Generation
Field, flux

Dynamos – A Very Short Introduction

– needs a course. see
  – Moffatt (old, new)
  – Kulsrud
  – O.H.
  – Hughes / Tobias Review

– can't really separate MHD T, dynamo

– solar: High Rm, Re
  Convective
  weak rotation
  Tachocline

geodynamo: Rm > 1 modest
  Re >>> 1
  strong rotation:
  High Ta.
  Ekman Layers
  Constrained $\left\{ \begin{array}{c} B \\ \Omega \end{array} \right\}$

all have asymptotic challenge

Galactic: Rm >> 1
  Pm >> 1.
  Differential rotation.

– General Questions
  – how $\langle B \rangle$ formed? {mean field}

* L > l, akin QL.
  ↓ ↓
  $\langle B \rangle$ fluctus

  concr. flux

- how $\langle \widetilde{B}^2 \rangle$ formed ?
  - ~ small scale $L \gtrsim \underline{\underline{l_d}}$ (why ?)

generated in turbulent / chaotic flow ?

~ obviously inseparable from MHDT

Thm: Need one $\lambda_h > 0$

(hold) (Lyapunov exp. of flow)

for field amplification

(L.S. Young) → stretching

- Dynamos - 3D

Recall,

$$\frac{\partial A}{\partial t} = \underline{B} \cdot \nabla \phi + B_0 \partial_z \phi + \cdots \eta \nabla^2 A$$

→ 2D:

i.e. 2D conserves magnetic potential

$$\partial_t A + \underline{V} \cdot \underline{\nabla} A = \eta \nabla^2 A$$

$$\frac{dA}{dt} = \eta \nabla^2 A$$

$$\frac{d}{dt} A^2 = \eta A \nabla^2 A$$

$$\frac{d}{dt}\langle A^2\rangle = -2\eta\langle B^2\rangle$$

$$\int_0^t dt\,\langle B^2\rangle = \frac{\langle A^2(0)\rangle - A^2(t))}{2\eta}$$

$$< \frac{\langle A^2(0)\rangle}{2\eta}$$

$\downarrow$

finite

$t \to \infty$

$\langle B^2\rangle \sim 1/t^\alpha \qquad \alpha > 1$

decay.

No field growth from fluid motion alone in 2D.

$\hookrightarrow$ Cowling's Thm.

— Traditional focus $\to \langle B\rangle$

mean field

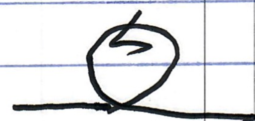
→ The canonical cartoons:

– Parker '55:

$\xrightarrow[\underline{J}]{}$ →  $\uparrow \underline{u}$

deform to loop

form $\underline{B}$ along $\underline{J}$ $(\nabla \times \underline{B} = 0)$

→  $\uparrow \underline{\omega}$

helical flow $\langle \underline{v} \cdot \underline{\omega} \rangle$ to twist loop

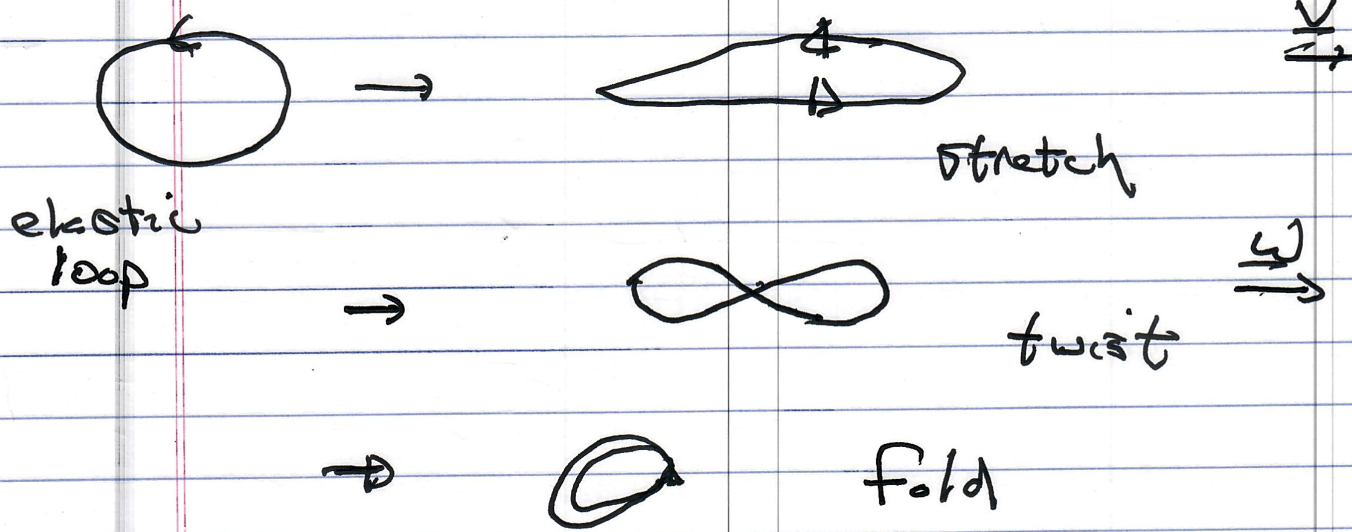
⇒  (coherence)

