Physics 239

MHD 2

Lecture 6

- Freezing-in Law
- Stress, Energetics
- Virial Theorem
- Waves, especially Alfven

  Magnetic Braking
- Energy Principle

(i.) Freezing-in Laws.

$$\frac{d}{dt}\underline{l}_{12} = \underline{l}_{12}\cdot\underline{\nabla}\,\underline{V}$$

and induction $\Rightarrow$

$$\partial_t \underline{B} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} + \eta\,\nabla^2\underline{B}$$

$\underline{B}$ frozen in.

$\underline{B}/\rho$ frozen in.

and,

Flux Freezing:

$$\frac{d}{dt}\int \underline{B}\cdot d\underline{a} = 0$$

ii) Stress / Energetics

Momentum conservation $\Rightarrow$

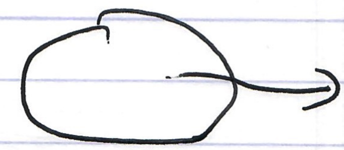
$$\partial_t(\rho \underline{v}) + \underbrace{\underline{\nabla}\cdot(\rho \underline{v}\,\underline{v})}_{\text{Reynolds stress tensor}} = \underbrace{-\underline{\nabla}\left(p + \frac{B^2}{8\pi}\right)}_{\text{Magnetic pressure}} + \underbrace{\underline{\nabla}\cdot\frac{\underline{B}\,\underline{B}}{4\pi}}_{\text{Magnetic stress, Maxwell stress.}} + \underline{f}_b$$

$$\partial_t(\rho \underline{v}) = -\underline{\nabla}\cdot\underline{\underline{T}} + \rho\underline{g}$$

$$T_{ij} = \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho v_i v_j$$

blob $\Rightarrow$
change in momentum is flux thru surface

$$\frac{d\underline{P}}{dt} = -\int d\underline{s}\cdot \underline{\underline{T}}$$



$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho\,\underline{v}\,\underline{v}$$

$\underline{\underline{T}}_R \cdot d\underline{s} = \rho\,\underline{v}\,\underline{v}\cdot d\underline{s}$ → flux of momentum density thru surface

$\underline{\underline{T}}_{P_{Tot}} \cdot d\underline{s} = \left(p + \frac{B^2}{8\pi}\right)\cdot d\underline{s}$ → pressure forces

$\underline{\underline{T}}_{Mag\,Ten} \cdot d\underline{s} = -\frac{\underline{B}}{4\pi}\underbrace{(\underline{B}\cdot d\underline{s})}_{\text{tension}}$ → $\boxed{\text{Magnetic tension}}$ on $+\underline{B}$ direction, thru surface.

\# lines out

dyad (sign order.)

Analogy

T	$B/4\pi$
μ	$\frac{\#\text{string}}{\text{area}} \sim B$ so $T = \rho/B$ mass/length
$v_{ph}^2 = T/\mu$	$v_{ph}^2 = \frac{B}{4\pi\ \rho/B} = V_A^2 = \frac{B^2}{\boxed{4\pi}\rho}$

Alfven speed.

Magnetic tension ⇒ Alfven wave

[fundamental idea!]

and for disks;

- recall $T_{r\phi}$ ⇒ radial flux of azimuthal momentum

⇝ key quantity

- for MHD flows

$$\Pi_{r\phi} = \rho r \left[\langle \tilde{v}_r \tilde{v}_\phi \rangle - \frac{\langle \tilde{B}_r \tilde{B}_\phi \rangle}{4\pi\rho} \right]$$

-- Radial flux of angular momentum.

- Note role of Maxwell stress!

[Dominant for MRI] (sign → Ω!)

