Lecture 4

→ Accretion :- 2 particle model, completed

- Elastic coupling → MRI

→ to MHD

Seek:

→ Lower energy

→ Conserve total angular momentum
and
conserve total mass

N.B: Important:

Rayleigh: Conserves angular momentum (mass) of each particle

L-B & P: Conserves total angular momentum and mass of pair

⇒ Pair can interact, so long as total angular momentum conserved.

Why 2 ?

→ account for mass and angular momentum

→ interaction anticipates MRI.

$L_0/m \equiv h \Rightarrow$ specific angular momentum

$E/m \equiv \epsilon \rightarrow$ specific energy.

grav pot. (object)

$$\epsilon = \frac{1}{2}\left(u_r^2 + u_z^2\right) + \frac{h^2}{2r^2} - \Psi(r,z)$$

maximal on $z = 0$.

So, minimal ϵ for given h (express ϵ in terms h)

$\left\{ \begin{array}{l} u_r = u_z = 0 \\ \text{eval. at} \\ z = 0 \end{array} \right.$: $\frac{h^2}{2r^2} - \Psi(r,0)$ minimal

So $\epsilon(h)$ s/t $\frac{\partial}{\partial r}\left\{ h^2/2r^2 - \Psi(r,0) \right\} = 0$

$\Rightarrow R_h \rightarrow$ radius of minimum energy circular orbit

$$\Rightarrow \quad \epsilon(h) = \frac{1}{2}\, h^2/r_h^2 - \Psi(r_h,0)$$

$$\Rightarrow \quad \boxed{\mathcal{E}(h) = \frac{v^2}{2} - \varphi \qquad v = h/r_h}$$

$\rightsquigarrow$ minimum energy.

Then:

$$\frac{d\mathcal{E}}{dh} = \mathcal{E}'(h) = \frac{\partial \mathcal{E}}{\partial h} = \frac{h}{r_h^2} = \Omega$$

rot freq.

NB: $\mathcal{E}(h)$ already stationary w/r variations

i.e. $\frac{d\mathcal{E}}{dh} = \frac{\partial \mathcal{E}}{\partial h} + \cancel{\frac{\partial \mathcal{E}}{\partial r}\frac{dr}{dh}}$ ✓

Now:

First ①

→ Minimize energy of 2 particles, keeping total ~~[illegible]~~ angular momentum constant.

nb
- minimize energy of each at L const, $\rightsquigarrow$ 2 circles
- lower → minimize via exchange?

each particle : $\epsilon(h)$

$$E = m\,\epsilon(h)$$

$\therefore$

$$E = m_1 \epsilon(h_1) + m_2 \epsilon(h_2)$$

$$H = m_1 h_1 + m_2 h_2$$

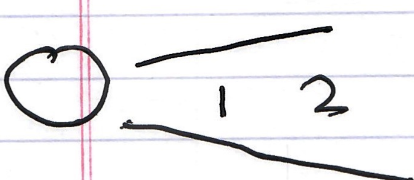
$dH = 0$ $\qquad m_1 dh_1 + m_2 dh_2 = 0$

$$dE = m_1 dh_1 \epsilon'(h_1) + m_2 dh_2 \epsilon'(h_2)$$

$$= m_1 dh_1 \left[\epsilon'(h_1) - \epsilon'(h_2) \right]$$

$m_1 = m_2$

$$dE = m_1 dh_1 (\Omega_1 - \Omega_2)$$



$\Omega_1 > \Omega_2$

so if $dE < 0$ → relaxn.

$dh_1 < 0$ so $dh_2 > 0$.

→ Orbit 2, of lower angular velocity, gains angular momentum

whole orbit 1, of higher angular velocity, lowers//loss angular momentum.

⇒

Minimization (relaxation) of energy by exchange of angular momentum to orbit of lower Ω ⇒ outward transfer angular momentum.

