Physics 239

Lecture 7

→ MHD waves, review

→ Magnetic Braking
+
Alfven Wave → Circular / Torsional

→ Energy Principle

→ Partially Ionized MHD + Ambipolar Diffusion.

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→ energetics → construct "Poynting theorem"

$$\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = \frac{B_0}{4\pi} \frac{\partial}{\partial z} \tilde{\underline{B}} \qquad (1)$$

$$\frac{\partial \tilde{\underline{B}}}{\partial t} = B_0 \frac{\partial}{\partial z} \tilde{\underline{V}} \qquad (2)$$

∴ construct energy evolution

Exercise

$$\underline{\mathcal{E} = \frac{\rho \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}} \quad \Rightarrow \text{energy density}$$

∴ $(1) \cdot \underline{V}$ and $(2) \cdot \underline{B}$ ⇒

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\tilde{\underline{V}}^2}{2} + \frac{\tilde{\underline{B}}^2}{8\pi}\right) = \frac{B_0}{4\pi}\left(\tilde{\underline{V}} \cdot \frac{\partial \tilde{\underline{B}}}{\partial z} + \tilde{\underline{B}} \cdot \frac{\partial \underline{V}}{\partial z}\right)$$

∴

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\tilde{\underline{V}}^2}{2} + \frac{\tilde{\underline{B}}^2}{8\pi}\right) = \frac{B_0}{4\pi} \frac{\partial}{\partial z} (\tilde{\underline{V}} \cdot \tilde{\underline{B}})$$

and have Poynting form: $\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0$

$$\boxed{\underline{S} = -\frac{B_0}{4\pi} (\underline{V} \cdot \underline{B})}$$

→ [wave energy density flux

$\int d^3x \, \underline{V} \cdot \underline{B}$ → cross helicity

↓

conserved in ideal MHD

N.B. $\underline{S} = \frac{c}{4\pi} \underline{E} \times \underline{B}$, $\underline{P} = \underline{S}/c^2$

wave energy density flux ↳ wave momentum density

$\underline{E} = -\frac{\underline{V} \times \underline{B}_0}{c}$

$$\underline{S} = -\frac{1}{4\pi} (\tilde{\underline{V}} \times \underline{B}_0) \times \tilde{\underline{B}} = \frac{1}{4\pi}\left[\cancel{(\tilde{\underline{B}} \cdot \underline{B}_0)} \tilde{\underline{V}} - (\tilde{\underline{V}} \cdot \tilde{\underline{B}}) \underline{B}_0\right]$$

$$= -\frac{\underline{B}_0}{4\pi} (\tilde{\underline{V}} \cdot \tilde{\underline{B}})$$

$\underline{S} = -\frac{\underline{B}_0}{4\pi} \underline{V} \cdot \underline{B}$

i.e. energy flows along field

$\rightarrow \underline{S} \sim \underline{V} \cdot \underline{B}$

Show

$H_c = \int d^3x \, \tilde{\underline{V}} \cdot \tilde{\underline{B}}$ → cross helicity
→ conserved in ideal MHD

Ex.: Show H_c conserved.

→ another way to formulate shear Alfven wave

velocity, es potential

since $\tilde{\underline{V}} \perp \underline{B}_0$, $\tilde{\underline{B}} \perp \underline{B}_0$ write $\tilde{\underline{V}} = \nabla\phi \times \hat{z}$

$\tilde{\underline{B}} = \nabla A \times \hat{z}$

↳ magnetic potential

i.e. $\underline{E} = \underline{E}_\perp$ so $\tilde{\underline{V}} = \frac{c}{B_0^2} \underline{E} \times \underline{B}_0$ in shear Alfven

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now,
$$\frac{\partial \underline{v}}{\partial t} = -\frac{1}{\rho_0}\nabla\left(\tilde{p + \frac{B^2}{8\pi}}\right) + \frac{\underline{B_0}\cdot\nabla\,\underline{B}}{4\pi\rho_0}$$

as $\tilde{\underline{v}}, \tilde{\underline{B}} \perp B_0$, take $\hat{z}\cdot\nabla\times$ $\Rightarrow$

$$\hat{z}\cdot\frac{\partial \underline{\omega}}{\partial t} = 0 + \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\,\hat{z}\cdot(\nabla\times\underline{B})$$