If slab/shearing sheet:

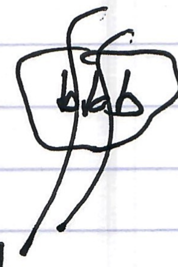
$$\Pi_{x,y} = \rho \left[\langle \tilde{v}_x \tilde{v}_y \rangle - \frac{\langle \tilde{B}_x \tilde{B}_y \rangle}{4\pi\rho} \right]$$

N.B. Turbulent viscosity $\leftrightarrow$ Turbulent Reynolds Maxwell stresses !

i.e. $\langle \tilde{v}_x \tilde{v}_y \rangle \rightarrow -\nu \dfrac{\partial \langle V_y \rangle}{\partial x}$

Energy:

$$E = E_{kin} + E_p + E_B + E_{grav.}$$



$$= \int_V d^3x \left[\underbrace{\frac{1}{2}\rho V^2}_{\text{kinetic}} + \underbrace{\frac{P}{\gamma-1}}_{\text{Th.}} + \underbrace{\frac{B^2}{8\pi}}_{\text{Magnetic}} + \underbrace{\frac{\rho\phi}{2}}_{\text{Grav.}} \right]$$

$\underline{g} = -\nabla\phi$

$\nabla^2\phi = 4\pi G\rho$

can show:

$$\frac{dE}{dt} = -\int d\underline{S}\cdot\left[\underbrace{\rho\underline{V}\frac{V^2}{2}}_{\substack{\text{flux kinetic energy}\\ \text{flux grav}}} + \underbrace{\frac{\gamma P\underline{V}}{\gamma-1}}_{\substack{\text{Conv. Thermal}\\ +\ PV\ \text{work}}} - \underbrace{\frac{\overbrace{(\underline{V}\times\underline{B})}^{-E}\times\underline{B}}{4\pi}}_{\substack{\underline{E}\times\underline{B}\ \to\ \text{Poynting}\\ +\ \text{(Ohm Law)}}} + \rho\underline{V}\phi \right]$$

n.b. energy change $\Rightarrow$ surface terms this brings us to ....

→ $\boxed{\text{Virial Theorems in MHD}}$

- what is a virial theorem
- why yet another theorem?

[Beloved in Astrophysics]
↓
useful in galactic dynamics

→ Virial Theorems are:

- space/time averaged energy theorems
- $\boxed{\text{"lumped parameter" relations for energies in complex, multi-element interacting systems}}$
- useful for 'back-of-envelope' estimates, etc.
- logically extend the moment program:

$$\underbrace{f(\underline{x}, \underline{v}, t)}_{\text{phase space fluid}} \xrightarrow[\underline{v}\text{ moments}]{} \underbrace{n(\underline{x}, t), \underline{u}, T}_{\text{position space fluid}} \xrightarrow[\text{virial integrals}]{} \underbrace{E_u, E_B}_{\#}, \text{ etc.}$$

virial integrals → space integrals

THE Question for Virial Theorem

Before proceeding:

$\boxed{\text{Can an } \underline{\text{isolated}} \text{ blob of MHD plasma } \underline{\text{confine}} \ \underline{\text{itself}} \text{ without self gravity?}}$

Easily answered by $\boxed{\text{Virial Theorem}}$...

Recall, for system of particles, Virial theorem (Mechanics) derived by considering:

$$\frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right) = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= \underbrace{2T}_{\text{kinetic energy}} + \underbrace{\sum_i \left(-\frac{\partial U}{\partial \underline{x}_i}\right) \cdot \underline{x}_i}_{\text{via Newton's Law}}$$

$$\langle \ \rangle = \int_0^T \frac{dt}{T}, \quad T \to \infty$$

blob

Now, if $\boxed{\sum_i \underline{p}_i \cdot \underline{x}_i \text{ bounded}}$,

$$\left\langle \frac{d}{dt} \sum \underline{p}_i \cdot \underline{x}_i \right\rangle = \frac{1}{T} \int_0^T dt \, \frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right) \underset{T \to \infty}{\longrightarrow} 0$$

so ...

