beta \sim 1$ (strongly magnetized — equipartition) $\Rightarrow$ stable

— MRI is weak field instability

$\Rightarrow$ MRI is strong!

$(\omega^2)^2 - \omega^2 B + C = 0$

$$2(\omega^2)\frac{d(\omega^2)}{d(kV_A)} - B\frac{d(\omega^2)}{d(kV_A)} - \omega^2\frac{dB}{dkV_A} + \frac{dG}{dkV_A} = 0$$

$$\frac{d(\omega^2)}{d(kV_A)} = \frac{\omega^2 dB/dkV_A - \frac{dG}{dkV_A}}{2\omega^2 - B} = 0$$

Crank $\Rightarrow$

$$(kV_A)_{max} = \sqrt{15/4}\,\Omega$$

kV_A for max growth

$$\boxed{\gamma_{max} = 3/4\,\Omega}$$

$i\gamma = \omega$

$\hookrightarrow$ e-folding ~~a~~ rotation period!

$\hookrightarrow$ strong.

MRI is robust instability.

$\Rightarrow$ Instability for:

$$\boxed{(k_z v_A)^2 + \frac{d\Omega^2}{d\ln r} < 0}$$

- instability possible for $\frac{d\Omega^2}{d\ln r} < 0$

$\rightarrow$ Keplerian ✓ possible

$\Rightarrow$ contrast to Rayleigh:

$$\Phi = \frac{2\Omega}{r}\frac{\partial}{\partial r}(r^2\Omega) < 0$$

$\Rightarrow$ differential rotation drive

Also observe;

$\rightarrow$ not relevant to strongly magnetized system

i.e. consider $(k_z v_A)^2 + d\Omega^2/d\ln r < 0$

condition.

→ Remaining questions:

① Linear Theory:

→ effects lower temp, weak ionization?

→ resistivity
→ ambi-polar diffusion ⇒ damp $\tilde{B}$

⋮

② Angular momentum transport and profile back-reaction

i.e. $\nu = \alpha c_s$ [redacted] H

↓ [redacted]

Shakura-Sunyaev alpha

$$\boxed{\alpha = \frac{\langle \tilde{v}_r \tilde{v}_\phi \rangle}{c_s^2} - \frac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi\rho c_s^2}}$$

→ calculate

N.B. Physics ⇒ magnetic stress dominant, show!

- Can relate to displacement $\underline{\xi}$.

$d\Omega^2/d\ln r < 0$ is drive

$\underline{\underline{so}}$

- expect instability relaxes Ω'.
- departure Keplerian profile?
  How much? - $\underline{\underline{\text{Gravity}}}$

$\Rightarrow$ Interesting QL dynamics problem.

③ $\underline{\text{Turbulence}}$

- Alfvenic + strong differential rotation.
- $\underline{\underline{\text{not}}}$ simple MHDT.

- some controversy about accuracy of sims.

$\Rightarrow$ what is NL state like?