Now,
$$\underline{v} = \nabla\phi\times\hat{z}$$
$$= (\partial_y\phi, -\partial_x\phi, 0)$$

$$\hat{z}\cdot\nabla\times\tilde{\underline{B}} = \frac{4\pi}{c}\tilde{J}_z$$

$$\cancel{\nabla(\nabla\cdot\underline{A})} - \nabla^2 A_z = +\frac{4\pi}{c}\tilde{J}_z$$

$$\underline{\omega}_z = \hat{z}\cdot\underline{\omega} = -\nabla_\perp^2\phi \rightarrow \hat{z}\text{ component vorticity}$$

$\Rightarrow$ magnetic torque

$$\underbrace{\frac{\partial}{\partial t}\nabla_\perp^2\phi}_{\text{vorticity evolution}} = \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\nabla_\perp^2 A$$

$\nabla\times(\nabla\times\underline{A})$

and
$$\frac{\partial\tilde{\underline{B}}}{\partial t} = B_0\frac{\partial}{\partial z}\underline{v}$$
and $\hat{z}\cdot\nabla\times$ $\Rightarrow$

$$\underbrace{\frac{\partial}{\partial t}\nabla_\perp^2 A}_{\text{current evolution}} = B_0\underbrace{\frac{\partial}{\partial z}\nabla_\perp^2\phi}_{\parallel\text{ vorticity gradient}}$$

observe if "un- $\nabla_\perp^2$", have:

$$\frac{\partial A}{\partial t} - B_0 \frac{\partial}{\partial z} \phi = 0$$

$\Rightarrow$ basically means $E_{\parallel} = 0$ for Alfven waves.

$$\underline{E} = -\frac{\underline{V} \times \underline{B_0}}{c} \quad , \therefore \quad \hat{z} \cdot \frac{\tilde{V} \times B_0 \hat{z}}{c} = 0 \ \checkmark$$

$\therefore$ can write shear Alfven wave equations as

$$\left.\begin{aligned} E_{\parallel} = 0 = \frac{\partial A}{\partial t} - B_0 \frac{\partial \phi}{\partial z} &= 0 \\ \frac{\partial}{\partial t} \nabla_\perp^2 \phi &= \frac{B_0}{4\pi\rho_0} \frac{\partial}{\partial z} \nabla_\perp^2 A \end{aligned}\right\}$$

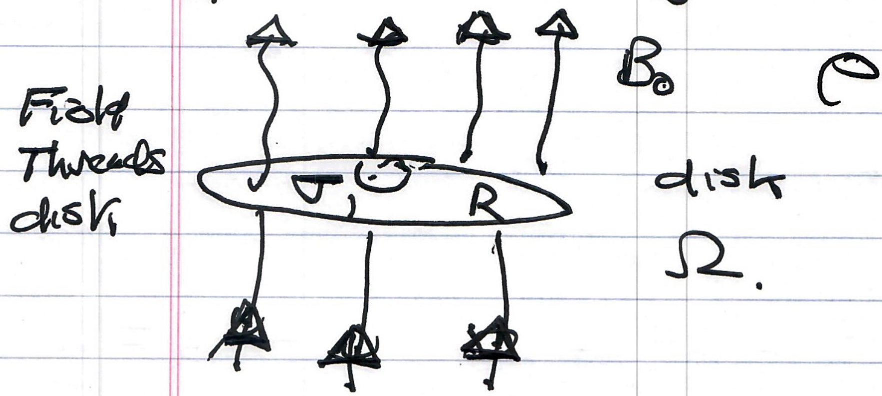
$\Rightarrow$ example of 'reduced equations'.

~~Now, need also consider:~~

~~$\rightarrow \underline{k} = k\hat{z}$, $\underline{\nabla} \cdot \underline{V} \neq 0$~~

~~What happens?~~

An Application : Magnetic Braking



- field threads disk
- treat as ideal

$1/\tau > \eta/L^2$

what happens ?

- field lines anchored in disk and frozen into plasma
- disk excites (torsional) Alfven waves. (vs shear)

$\downarrow$

- disk radiates angular momentum, which spins up plasma column.