Aside,

Aside: $\boxed{\text{simplest realization of negative specific heat 'paradox', i.e.}}$

(R) $\rightarrow$ consider 'blob' of self gravitating matter

$$E \sim -1/R$$

if radiation  $\rightarrow$ E decreases $\rightarrow$
R decreases

$R \downarrow$

$\therefore$ $(-E)$ increases $\Rightarrow$ $\langle T \rangle$ increases — Virial Theorem
$(-E) \rightarrow$ kinetic energy

but $\langle T \rangle \sim$ temperature, so have cycle of: radiative cooling $\Rightarrow$ [temperature increase] !

$$\Rightarrow \underbrace{c}_{\text{specific heat}} < 0 \ !!?$$

In the days before the discovery of nuclear fusion, this was thought to be what 'heated' stars. Kelvin, in particular, was a ~~propon~~ proponent.

Now, proceeding to full Virial theorem ...

(first) Virial of system

$$2\langle T\rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i \right\rangle$$

Further, if $U = U(\underline{x}_1, \underline{x}_2, \cdots \underline{x}_n)$

where $U(\alpha \underline{x}_1, \alpha \underline{x}_2, \ldots \alpha \underline{x}_n) = \alpha^k U(\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n)$

(scaling ⟺ structure of power law potentials → i.e. h.o. → $k=2$, Coulomb → $k=-1$)

scaling!

homogeneous function

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle}$$

but of course:

$$T + U = \langle T\rangle + \langle U\rangle = E$$

$$\boxed{\underset{\text{gravity}}{\langle T\rangle = -E}}$$

then

$$\left(\frac{k}{2}+1\right)\langle U\rangle = E$$

$$\boxed{\langle U\rangle = \frac{2}{k+2}E}, \quad \boxed{\langle T\rangle = \frac{k\,E}{k+2}}$$

check: $k=2$, $\langle U\rangle = (1/2)E$, $\langle T\rangle = (1/2)E$ ✓

$\underset{\text{gravity}}{k=-1}$, $\boxed{\begin{matrix}\langle T\rangle = -E \\ (\Rightarrow E<0)\end{matrix}}$ ✓ ⇒ bounded motion only if total energy negative (i.e. bound state)

→ Consider equation of motion:

$$\begin{cases} \underbrace{\frac{\partial}{\partial t}(\rho v_i)}_{\text{momentum}} = -\frac{\partial}{\partial x_j}\underbrace{T_{ij}}_{\text{\underline{full} stress tensor}} \\ T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \cancel{\rho \phi \delta_{ij}} \end{cases}$$

Now, recalling relation of **Virial to $\frac{d}{dt}(\underline{p} \cdot \underline{x})$**
⇒ consider:

Moment inertia / distrib.

$$\boxed{I_{ij} = \int d^3x \, \rho \, x_i x_j}$$

(~ $\underline{\text{moment of inertia}}$) → distrib

Virial theorem is for tensor ...

and

$$\frac{d}{dt} I_{ij} = \int d^3x \, \frac{\partial \rho}{\partial t} x_i x_j$$

<u>Continuity</u>

$$= -\int d^3x \, \frac{\partial}{\partial x_t}(\rho v_t) \, x_i x_j$$

integrating by parts assuming ρ <u>compact</u> (i.e. 'blob' of interest)

$$= \int d^3x \left[\rho x_i v_j + \rho x_j v_i\right]$$

<u>so</u>

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i \left(\frac{\partial}{\partial t}\rho v_j\right) + x_j \frac{\partial}{\partial t}(\rho v_i)\right]$$

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but $\frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_k} T_{ik}$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \left[x_i \frac{\partial T_{j,t}}{\partial x_t} + x_j \frac{\partial T_{i,t}}{\partial x_t} \right]$$

and integrating by parts, assuming compact blob, no external linkage

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = +\int d^3x \left[\delta_{it} T_{j,t} + \delta_{jt} T_{i,t} \right]$$

$\partial x_i / \partial x_t = 0$ unless $i = t$

$$= +\int d^3x \left[T_{j,i} + T_{i,j} \right]$$

and as $\underline{T_{ij}}$ manifestly symmetric $\Rightarrow$

$$\boxed{\begin{aligned} \frac{1}{2} \frac{d^2 I_{ij}}{dt^2} &= +\int d^3x \, T_{ij} \\ T_{ij} &= \rho V_i V_j + \left(P + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\delta_{ij} \end{aligned}}$$

— tensor virial theorem.