solenoid

$\xrightarrow[\underline{B}]{}$

Message: → need helicity, [breaking of reflection symmetry] in turbulence

— Zeldovich '72



"stretch, twist fold"

⇒ points to helical flow.

and

→ reconnect

i.e. must somehow prevent loop from coming undone.

Message : — stretching
— symmetry breaking

– locking-in, origin irreversibility

$\Rightarrow$ rate $\sim \eta^{\alpha}$? (fast dynamo $\rightarrow$ rate $\sim \eta^{(0)}$)

Really stretch - twist - fold - reconnect.

$\rightarrow$ ~~how estim~~ Something rigorous?

$\rightarrow$ Young's Theorem

Need stretching flow for fast dynamo

$\Rightarrow$ at least 1 positive Lyapunov exponent in flow.

$\rightarrow$ How calculate something?

– welcome to mean field electrodynamics

– Consider weak $\langle \underline{B} \rangle$. Can fluid motion amplify it?

Mean Field Electrodynamics - A Brief Introduction

recall

→ discussion of relaxation ⇒

$$-\langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle = \frac{\langle \underline{B} \rangle}{\langle B \rangle^2} \underline{\nabla} \cdot \Gamma_H \, ; \quad \Gamma_H = -\lambda \nabla \left(\langle J_{\parallel} \rangle / \langle B \rangle \right)$$

for consistency with Taylor hypothesis.
→ from eqns.

⇒ but, how calculate $\langle \tilde{V} \times \tilde{B} \rangle$ - i.e. what form does mean field EMF actually have? EMF

$$\partial_t \langle B \rangle = \nabla \times \langle \tilde{V} \times \tilde{B} \rangle + \nabla \times \langle V \rangle \times \langle B \rangle + \eta \nabla^2 \langle B \rangle$$

→ problem in mean field electrodynamics (i.e. closure, akin Q.L.T.).

→ some simple cases:

- fluid turbulence + weak $\langle B_0 \rangle$
- $R_m < 1$ → doable

kinematic +

$$\langle \tilde{V} \times \tilde{B} \rangle = \langle \underline{V} \times \tilde{B}^{(1)} \rangle$$

response of $\tilde{B}$ to $\tilde{V}$, in presence $\langle B_0 \rangle$

why?

kinematics thinking

→ [is both turb. unstable to test $\langle B \rangle$? smooth

$\frac{1}{\tau}\delta B$ ① $\qquad \frac{\tilde{V}}{\ell}\delta B$ ② $\qquad \frac{\eta}{\ell^2}\delta B$ ③ $= \langle B\rangle\cdot\nabla\tilde{V} - \tilde{V}\cdot\nabla\langle B\rangle$ ④

spec $\tilde{v}$.