$\Rightarrow$

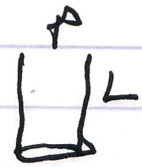
- disk slows down. $\Rightarrow$ magnetic braking

(i.e. Alfvenic radiative damping).

$\therefore$ Calculate time to spin up column above & below disk.

signal speed $\rightarrow$ V_A !

$V_A^2 = B_0^2 / 4\pi\rho$



So $L_{gas} \cong 2\, V_A \tau\, \underbrace{\pi R^2}_{A}\, \rho\, \Omega R^2$

± ; L (ht col.) ; A area ; density ; spec. ang mom. frozen in

$L_{disk} \cong \pi R^2 \sigma \Omega R^2$

$L_{gas} \sim L_{disk} \Rightarrow$ disk loses angular momentum.

$2 V_A \tau \cancel{\pi R^2} \rho \cancel{\Omega R^2} \sim \cancel{\pi R^2} \sigma \cancel{\Omega R^2}$

$\tau \sim \sigma / 2\rho V_A \sim \left(\frac{\sigma}{2\rho}\right) \frac{(4\pi\rho)^{1/2}}{B_0}$

τ: time for disk to lose angular momentum. $\rightarrow$ $\boxed{\text{braking time}}$

Message: – Alfven waves (torsional) transport angular momentum.

– Magnetic linkage is important!

Torsional Alfven Waves (cf Shu)

- previous → linear
- torsional → angular momentum transport

Consider: Finite amplitude torsional wave

- cylindrical coordinates : r, ϕ, z
- $\rho = \text{const.}$ $(\underline{\nabla} \cdot \underline{V} = 0)$

$\underline{V} = r\,\Omega(z,t)\,\hat{e}_\phi$

velocity as rotation, as fctn. $\hat{z}$

$\underline{B} = B_0 \hat{e}_z + \beta \rho \underline{V}$

used.

structure assures no nonlinearity (expect pressure balance)

then,

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times \underline{V} \times \underline{B}$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{B} \cdot \underline{\nabla}\, \underline{V} \Rightarrow \frac{\partial}{\partial t}\left[\beta \rho r \Omega(z,t)\right] = B_0 \frac{\partial}{\partial z} r\Omega$$

$$\therefore \quad \beta \rho \frac{\partial}{\partial t} \Omega(z,t) = B_0 \frac{\partial}{\partial z} \Omega$$

$$\boxed{B\rho \frac{\partial}{\partial t}\Omega(z,t) = B_0 \frac{\partial}{\partial z}\Omega}$$

nonlinearities vanish!

of course; $B\rho \rightarrow (4\pi\rho)^{1/2}$

$$\partial_t \Omega = V_A \partial_z \Omega .$$

Formally,

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right] = \cancel{-\rho\nabla\phi} - \underline{\nabla}p + \cancel{\underline{\nabla}\cdot\underline{\underline{\Pi}}_{vis}} + \frac{1}{4\pi}\,\underline{J}\times\underline{B}$$

$$\underbrace{\underline{v}\cdot\underline{\nabla}\,\underline{v} = \nabla\left(\frac{v^2}{2}\right) - \underline{v}\times\underline{\omega}} \qquad -\nabla\frac{B^2}{2} + \underline{B}\cdot\underline{\nabla}\,\underline{B}$$

then noting $\underline{v}$ only in $\hat{e}_\phi$:

$$\cancel{\phi} - \rho r\Omega^2 = -\frac{\partial p}{\partial r} - \frac{B^2}{4\pi} 2 r\Omega^2$$

pressure balance vs dynamic + magnetic pressure.

$$\boxed{-\rho r\Omega^2 = -\frac{\partial p}{\partial r} - \frac{B}{4\pi} 2r\Omega^2} \quad (1)$$

and

$$\boxed{\rho r \frac{\partial \Omega}{\partial t} = \frac{B B_0}{4\pi} r \frac{\partial \Omega}{\partial z}} \qquad (2)$$

e_ϕ

$$\left(\frac{B_0 \cdot \nabla \tilde{B}}{4\pi}\right)_\phi \qquad \text{tension}$$

and $\hat{z}$:

$$\boxed{0 = -\frac{\partial P}{\partial z} - \frac{B^2}{4\pi} r^2 \Omega \frac{\partial \Omega}{\partial z}} \qquad (3)$$

so $\hat{z}$ eqn:

$$\boxed{P = -\frac{B^2}{8\pi} r^2 \Omega^2(z,t) + f(r,t)} \qquad (4)$$

$\Rightarrow$ perturbed pressure balance.