[note unlike simple pt particle example, time dependence remains.]

Now, to make contact with notions of energy, etc., useful to contract the tensor

$$I \equiv I_{i,i} = \mathrm{tr}\, I_{i,j}$$

repeated indexes summed

$\mathrm{tr}\,(\mathrm{V.T.}) \Rightarrow$

$$\mathrm{tr}\, \frac{1}{2}\frac{d^2 I_{i,j}}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x\, \frac{\rho x^2}{2}\right)$$

$$= \mathrm{tr} \int d^3x \left[\rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{i,j} - \frac{B_i B_j}{4\pi} + \rho\phi\, \delta_{ij}\right]$$

$$= \int d^3x \left[\rho v^2 + 3\left(p + \frac{B^2}{8\pi}\right) - \frac{B^2}{4\pi} + 3\rho\phi\right]$$

$\therefore \quad I \equiv \int d^3x\, \rho x^2/2 \quad \Rightarrow$

$$\boxed{\frac{d^2 I}{dt^2} = \int d^3x \left[\rho v^2 + 3p + \frac{B^2}{8\pi} + 3\rho\phi\right]}$$

$\rightarrow$ Scalar Virial Theorem

Now, first neglect self-gravitation $\Rightarrow$
↳ will return in Galaxies

$$\frac{d^2 I}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x \, \frac{\rho x^2}{2}\right)$$

$$= \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$$

Now → can an isolated blob of MHD fluid confine itself ??

If 'self-confined' $\Rightarrow$ $\frac{dI}{dt} \leq 0$

$\frac{d^2 I}{dt^2} \leq 0$

i.e. quiescent $\Rightarrow \dot{I}, \ddot{I} = 0$

stable pulsation $\Rightarrow \ddot{I} = -\Omega^2 I < 0$

but have $\ddot{I} = \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$

so even if $V^2 = 0$ (no fluid motion in blob) $\Rightarrow$

$p > 0$, $B^2/8\pi > 0$ $\Rightarrow \ddot{I} > 0$!

$\therefore$ $\boxed{\text{\underline{No} → isolated blob \underline{can't confine} itself.}}$

More generally, noting that

$$E_V = \int d^3x\, \rho V^2/2$$

$$E_P = \int d^3x \frac{P}{\gamma - 1} = \frac{3}{2}\int d^3x\, P \qquad \text{(gas)}$$

$$E_B = \int d^3x \frac{B^2}{8\pi}$$

can write scalar Virial theorem in form:

$$\boxed{\frac{d^2I}{dt^2} = 2E_V + 2E_P + E_B}$$

simple relation in terms energies.

Aside: → So, isolated blob can't confine itself

⇒ how is
- tokamak → $\underline{B_T}$ for stability; not transport
- or - better — macro-confinement
- RFP → weak external B_T guide (negligible)

confined ?? Confinement by wall is unacceptable ...

Lecture IV — Linear Waves in MHD

Linear Waves, Instabilities and Energy Principle

→ Contents

- this unit presents the linear structure, response theory and energetics for MHD
- proceed by: a) linear waves
  - b) Least Action and Energy Principle
  - c.) simple linear instabilities
- Later discuss nonlinear evolution, i.e.:
  i.e. a.) MHD shocks
  - b.) collisionless shocks
  - c.) MHD turbulence (later)

A) Linear Waves in MHD

$$\boxed{\text{Anisotropy is key!}}$$

i.) Simple Cases

- before proceeding with full crank useful to discuss some limiting cases in depth
- always have

$$\boxed{\left.\begin{array}{l} \underline{B}_0 = B_0 \hat{z} \\ \rho = \rho_0,\ P = P_0 \end{array}\right\} \text{uniform}}$$