∴ $\boxed{\text{Energy lowered if angular momentum 'flows' / 'transported' outward.}}$

② Now, consider that total mass fixed $\Rightarrow$ orbits exchange mass, too, as well as angular momentum.

$$-dE = d\left[m_1 \epsilon(h_1) + m_2 \epsilon(h_2)\right]$$

$$- dM = 0 \;, \quad dm_1 = -dm_2$$

$$- dH = 0 = dH_1 + dH_2$$

$$d(m_1 h_1) + d(m_2 h_2) = 0$$

Now,

$$dE = dm_1 \epsilon(h_1) + m_1 \epsilon'(h_1)\, dh_1 + dm_2 \epsilon(h_2) + m_2 \epsilon'(h_2)\, dh_2$$

$$= dm_1\left[\epsilon(h_1) - h_1 \epsilon'(h_1)\right] + d(m_1 h_1)\, \epsilon'(h_1) + dm_2\left[\epsilon(h_2) - h_2 \epsilon'(h_2)\right] + d(m_2 h_2)\, \epsilon'(h_2)$$

so

$$dE = dm_1 \left\{ \left[\epsilon(h_1) - h_1 \Omega_1 \right] - \left[\epsilon(h_2) - h_2 \Omega_2 \right] \right\} + dH_1 (\Omega_1 - \Omega_2)$$

so

$$dE = dH_1 \overset{①}{(\Omega_1 - \Omega_2)} + dm_1 \overset{②}{\left\{ \left[\epsilon(h_1) - h_1 \Omega_1 \right] - \left[\epsilon(h_2) - h_2 \Omega_2 \right] \right\}}$$

< 0 ?

① $\Omega_1 > \Omega_2$

$dH_1 < 0$.

② need profile $\epsilon - h\Omega$

$$\frac{d}{dr}(\epsilon - h\Omega) = \frac{d}{dr}\left(-\frac{1}{2} V^2 - \psi \right)$$

$$= -V \frac{dV}{dr} + \frac{V^2}{r}$$

cent bal.

$$= -V \left(\frac{dV}{dr} - \frac{V}{r} \right)$$

$$= -rV \frac{d}{dr}\left(\frac{V}{r}\right) > 0$$

$$\underbrace{\phantom{\frac{d}{dr}\left(\frac{V}{r}\right)}}_{<0}$$

so

$$\frac{d}{dr}(\epsilon - h\Omega) > 0$$

so

$$\boxed{\begin{aligned} dE &= dH_1 \underbrace{(\Omega_1 - \Omega_2)}_{}\!^{>0} \\ &\quad + dm_1 \left(\underbrace{\Delta(\epsilon - h\Omega)}_{<0}\right) \\ dE < 0 &\Rightarrow dH_1 < 0 \\ &\quad\;\; dm_1 > 0 \end{aligned}}$$

$dE < 0$ for:

→ ② gains angular momentum

↝ angular momentum coupled outward.

→ ① gains mass

i.e mass <u>accretes</u>.

So → Accretion with angular momentum transport outward — <u>is</u> energy-minimizing / relaxation process.

→ Deviation / Departure from solid body comes from mass accretion.

<u>So</u>

⇒ final, minimum energy state is

- full accretion
- 1 particle at ∞ to carry angular momentum

Quite a contrast to solid body ! ☺.

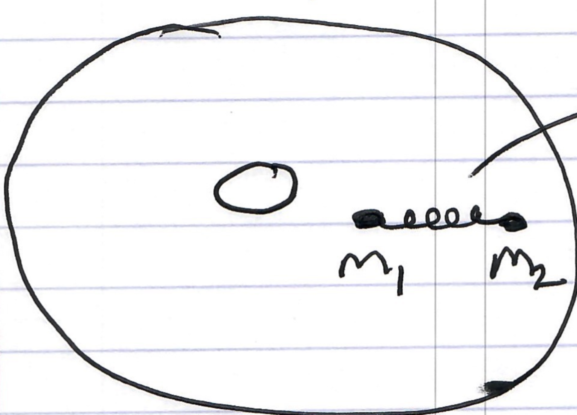
TBC.

→ Recall :- 2 particles:

- conserve angular momentum and mass of sum

→ interchange / etc.

→ Gedanken experiment:

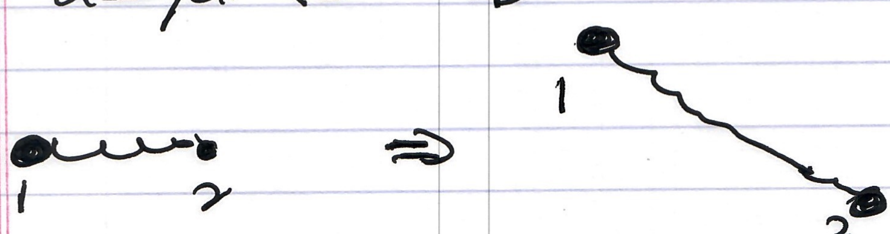


→ consider light, ~ massless spring

$$\frac{d\Omega}{dr} < 0$$

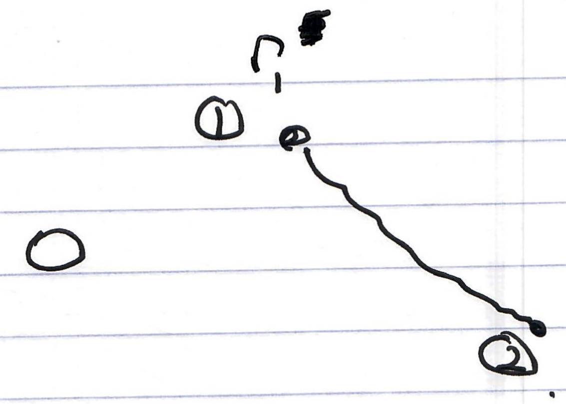
n.b. $d\Omega/dr = 0$ → no spring extension change

if $d\Omega/dr < 0$ ⇒



① - faster Ω → pulls ahead

② - slower Ω → falls behind



⇒ spring is extended,

but now

→ spring pulls back on ①

⇒ ① loses angular momentum

[redacted] as $L \simeq r^{1/2}$

① must drop to lower radius.

"Donkey" – L.B.

$r_1 \to r_1 - \Delta r_1$

Force ① to slow down, ① loses angular momentum drops to lower orbit where Ω larger → rotation increases.

→ spring pulls ② ahead

⇒ ② gains angular momentum

so

② must move to larger radius

$(L \sim r^{1/2})$

so $r_2 \to r_2 + \Delta r_2$

Ω_2 $\quad r_2 - r_1 = \Delta r_2 + \Delta r_1$

Ω_2 extension of spring increases

and repeat

$\Rightarrow$ spring extension re-enforced / expanded

$\Rightarrow$ instability !

$\therefore$

- 2 particles + elastic connection (spring) is unstable system
- free energy : $d\Omega/dr < 0$ (gradient)

  spring : facilitates angular momentum exchange
- replace spring by magnetic tension + freezing-in law.

⇒ M.R.I. – Magneto-rotational Instability

– mechanism most likely to generate "turbulent viscosity" ν in thin disk.

– obviously relevant to disks hot enough to be ionized

⇒ planet forming?

⇒ This brings us to MHD.

→ MHD - A Quick Introduction

- see 218b Fall '21.

- books - Goltier
  $\boxed{\text{Kulsrud}}$

- Intro is not a substitute for a full discussion.

What is it?

- Magnetohydrodynamics: Fluids + $\underline{B}$ field

- Single fluid (electrons + ions), net $\simeq$ neutral ($L > \lambda_D$)
(no electrostatic)

- Maxwell equations with
  - no displacement current
  (low frequency)

- no electrostatics.

- Equation of state

What are the equations? (Ideal)

① $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$ continuity

② $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla P + \frac{\underline{J}\times\underline{B}}{c}$ Newton's Law.

current

Lorentz force

$\frac{dS}{dt} = 0$ (isentropic)

③ $\partial_t S + \underline{v}\cdot\nabla S = 0$ $S = S_0 \ln\left(P/\rho^\gamma\right)$

[equation of state] - ideal gas.

→ ③ fluid equations

(or $\nabla\cdot\underline{v} = 0$)

(4) $\underline{\nabla} \cdot \underline{B} = 0$

(5) $\underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$

(6) $\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J}$

Pre-Maxwell as usual.

(3) eqns.

+

(7) $\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{J}$

η → departs ideal

– Ohm's Law.
(most sensitive part)

→ 7 eqns.

→ in some sense,

– Eqn. motion → "ions"
(avg velocity) → weighted by mass

– Ohm's Law → "electrons"
(velocity difference)

- Incompressible MHD is of importance → especially MRI

[$\underline{\nabla} \cdot \underline{V} = 0$ eliminates sound wave]

so

$$\boxed{\rho \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{V} \right) = -\underline{\nabla} p + \frac{\underline{J} \times \underline{B}}{c}} \quad (1)$$

and

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{J}$$

$$\boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}} \quad (2)$$

- 2 eqns — momentum balance (1)

[$\partial_t \rho + \underline{V} \cdot \underline{\nabla} \rho = 0$] — Induction Equation (2)

What does it mean?

→ 2 strongly coupled, interpenetrating fluids. (E. N. Parker)

$\underline{V}(\underline{x},t) \to$ E.O.M.