N.B. Fluid motions can develop in $\hat{z}$ direction, eventually.

Now, plug (4) into (1):

$\Rightarrow$

$P_0 = \text{const.}$

so $P_0 + B_0^2/8\pi = \text{const.}$ [redacted] pressure balance.

and $\quad \beta = (4\pi\rho)^{1/2} \quad \rightarrow$ as we knew, V_A with B_0

Then

$$\underline{B} = B_0 \left[\hat{z} \pm \frac{r\Omega(z,t)}{V_A} e_\phi \right]$$

and

$$\boxed{\frac{\partial}{\partial t} \Omega(z,t) = V_A \frac{\partial}{\partial z} \Omega(z,t)}$$

Alfven waves.

linear eqn!

$$\Omega(z,t) = F(z \pm V_A t)$$

Note:

- relevance to braking:

$$\Omega(z=0,t) = \Omega_{disk}, \text{ etc.}$$

- note torsional Alfven wave is nonlinear solution, but obeys linear wave equation.

- important for momentum transport

MHD Waves:

→ 7 eqns → 7 waves

$\underset{\varsigma}{3 \times 2} + 1$

$\omega^2 =$

$3 \times 2 \begin{cases} \text{Fast} \\ \text{Shear Alfven} \\ \text{Slow} \end{cases}$

$+1$. Entropy

→ speeds: C_s, V_A

$\underline{B} = B_0 \hat{z}$

	$\underline{\nabla} \cdot \underline{V} = 0$	$\underline{\nabla} \cdot \underline{V} \neq 0$
$\underline{k} = k \hat{z}$	Shear Alfven $\omega = \pm k_{\parallel} V_A$	Acoustic $\omega = \pm k_{\parallel} C_s$
$\underline{k} = k \hat{x}$	X	Magnetosonic $\omega = \pm (V_A^2 + C_s^2)^{1/2} k$ why?

+ Entropy ($\omega = 0$).

Here, we focus primarily on incompressible MHD.

→ **Energy Principle**

- stability problems → inhomogeneity ∇P, $\nabla \rho$, $\underline{J}$ -----
  → complex magnetic geometry

- Exploit self-adjointness MHD equations (without flow; usually w/o self-gravitation).

- usual { Sturm-Liouville games / Variational Principle → Instab.

$\Rightarrow \quad \delta W(\underline{\xi})$

- consider $\underline{\xi}(x,t)$ → displacement of plasma from equilibrium

$$\underline{\tilde{V}} = \partial_t \underline{\xi}$$

- then compute $\delta W(\underline{\xi})$ → change in energy of system, due displacement.

$\delta W < 0$ → instability
$\delta W > 0$ → stability
} for specific $\underline{\xi}$

$$\omega^2 = -\delta W \Big/ \int \rho_0 \frac{\underline{\xi}^2}{2} d^3x$$

$$-\delta W = \frac{1}{2}\int d^3x \Big\{ \overset{①}{\frac{\underline{Q}^2}{4\pi}} + \overset{③}{\underline{J}_0(\underline{x}) \cdot (\underline{\xi} \times \underline{Q})}$$

$$+ \overset{②}{\gamma P_0(\underline{x}) (\underline{\nabla}\cdot\underline{\xi})^2} + \overset{④}{(\underline{\xi}\cdot\underline{\nabla} P_0(\underline{x}))(\underline{\nabla}\cdot\underline{\xi})}$$

$$\overset{⑤}{- (\underline{\xi}\cdot\underline{\nabla}\phi)(\underline{\nabla}\cdot(\rho_0 \underline{\xi}))} \Big\}$$

→ ①, ② > 0 — stability

$\underline{Q} = \underline{B}_0 \cdot \underline{\nabla}\,\underline{\xi} - \underline{\xi}\cdot\underline{\nabla}\,\underline{B} - \underline{B}(\underline{\nabla}\cdot\underline{\xi})$ induction

① → magnetic energy

② → compression energy

→ ③, ④, ⑤ → sign indeterminate.