- consider

	$\nabla \cdot \underline{v} = 0$	$\nabla \cdot \underline{v} \neq 0$	
$\underline{k} = k\hat{z}$	Shear Alfven	Acoustic	– parallel propagation
$\underline{k} = k\hat{x}$	X	Magnetosonic	– perpendicular propagation

$B_0^2/4\pi\rho_0 \equiv V_A^2$ Alfven velocity

$\Rightarrow \begin{cases} \omega^2 = k_\parallel^2 V_A^2 & \rightarrow \text{dispersion relation for \underline{shear} \underline{Alfven} \underline{wave}} \\ V_{ph} = V_{gr} = V_A \hat{z} & \rightarrow \text{speed} \begin{cases} \text{phase} \\ \text{group} \end{cases} \end{cases}$

wave propagates along $\hat{z}$ at Alfven speed

→ wave is consequence of <u>magnetic</u> <u>tension</u>

$\dfrac{T}{M} \rightarrow \dfrac{B/4\pi}{\rho_0/B}$ ~ <u>tension</u> - <u>in</u> <u>line</u> $\Rightarrow V_A^2$

↳ <u>mass</u> - <u>per</u> - <u>line</u>

→ tension ↔ plucking $\Rightarrow \underline{\tilde{V}} \perp B_0$

$\left(\begin{matrix} \nabla \cdot \underline{V} = 0 \\ \text{parallel variation} \end{matrix} \right)$

<u>i.e.</u> $\begin{cases} \underline{\tilde{V}} = \tilde{V}_x \hat{x} \\ \underline{\tilde{B}} = \dfrac{\partial}{\partial z} (\tilde{V}_x B_0) = \tilde{B}_x \hat{x} \end{cases}$

in shear Alfven wave:

$\begin{cases} \underline{\tilde{V}}, \underline{\tilde{B}} \perp \underline{B_0} \\ \underline{\tilde{V}} \parallel \underline{\tilde{B}}, \text{ but out of phase} \end{cases}$

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$\rightarrow$ $\underline{k} = k\hat{z}$, $\boxed{\nabla\cdot\underline{V} = 0}$

Shear Alfven Wave

$$\left.\begin{aligned}\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} &= -\nabla\left(\tilde{p} + \frac{\tilde{B}^2}{8\pi}\right) + \underbrace{\frac{\underline{B}_0\cdot\underline{\nabla}\,\tilde{\underline{B}}}{4\pi}}_{\text{tension}} \\ \frac{\partial \tilde{\underline{B}}}{\partial t} &= \underline{B}_0\cdot\underline{\nabla}\,\tilde{\underline{V}}\end{aligned}\right\}\text{ linearized eqns.}$$

Now, $\underline{\nabla}\cdot\tilde{\underline{V}} = 0 \Rightarrow$

$$-\nabla^2\left(\tilde{p} + \frac{\underline{B}_0\cdot\tilde{\underline{B}}}{8\pi}\right) + \underline{B}_0\cdot\nabla\cancelto{0}{(\nabla\cdot\tilde{\underline{B}})} = 0$$

ρ_0, B_0 uniform

$$\therefore \boxed{\tilde{p} + \frac{\underline{B}_0\cdot\tilde{\underline{B}}}{8\pi} = 0}$$

$\rightarrow$ "perturbed pressure balance"

$\rightarrow$ holds for incompressible (and weakly compressible) modes *

$$\Rightarrow \quad \left.\begin{aligned}\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} &= \frac{B_0}{4\pi}\frac{\partial}{\partial z}\tilde{\underline{B}} \\ \frac{\partial \tilde{\underline{B}}}{\partial t} &= B_0 \frac{\partial}{\partial z}\tilde{\underline{V}}\end{aligned}\right]$$

$$\therefore \boxed{\frac{\partial^2 \tilde{\underline{V}}}{\partial t^2} = \frac{B_0^2}{4\pi\rho_0}\frac{\partial^2}{\partial z^2}\tilde{\underline{V}}}$$