PIN ?

then

$$\partial_t \tilde{B} + \underbrace{\nabla\times\tilde{v}\times\tilde{B}}_{NL \to \Delta\omega_k} - \underbrace{\eta\nabla^2\tilde{B}}_{\text{resistive diffn, } \eta k^2}$$

Pain-in-the-neck term.

$$= \langle B\rangle\cdot\nabla\tilde{V} - \tilde{V}\cdot\nabla\langle B\rangle$$

②/① ~ ku $\qquad$ ratios? $\qquad$ ②/③ ~ R_M

low R_M

$$\therefore \quad (-i\omega + \eta k^2)\tilde{B}_{k,\omega} = \underbrace{i\underline{k}\cdot\langle B\rangle\tilde{V}_{k,\omega}}_{\text{bending}} - \underbrace{\tilde{V}_{k,\omega}\cdot\nabla\langle B\rangle}_{\text{field advection}}$$

$$\tilde{B}_{k,\omega} = \frac{i\underline{k}\cdot\langle B\rangle\tilde{V}_{k,\omega} - \tilde{V}_{k,\omega}\cdot\nabla\langle B\rangle}{-i\omega + \eta k^2}$$

so

generator → odd ① $\qquad$ even ② turb. resist

$$\langle\tilde{V}\times\tilde{B}\rangle = \sum_{k,\omega}\tilde{V}_{-k,-\omega}\times\frac{\left[i\underline{k}\cdot\langle B\rangle\tilde{V}_{k,\omega} - \tilde{V}_{k,\omega}\cdot\nabla\langle B\rangle\right]}{-i\omega + \eta k^2}$$

→ linear $\qquad$ reson. ? flow.

turb. mixing of field.

② → even in k

⇒ advection of $\langle B\rangle$

i.e. turbulent resistivity

random advection

$$\frac{②}{①} \cong \frac{v}{\ell}\tau = Ku \qquad \text{Kubo \#}$$

$$\frac{②}{③} \simeq Rm.$$

$$\frac{④}{②} \simeq \frac{\langle B\rangle}{\ell}\frac{v}{\cancel{v}\,\frac{\delta B}{\ell}} \sim \frac{\langle B\rangle}{\delta B}$$

often: $\boxed{\langle \delta B^2\rangle^{1/2} > \langle B\rangle}$ / what does mean field theory mean? → embedded in MHD T.

Message:

~~for~~ time-honored crank ala Moffatt, need:

$Rm < 1$

$Ku < 1$ → memory in flow! Turb. $Ku \sim 1$ (MHD large scales?)

$\frac{\delta B}{\langle B\rangle} < 1$

all grossly ~~violated~~ violated in reality.

① → odd in $\underline{k}$ ↔ bending → kink
symmetry breaking, for contribution → physics?
→ ?

N.B.: In both cases irreversibility provided by resistive diffusion ⇒ otherwise difficulty

contrast QLT
→ irrev. from {stochast., resonance

For isotropic velocity spectrum: turbulence?

$$\langle \tilde{v}_i(\underline{k},\omega)\, \tilde{v}_j^*(\underline{k}',\omega') \rangle = \delta(\underline{k}-\underline{k}')\,\delta(\omega-\omega')\,\Phi_{ij}(\underline{k},\omega)$$

① even, $\underline{\nabla}\cdot\underline{v} = 0$

$$\Phi_{i,j}(\underline{k},\omega) = \frac{E(k,\omega)}{4\pi k^2}\left(k^2\delta_{ij} - k_i k_j\right)$$

② odd

$$+\frac{iF(k,\omega)}{8\pi k^2}\,\epsilon_{i,j,k}\,k_k$$

↳ helicity

① → energy density, even power
→ $\underline{\nabla}\cdot\underline{v} = 0$

② Now,

$$F(k,\omega) = i\int \epsilon_{ike}\, k_k\, \Phi_{ie}(k,\omega)\, dS_{\underline{k}}$$

so

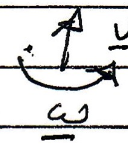
$$\langle \underline{v}\cdot\underline{\omega}\rangle = i\epsilon_{ij\ell}\iint k_m\,\Phi_{j\ell}(\underline{k},\omega)\,d\underline{k}\,d\omega$$

spectral helicity $= \iint dk\,d\omega\,F(k,\omega)$

⇒

(2) $\sim \langle \underline{v}\cdot\underline{\omega}\rangle$

⇒ $\boxed{\text{turbulence helicity (mean projection } \underline{v} \text{ on } \underline{\omega})}$



so after some crank (see Moffat: available free, online): TBC

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \alpha \langle \underline{B} \rangle - \beta \langle \underline{J} \rangle$$

$$\alpha = -\frac{1}{3}\eta \iint dk\,d\omega\,\frac{k^2 F(k,\omega)}{\omega^2 + (\eta k^2)^2}$$

⇒ $\boxed{\alpha \text{ is weighted integral of helicity spectrum}}$

i.e.