→ kinks

∇P relaxation — interchanges

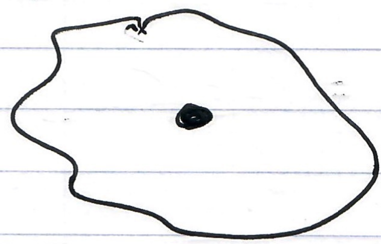
→ Rayleigh Taylor.

→ Very useful for insight into complex stability problems.

Ambipolar Diffusion and Partially Ionized MHD

Now – consider plasma + (lots) neutrals.

– classic example:



GMC collapse to core.

mostly cold, neutral gas

$\underline{V}_n \neq \underline{V}_i \sim \underline{V}_e \to \underline{V}$

$\downarrow$ plasma

$\boxed{\text{How does } \underline{B} \text{ field affect mostly neutral gas?}}$

$\Rightarrow$ change M/Φ of mostly neutral system?

Why? core : Virial Thm $\Rightarrow$ (see 28b)

$$R^3 P_{ext} \sim \left(\underbrace{\beta \frac{\Phi^2}{R}}_{\text{magnetic pressure}}^{\text{Flux - B pressure}} - \underbrace{\alpha \frac{G M^2}{R}}_{\text{gravity}}^{\text{gravity}} + \underbrace{\frac{3}{2} c_s^2 M}_{\text{Jeans scale}}^{\text{Pressure}} \right)$$

$$\underbrace{\Phi \sim R^2 B}_{\text{frozen} \Rightarrow \text{Alfven}}$$

$\therefore \Phi / M$ determines collapse!

$\Rightarrow$ of interest to explore physics which changes Φ/M!

$\Rightarrow$ hence ambipolar diffusion....

N.B. Also relevant to cool, planet forming disks - weak ionization.

The point: To change Φ/M, and to understand how $\underline{B}$ field affects neutrals

$\Rightarrow$ ion - neutral $\underline{\underline{\text{drag}}}$ (charge exchange)
(plasma)

$\underline{f_d}$ → force drag by ions / plasma on neutrals

$$\underline{f_d}_{i-n} = \gamma \rho_n \rho_i (\underline{V_0} - V_n)$$

↳ coeff from ion-neutral collisions

$$\underline{f_d}_{n-i} = \gamma \rho_n \rho_i (\underline{V_n} - \underline{V_0}) = -\underline{f_d}_{i-n}$$

And, for weakly ionized system:

→ ions and neutrals 'feel' same pressure, gravity etc.

→ ions will feel $\underline{\underline{J \times B}}$

$$\rho_i \frac{d\underline{v_i}}{dt} = -\underline{\nabla} P + \frac{\underline{J} \times \underline{B}}{c} + \underline{f_d}$$

$$\rho_n \frac{d\underline{v_n}}{dt} = -\nabla P + \underline{f_d}$$

Mom conservation

~ allows energy exchange, transfer etc.

- $\nu_{i-n} = n_i \langle v_{rel} \sigma_{i,n} \rangle$

- rate momentum transfer:

$$\sim n_n \left(\frac{m_n m_i}{m_i + m_n} \right) (\underline{v}_i - \underline{v}_n)\, \nu_{i-n}.$$

so in steady state:

$$\frac{\underline{J} \times \underline{B}}{c} = \underline{F}_d = \gamma \rho_n \rho_i (\underline{V}_i - \underline{V}_n)$$

$$= \gamma \rho_n \rho_i \underline{V}_d$$

drift velocity of ions relative to neutrals.

$$\underline{V}_d \sim \frac{\underline{J} \times \underline{B}}{c\, \gamma \rho_n \rho_i} \sim \frac{B^2}{4\pi \gamma \rho_n \rho_i L_B}$$

ion-neutral drift

$$\sim \frac{V_A^2}{V}$$

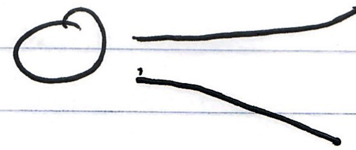
[Alfven with neutrals]

$$V = \gamma \rho_i L$$

typically $V_d < V_A$.