→ energetics → construct "Poynting theorem"

$$\rho_0 \frac{\partial \tilde{V}}{\partial t} = \frac{B_0}{4\pi} \frac{\partial}{\partial z} \tilde{B} \qquad (1)$$

$$\frac{\partial \tilde{B}}{\partial t} = B_0 \frac{\partial}{\partial z} \tilde{V} \qquad (2)$$

∴ construct energy evolution

Exercise

$$\underline{\mathcal{E} = \frac{\rho \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}} \rightarrow \text{energy density}$$

∴ (1) · V and (2) · B ⇒

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi}\left(\underline{V} \cdot \frac{\partial \underline{B}}{\partial z} + \underline{B} \cdot \frac{\partial \underline{V}}{\partial z}\right)$$

∴

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi} \frac{\partial}{\partial z} (\tilde{\underline{V}} \cdot \underline{B})$$

and have Poynting form: $\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \underline{S} = 0$

$$\boxed{\underline{S} = -\frac{B_0}{4\pi}(\underline{V} \cdot \underline{B})}$$

→ [wave energy density flux

$\int d^3x \, \underline{V} \cdot \underline{B}$ → cross helicity

↓

conserved in ideal MHD

N.B. $\underline{S} = \frac{c}{4\pi} \underline{E} \times \underline{B}$, $\underline{P} = \underline{S}/c^2$

wave energy density flux → wave momentum density

$\underline{E} = -\frac{\underline{V} \times \underline{B}_0}{c}$

$$\underline{S} = -\frac{1}{4\pi} (\tilde{\underline{V}} \times \underline{B}_0) \times \tilde{\underline{B}} = \frac{1}{4\pi} \left[\cancel{(\tilde{\underline{B}} \cdot \underline{B}_0)} \tilde{\underline{V}} - (\tilde{\underline{V}} \cdot \tilde{\underline{B}}) \underline{B}_0 \right]$$

$$= -\frac{\underline{B}_0}{4\pi} (\tilde{\underline{V}} \cdot \tilde{\underline{B}})$$

$\underline{S} = -\frac{\underline{B}_0}{4\pi} \underline{V} \cdot \underline{B}$.

i.e. energy flows along field

→ $\underline{S} \sim \underline{V} \cdot \underline{B}$

shows

$H_c = \int d^3x \, \tilde{\underline{V}} \cdot \tilde{\underline{B}}$ → cross helicity
→ conserved in ideal MHD

Ex.: Show H_c conserved.

→ another way to formulate shear Alfven wave

→ velocity potential

since $\tilde{\underline{V}} \perp B_0$, $\tilde{\underline{B}} \perp B_0$ write $\tilde{\underline{V}} = \nabla\phi \times \hat{z}$, $\underline{B} = \nabla A \times \hat{z}$

→ magnetic potential

i.e $\underline{E} = \underline{E}_\perp$ so $\tilde{\underline{V}} = \frac{c}{B_0^2} \underline{E} \times \underline{B}_0$ in shear Alfven

73.

now, $\frac{\partial \underline{v}}{\partial t} = -\frac{1}{\rho_0}\underline{\nabla}\left(p + \frac{\tilde{B}^2}{8\pi}\right) + \frac{\underline{B_0}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho_0}$

as $\tilde{\underline{v}}, \tilde{\underline{B}} \perp B_0$, take $\hat{z}\cdot\underline{\nabla}\times$ $\Rightarrow$

$$\hat{z}\cdot\frac{\partial \underline{\omega}}{\partial t} = 0 + \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\,\hat{z}\cdot(\underline{\nabla}\times\underline{B})$$

Now, $\underline{v} = \underline{\nabla}\phi\times\hat{z}$ $\qquad$ $\hat{z}\cdot\underline{\nabla}\times\tilde{\underline{B}} = \frac{4\pi}{c}\tilde{J}_z$

$= (\partial_y\phi, -\partial_x\phi, 0)$ $\qquad$ $\cancel{\underline{\nabla}(\underline{\nabla}\cdot\underline{A})} - \nabla_\perp^2 A_z = +\frac{4\pi}{c}\tilde{J}_z$