$\underline{B}(\underline{x},t) \to$ think of $\underline{B}$ as a fluid field.

↓

Induction Eqn.

Coupling: $\begin{cases} \underline{J}\times\underline{B} \to \underline{V} \\ \underline{V}\times\underline{B} \to \underline{B} \end{cases}$

Contrast two approaches:

Parker → field ($\underline{B}$) fundamental

Alfven → $\underline{J}$ fundamental.

→ How strongly coupled?

→ Alfven Theorem / Freezing-in law.

(ideal)

→ Recall: Pure Fluid (Isentropic)

Kelvin Circulation Theorem:

$$\oint \underline{v} \cdot d\underline{l} = \text{const} = \Gamma$$

$$\Updownarrow$$

$$\underline{\omega} = \nabla \times \underline{v} \qquad \int d\underline{a} \cdot \underline{\omega} = \text{const}$$

Vorticity

Prove:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\frac{\nabla p}{\rho}$$

$$dE = T\,dS - p\,dV$$

$$dH = T\,dS + V\,dP = T\,dS + \frac{dP}{\rho}$$

$$dS = 0 \quad \text{(isentropic)}$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla H$$

$$\frac{d}{dt}\oint \underline{v}\cdot d\underline{\ell} = \oint \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint \underline{v}\cdot \frac{d}{dt} d\underline{\ell}$$

$$= \underbrace{\cancel{\oint -\nabla H \cdot d\underline{\ell}}}_{0} + \int \underline{v}\cdot \underbrace{\frac{d\,d\underline{\ell}}{dt}}_{\underline{v}}$$

$$= \oint \underline{v}\cdot d\underline{v} = 0$$

- Circulation conserved in ideal fluid.
- viscosity breaks circulation conservation

i.e. $\frac{d\underline{v}}{dt} = -\nabla H + \nu \nabla^2 \underline{v}$

- for vorticity:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\nabla H$$

$$\underline{v}\cdot\nabla\underline{v} = +\nabla\frac{v^2}{2} - \underline{v}\times\underline{\omega}$$

Magnus force

$$\partial_t \underline{V} = \underline{V} \times \underline{\omega} - \nabla\left(H + \frac{\underline{V}^2}{2}\right)$$

$$\partial_t \underline{\omega} = \nabla \times \underline{V} \times \underline{\omega} \qquad \cancel{+ \nu \nabla^2 \underline{\omega}}$$

Vorticity obeys induction equation (like $\underline{B}$!)

$\Rightarrow$ $\nabla \cdot \underline{V} = 0$ then

$$\frac{d\underline{\omega}}{dt} = \partial_t \underline{\omega} + \underline{V} \cdot \nabla \omega = \underline{\omega} \cdot \nabla \underline{V} \qquad \cancel{+ \nu \nabla^2 \underline{\omega}}$$

$\nabla \cdot \underline{V} \neq 0$

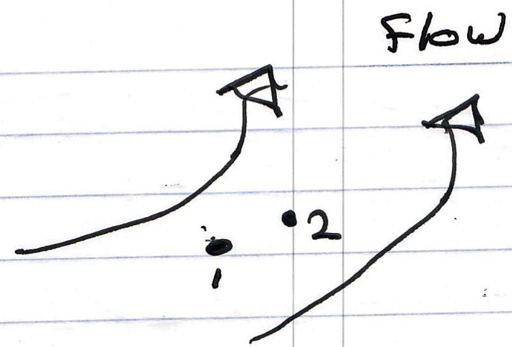
$$\frac{d}{dt}\, \underline{\omega}/\rho = \frac{\underline{\omega}}{\rho} \cdot \nabla \underline{V}$$

Means "$\underline{\omega}$ frozen into flow"

"$\underline{\omega}/\rho$ frozen into flow"

Why?

Consider



2 pts.

$$\frac{d\underline{x}_1}{dt} = \underline{V}(\underline{x}_1)$$

i.e. particles go with the flow.

$$\frac{d\underline{x}_2}{dt} = \underline{V}(\underline{x}_2)$$

$$\frac{d}{dt}\underbrace{\underline{\ell}_{1,2}}_{\underline{x}_2-\underline{x}_1} = \underline{\ell}_{1,2}\cdot\underline{\nabla}\,\underline{V} \quad , \quad \underline{V} = \underline{V}(\underbrace{\underline{x}_+}_{\text{avg}})$$

and note:

$$\frac{d}{dt}\underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V} + \cancel{\nu\nabla^2\omega}$$

and

$$\frac{d}{dt}\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} + \cancel{\eta\nabla^2\underline{B}}$$

compr. $\rightarrow$ $\underline{\omega}/\rho$, $\underline{B}/\rho$

same structure!