$\sim \langle \underline{v}\cdot\underline{\omega}\rangle$

$$\boxed{\beta = \frac{2}{3}\eta \iint dk\, d\omega \, \frac{k^2 E(k,\omega)}{\omega^2 + (\eta k^2)^2}}$$

$\Rightarrow$ β is weighted integral of energy spectrum

i.e. $\sim \langle \tilde{v}^2 \rangle$ business as usual.

so

$$\frac{\partial \langle \underline{B} \rangle}{\partial t} - \eta \nabla^2 \langle \underline{B} \rangle = \underline{\nabla} \times \underbrace{\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle}_{\text{mean EMF}}$$

$$\underbrace{\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle}_{\text{mean EMF}} = \underbrace{\alpha \langle \underline{B} \rangle}_{\alpha\text{-effect}} - \underbrace{\beta \langle \underline{J} \rangle}_{\beta\text{-effect}} + \ldots$$

$\therefore$

$$\frac{\partial \langle \underline{B} \rangle}{\partial t} - \eta \nabla^2 \langle \underline{B} \rangle = \alpha \underline{\nabla} \times \langle \underline{B} \rangle + \beta \nabla^2 \langle \underline{B} \rangle + \eta \nabla^2 \langle B \rangle$$

$$= \alpha \langle \underline{J} \rangle + \eta \nabla^2 \langle B \rangle + \beta \nabla^2 \langle B \rangle.$$

$\therefore$

$\rightarrow$ β as turbulent resistivity $\Rightarrow$ random advective mixing of $\langle B \rangle$.

like PL

random B frozen in $\langle B \rangle \rightarrow$ coarse graining

→ α ?

Further interesting to note:

→ look for force-free condition fields:

$$\nabla \times \langle \underline{B} \rangle = \lambda \langle \underline{B} \rangle$$

$$\partial_t \langle \underline{B} \rangle - (\eta+\beta)\nabla^2 \langle B \rangle = \alpha \lambda \langle \underline{B} \rangle$$

⇒ $\boxed{\gamma_B = \alpha\lambda - (\eta+\beta)\lambda^2}$

→ critical Rm.
→ $\tau_c \langle \tilde{v}^2 \rangle / L$ vs. η / L

∴ α can amplify field → depending on λ (scale)

⇒ dynamo ↔ via α-effect!

$\alpha\lambda$ vs $\eta\lambda^2$

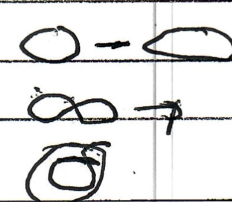
→ critical Rm
→ ~~spectral dist~~
α ?

→ Physics:

α → helicity ↔ $\langle \underline{\hat{v}} \cdot \underline{\hat{\omega}} \rangle$

Then:

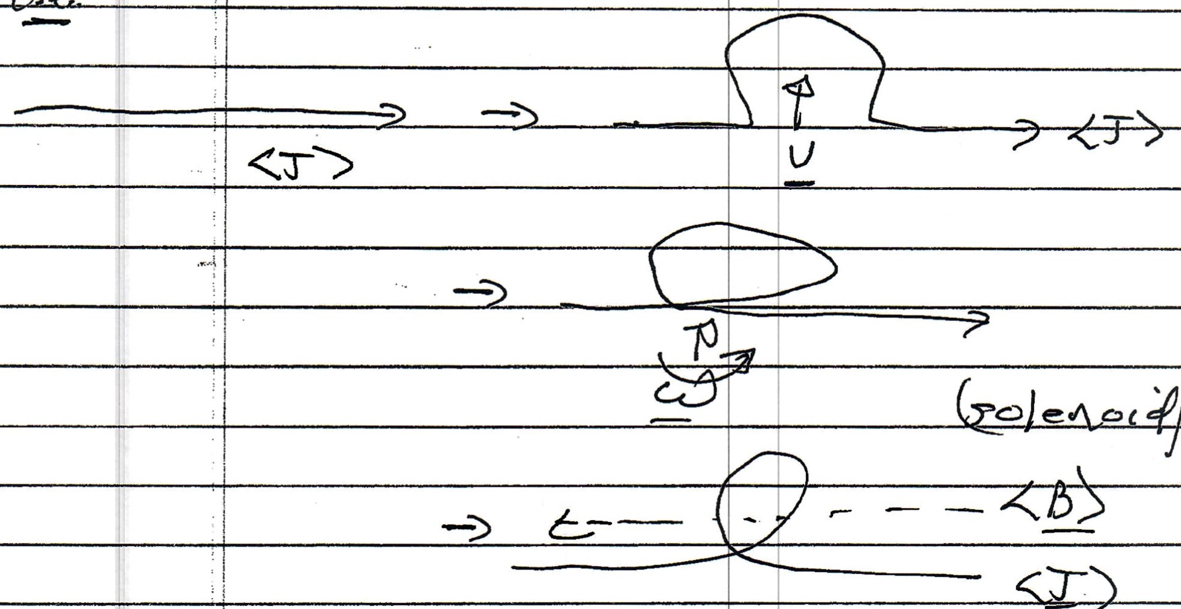
→ can stetch, twist, fold lines,



to amplify field.

$\Rightarrow \alpha : \langle J \rangle \rightarrow \langle \underline{B} \rangle$

i.e.



$\langle \underline{B} \rangle \parallel \langle J \rangle$.

generates $\langle B \rangle \parallel \langle J \rangle$ → twists one direction into another.

repeat ⇒ amplify field.

i.e. $\gamma_B = \alpha\lambda - (\eta+\beta)\lambda^2$

$$d\gamma/d\lambda = \alpha - 2(\eta+\beta)\lambda = 0$$

$$\boxed{\lambda_{\substack{max \\ growth}} = \alpha/2(\eta+\beta)}$$

$$\gamma_B\Big|_{max} = \alpha^2 / 4(\eta+\beta)$$

$$\not\uparrow$$

$\sim$ $\underline{\alpha^2 \text{ dynamo}}$. also $\alpha\Omega$ shear.

N.B:

→ locking-in (reconnection!) crucial!
⇒ role of η as cross-phase.
⇒ high Rm → problematic.

→ non-linearity, especially high Rm
⇒ a field in itself!

N.B.: Impact/role of magnetic helicity in dynamo control.

Some thoughts:
→ shear
→ α-Ω dynamo $\quad \underline{\gamma \sim \sqrt{\alpha \Omega'}}$
→ back helicity PEL
→ small scale → dynamo process couples to cascade

→ Tachocline - interface dynamo

→ Solar B-field story
Tach + lower zone
Tube rise
OD-PP, etc. cycle.

→ Sheared Rotation / Flows Helps

α-ω dynamo: synergy of shear and α-effect.

$$\frac{\partial A}{\partial t} + \underline{U}_p \cdot \underline{\nabla} A = \alpha B + \eta \nabla^2 A$$

$$\frac{\partial B}{\partial t} + \underline{u}_p \cdot \underline{\nabla}\, \underline{B} = \underbrace{\underline{B}_p \cdot \underline{\nabla}\, \underline{u}}_{\text{shear}} + \eta \nabla^2 B$$

or

$$\frac{\partial A}{\partial t} = \alpha B + \eta \nabla^2 A \qquad B_r.$$

$$\frac{\partial B}{\partial t} = -G \frac{\partial A}{\partial z} + \eta \nabla^2 B \qquad B_\theta$$

$$\gamma \sim (\alpha V')^{1/2}$$

→ shear.

vs.

$$\gamma \sim \alpha$$

→ Current Issues

- is kinematic theory meaningful? → no
- beyond kinematics:

  $\alpha_K - \alpha_M$.

  Helicity conservation constraints.

- is mean field theory meaningful?
  - mean vs. small scale dynamo.

- Tachocline
  - → discuss
  - → shear + stable strat.
  - → in → overshoot
  - → out → buoyancy.

- How sun works.