$\underline{\omega}_z = \hat{z}\cdot\underline{\omega} = -\nabla_\perp^2\phi \rightarrow \hat{z}$ component vorticity

$\Rightarrow$

$$\underbrace{\frac{\partial}{\partial t}\nabla_\perp^2\phi}_{\text{vorticity evolution}} = \underbrace{\frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\nabla_\perp^2 A}_{\rightarrow\ \text{magnetic torque}}$$

$\underline{\nabla}\times(\underline{v}\times\underline{B})$

and $\frac{\partial\tilde{\underline{B}}}{\partial t} = B_0\frac{\partial}{\partial z}\underline{v}$ $\qquad$ and $\hat{z}\cdot\underline{\nabla}\times$ $\Rightarrow$

$$\underbrace{\frac{\partial}{\partial t}\nabla_\perp^2 A}_{\text{current evolution}} = \underbrace{B_0\frac{\partial}{\partial z}\nabla_\perp^2\phi}_{\parallel\text{ vorticity gradient}}$$

observe if "$\omega \sim \nabla_\perp^2$", have:

$$\frac{\partial A}{\partial t} - B_0 \frac{\partial}{\partial z} \phi = 0$$

⇒ basically means $E_\parallel = 0$ for Alfven waves.

$$\underline{E} = -\frac{\underline{V} \times \underline{B}_0}{c} \quad , \therefore \quad \hat{z} \cdot \frac{\tilde{\underline{V}} \times B_0 \hat{z}}{c} = 0 \; \checkmark$$

∴ can write shear Alfven wave equations as

$$\left. \begin{aligned} E_\parallel &= 0 = \frac{\partial A}{\partial t} - B_0 \frac{\partial \phi}{\partial z} = 0 \\ \frac{\partial}{\partial t} \nabla_\perp^2 \phi &= \frac{B_0}{4\pi \rho_0} \frac{\partial}{\partial z} \nabla_\perp^2 A \end{aligned} \right\}$$

→ example of 'reduced equations'.

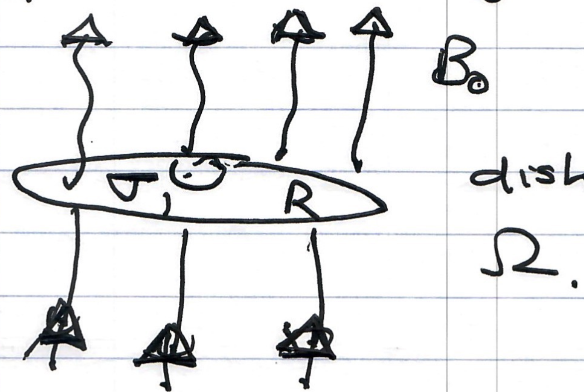
~~Now, need also consider:~~

~~→ $\underline{k} = k\hat{z}$, $\underline{\nabla} \cdot \underline{v} \neq 0$~~

~~What happens?~~

An Application : Magnetic Braking

Field threads disk



- field threads disk
- treat as ideal

$1/\tau > \eta/L^2$

What happens?

- field lines anchored in disk and frozen into plasma
- disk excites (torsional) Alfven waves. (vs shear)

$\downarrow$

- disk radiates angular momentum, which spins up plasma column

$\Rightarrow$

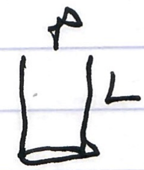
- disk slows down. $\Rightarrow$ magnetic braking

(i.e. Alfvenic radiative damping).

$\therefore$ Calculate time to spin up column above & below disk.