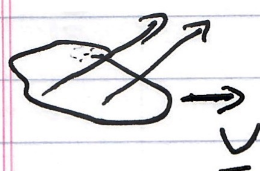
$\Rightarrow$ $\underline{\ell}_{12}$, $\underline{\omega}$, $\underline{B}$ all frozen in.

(recall masses on spring — MRI cartoon)

Related: Alfven's Theorem.

$$\frac{\partial}{\partial t}\underline{B} = \nabla \times \underline{v} \times \underline{B}$$

$\Rightarrow$



: loop, moving thru field

$$\frac{d}{dt}\int \underline{B}\cdot d\underline{a} = 0$$

Φ

[flux-frozen-in
Kelvin counterpart

$$\frac{d\Phi}{dt} = \int d\underline{a}\cdot\frac{\partial \underline{B}}{\partial t} + \int \frac{d\underline{a}}{dt}\cdot\underline{B}$$

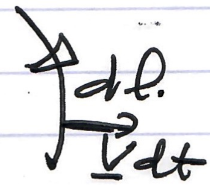
$$① = \int d\underline{a}\cdot\nabla\times\underline{v}\times\underline{B}$$

$$= \oint d\underline{\ell}\cdot(\underline{v}\times\underline{B})$$

(2) $= \int \frac{d\underline{a}}{dt} \cdot \underline{B}$

$= \int \underline{B} \cdot (\underline{V} \times d\underline{\ell})$

$= -\int d\underline{\ell} \cdot \underline{V} \times \underline{B}$



(1) + (2) = 0.

$$\boxed{\frac{d}{dt}\left(\int \underline{B} \cdot d\underline{a}\right) = 0}$$

flux freezing

broken by η.

N.B. – freezing in → "local"

– Alfven → "global" at loop level.

→ Magnetic topology conserved.

→ Major constraint on MHD..

Breaking → Magnetic Reconnection

Key topic.

→ Another identity:

$$\frac{\underline{J}\times\underline{B}}{c} = \frac{(\underline{\nabla}\times\underline{B})\times\underline{B}}{4\pi} = -\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}$$

$$\rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla}P - \underline{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}$$

$$= -\underline{\nabla}\underbrace{\left(P + \frac{B^2}{8\pi}\right)}_{\text{Magnetic Pressure}} + \underbrace{\frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}}_{\text{Magnetic Tension}}$$

Why magnetic tension?

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = B\,\underset{①}{\hat{b}}\cdot\underline{\nabla}(B\hat{b}) \quad ②$$

$$= B^2\underbrace{\hat{b}\cdot\underline{\nabla}\,\hat{b}}_{\substack{\text{Curvature of } \hat{b}\\ \text{(rt. change along itself).}}} + \hat{b}\,\hat{b}\cdot\underline{\nabla}(B^2/2)$$

② ?

if re-combine :

$$-\nabla \frac{B^2}{8\pi} + B^2\,\frac{\hat{b}\cdot\nabla\hat{b}}{4\pi} + \hat{b}\,\hat{b}\cdot\nabla\left(\frac{B^2}{8\pi}\right)$$

$$= -\nabla_\perp\left(\frac{B^2}{8\pi}\right) - \hat{b}\,\hat{b}\cdot\nabla\left(\frac{B^2}{8\pi}\right) + \hat{b}\,\hat{b}\cdot\nabla\left(\frac{B^2}{8\pi}\right) + \frac{B^2\,\hat{b}\cdot\nabla\hat{b}}{4\pi}$$

$$\frac{\underline{J}\times\underline{B}}{c} = -\nabla_\perp\left(\frac{B^2}{8\pi}\right) + B^3\,\frac{\hat{b}\cdot\nabla\hat{b}}{8\pi}$$

only cross field gradient of P_{mag} contributes to magnetic pressure.

$\Rightarrow$ Magnetic tension is key new effect in MHD. $\rightarrow$ Alfven Wave.

N.B. — Incompressible, uniform hydro
$\Rightarrow$ no waves.

— Incompr uniform MHD (with $\underline{B}_0$) $\rightarrow$
Alfven waves.