N.B. - Fractional ionization "assured" by cosmic rays, in GMC.

- In disk
  protoplanetary



⇒ Central object can photoionize at least portion of disk.

Ambipolar diffusion relevant to $\underline{B}$ - field dynamics.

Now, eqns:

$$\rho_n\left[\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n\cdot\underline{\nabla}\,\underline{v}_n\right] = -\underline{\nabla} P_n + \underline{f}_q$$

but $\underline{f}_q = \frac{\underline{J}\times\underline{B}}{c}$!

⇒

$$\boxed{\rho_n\left[\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n\cdot\underline{\nabla}\,\underline{v}_n\right] = -\underline{\nabla} P_n + \frac{\underline{J}\times\underline{B}}{c}}$$

$\therefore$ by exploiting balance

$$\frac{\underline{J} \times \underline{B}}{c} = \gamma \rho_i \rho_n (\underline{V}_0 - \underline{V}_1)$$

can "appear" that $\underline{J} \times \underline{B}$ force affects neutrals.

Also, freezing-in:

$$\partial_t \underline{B} = \underline{\nabla} \times (\underline{V}_i \times \underline{B})$$

but

$$\frac{\underline{J} \times \underline{B}}{c} = \gamma (\rho_i \rho_n)(\underline{V}_i - \underline{V}_1)$$

$$\boxed{\partial_t \underline{B} = \underline{\nabla} \times (\underline{V}_1 \times \underline{B}) + \underbrace{\left[\frac{\underline{\nabla} \times \left[\underline{B} \times (\underline{\nabla} \times \underline{B})\right]}{4\pi \gamma \rho_i \rho_n}\right]}_{\text{ambipolar diffusion of } \underline{B}}}$$

nonlinear

- $\underline{B}$ field frozen onto neutrals, with NL diffusion,

Schematically:

$$\frac{\partial}{\partial t}\underline{B} = \underline{\nabla}\times(\underline{V}_n \times \underline{B}) + \frac{D}{L^2}B$$

$$\begin{cases} D \cong V_A^2/(\gamma\rho_i) \;\hookrightarrow\; \sim B^2 \\ (\gamma\rho_i)^{-1} \sim \tau_{ni} \equiv \text{mean collision time in sea of ions.} \end{cases}$$

$\boxed{\text{NL diffusion}}$

and

$$1/\tau_{amb} \sim \frac{D}{L^2} \sim \frac{V_A^2}{\gamma\rho_i L^2} \sim \frac{V_d}{L}.$$

Characteristic time is drift time.

Can also examine heat:

spec. entropy

heating

!

$$\rho_n T_n \left(\frac{\partial s_n}{\partial t} + \underline{V}_n \cdot \nabla s_n \right) = \Gamma_n - \Lambda_n + \Gamma_{AD}$$

ds/dt

cooling

heating due ion-neutral drag.

power due drag

$$\Gamma_{AD} \simeq \underline{f}_d \cdot (\underline{V}_i - \underline{V}_n)$$

$$\cong \gamma \rho_i \rho_n |\underline{V}_i - \underline{V}_n|^2$$

Finally, 1D ambi-polar MHD:

all n

$$\partial_t \rho + \partial_z (\rho v) = 0$$

$$\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} \right) = g - \frac{\partial}{\partial z} \left(P + \frac{B^2}{8\pi} \right)$$

magn. pressure only.

$$P = c_s^2 \rho$$

$$\frac{\partial g}{\partial z} = -4\pi G \rho$$

and

$$\boxed{\begin{aligned} &\frac{\partial B}{\partial t} + \frac{\partial}{\partial z}(BV) = \frac{\partial}{\partial z}\left(\frac{B^2}{4\pi\gamma\rho_i\rho}\frac{\partial B}{\partial z}\right) \\ &\frac{d}{dt}\left(B/\rho\right) = \frac{1}{\rho}\frac{\partial}{\partial z}\left[\frac{B^2}{4\pi\gamma\rho\rho_i}\frac{\partial B}{\partial z}\right] \end{aligned}}$$

→ More generally: 2 fluids $\begin{cases} p \\ n \end{cases}$

coupling vs dynamical rates.