Signal speed $\rightarrow$ V_A!

$$V_A^2 = B_0^2 / 4\pi\rho$$



So $L_{gas} \cong 2\, V_A \tau\, \pi R^2\, \rho\, \Omega R^2$

$2 \to \pm$; $V_A \tau \to L$ (ht col.); $\pi R^2 \to A$ (area); $\rho \to$ density; $\Omega R^2 \to$ spec. ang mom. frozen in

$$L_{disk} \cong \pi R^2 \sigma \Omega R^2$$

$L_{gas} \sim L_{disk} \Rightarrow$ disk loses angular momentum.

$$2 V_A \tau \cancel{\pi R^2} \rho \cancel{\Omega R^2} \sim \cancel{\pi R^2} \sigma \cancel{\Omega R^2}$$

$$\tau \sim \sigma / 2\rho V_A \sim \left(\frac{\sigma}{2\rho}\right) \frac{(4\pi\rho)^{1/2}}{B_0}$$

$\tau \to$ time for disk to lose angular momentum. $\to$ $\boxed{\text{braking time}}$

Message: — Alfven waves (torsional) transport angular momentum.

— Magnetic linkage is important!

MHD Waves:

→ 7 eqns → 7 waves

3×2 { Fast, Shear Alfven, Slow }

$3 \times \underset{\omega^2 =}{\underbrace{2}} \quad +1$

+1 . Entropy

→ speeds: c_s, V_A

$\underline{B} = B_0 \hat{z}$

	$\nabla \cdot \underline{V} = 0$	$\nabla \cdot \underline{V} \neq 0$
$\underline{k} = k\hat{z}$	Shear Alfven $\omega = \pm k_{\parallel} V_A$	Acoustic $\omega = \pm k_{\parallel} c_s$
$\underline{k} = k\hat{x}$	X	Magnetosonic $\omega = \pm (v_A^2 + c_s^2)^{1/2} k$

why?

+ Entropy $(\omega = 0)$.

Here, we focus primarily on incompressible MHD.

→ Energy Principle

- stability problems
  - inhomogeneity ∇P, $\nabla \rho$, $\underline{J}$ ....
  - complex magnetic geometry

- Exploit self-adjointness MHD equations (without flow; usually w/o self-gravitation).

- usual { Sturm-Liouville games / Variational Principle } → Insb

$\Rightarrow \delta W(\underline{\xi})$

- consider $\underline{\xi}(x, t)$ → displacement of plasma from equilibrium

$$\underline{\tilde{V}} = \partial_t \underline{\xi}$$

- then compute $\delta W(\underline{\xi})$ → change in energy of system, due displacement.

$\delta W < 0$ → instability
$\delta W > 0$ → stability

} for specific $\underline{\xi}$

$$\omega^2 = -\delta W \Big/ \int \rho_0 \frac{\xi^2}{2} d^3x$$

$$-\delta W = \frac{1}{2}\int d^3x \left\{ \frac{Q^2}{4\pi} + \underline{J_0}(\underline{x}) \cdot (\underline{\xi} \times \underline{Q}) + \gamma P_0(\underline{x}) (\underline{\nabla} \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \underline{\nabla} P_0(\underline{x}))(\underline{\nabla} \cdot \underline{\xi}) - (\underline{\xi} \cdot \underline{\nabla} \phi)(\underline{\nabla} \cdot (\rho_0 \underline{\xi})) \right\}$$

(1) $Q^2/4\pi$, (2) $\gamma P_0 (\nabla\cdot\xi)^2$, (3) $J_0 \cdot (\xi \times Q)$, (4) $(\xi\cdot\nabla P_0)(\nabla\cdot\xi)$, (5) $(\xi\cdot\nabla\phi)(\nabla\cdot(\rho_0\xi))$

→ (1), (2) > 0 – stability

$Q = \underline{B_0} \cdot \underline{\nabla}\, \underline{\xi} - \underline{\xi} \cdot \underline{\nabla} B - \underline{B} (\underline{\nabla} \cdot \underline{\xi})$ induction

(1) → magnetic energy

(2) → compression energy

→ (3), (4), (5) → sign indeterminate.

(3) → kinks

(4) → ∇P relaxation – interchanges

(5) → $\nabla \rho$ → Rayleigh Taylor.

→ Very useful for insight into complex stability problems.