Anomalous Resistivity from the Ion-Acoustic Instability

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Abstract

Instabilities in a plasma can arise when the drift velocity of the electron distribution with respect to ion distribution surpasses a critical value. These instabilities excite waves which are damped through linear and non-linear wave-particle interactions. The Anomalous resistivity then emerges due to momentum transfer of electrons through their interaction with the excited waves. It is important to relate the anomalous resistivity to phenomena that produces the instability. This paper will discuss the anomalous resistivity due to current driven ion-acoustic instability.

I Introduction

The evolution of magnetic field in the resistive-MHD is governed by the induction equation.

$$\frac{\partial B}{\partial t} = \nabla \times (V \times B) + \frac{\eta c}{4\pi}\nabla^2 B \qquad S = 4\pi \frac{L v_A}{\eta c}$$

the inclusion of the resistive diffusion term allows the flux freezing to be violated. In particular if the resistive diffusion term is much greater then convection term, $S \ll 1$ then the topology of the magnetic field is allows to change. The phenomenon of the magnetic field lines breaking and reconnecting is called magnetic reconnection. During magnetic reconnection the magnetic field gets dissipated and the magnetic energy is converted to plasma kinetic energy. This causes the Ohmic heating of the plasma and can cause instabilities to arise.

I.1 The Sweet-Parker Model of Magnetic Reconnection and Faster Reconnections

The Sweet-Parker models the steady state magnetic reconnection of oppositely align magnetic fields due to micro-resistivity. The magnetic reconnection speed in the Sweet-Parker model is given by $v_R = \frac{v_A}{\sqrt{S}} \propto \sqrt{\eta}$. This reconnection speed is observed to be too slow compared to observational results [1]. In order agree with the observe reconnection rates we must modify the Sweet-Parker model to get a faster reconnection rate. One approach to seek faster reconnection rates is to consider the current-driven micro-instabilities arises due to Ohmic heating. The momentum transfer between the excited waves due to instability and the particles provides an effective resistivity to the system.

I.2 Experimental Tests of Sweet Parker with Anomalous Resistivity

The magnetic reconnection experiments in (MRX) indicates by inclusion of Anomalous Resistivity to the Sweet-Parker model. The new magnetic reconnection rate agrees with cases of magnetic reconnection simulated in laboratory.

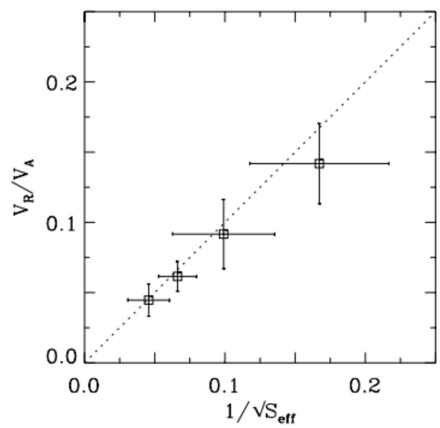

Figure 1: Magnetic reconnection in generalized Sweet-Parker including anomalous resistivity compare to magnetic reconnection in Laboratory plasma. Taking from reference [2]

II Fluctuations in Warm Plasma

II.1 Evolution of Distribution Function and Electrostatic waves

Consider a collisionless plasma with zero guide field and take the constant current density to be along $\hat{x}$. Denote the electron and ion distribution functions as $f_e(r,v,t)$ and $f_i(r,v,t)$ respectively. Since our problem is one dimensional it is useful to integrate f over the orthogonal components to k and denote it as F.Furthermore we restrict ourselves to the electrostatic perturbations hence $\mathbf{E} = E\mathbf{k}/k$ so there is no perturbed magnetic field. The evolution of the distribution functions is governed by the Vlasov equation

$$\frac{\partial f_e}{\partial t} + v\frac{\partial f_e}{\partial x} - \frac{e}{m_e}E\frac{\partial f_e}{\partial v} = 0 \qquad \frac{\partial f_i}{\partial t} + v\frac{\partial f_i}{\partial x} + \frac{e}{m_e}E\frac{\partial f_i}{\partial v} = 0$$

and the Poisson‘s equation takes the form

$$\nabla^2\phi = 4\pi(n_i - n_e)e = 4\pi e\int (f_i - f_e)d^3v$$

Suppose a small perturbation of the distribution function $f(r,v,t) = f_0(v) + f_1(r,v,t)$ where f_0 is the equilibrium distribution and f_1 is the non-stationary perturbation. Then the linearized equations are given by

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{q}{m}E\frac{\partial f_{0e}}{\partial v} = 0 \qquad \nabla^2\phi = 4\pi e\int (f_{1i} - f_{1e})d^3v$$

where $q = \pm e$ and we dropped the subscripts. Therefore perturbations of the distribution function result in the perturbation of electric field which in turn feed backs to the perturbation of the distributions. Now we consider the initial value problem at $t = 0$ for the perturbed fields and Fourier transform with respect to the spatial dimension

$$E(x,t) = \int_{-\infty}^{\infty} \hat{E}_1(k,t)e^{ikx}dk$$

Hence the linearized equations take the form

$$\frac{\partial \hat{F}_1}{\partial t} + ikv\hat{F}_1 + \frac{q}{m}\hat{E}\frac{\partial F_0}{\partial v} \qquad ik\hat{E} = 4\pi e\int_{-\infty}^{\infty}(\hat{F}_{1i} - \hat{F}_{1e})dv$$

To solve the initial value problem we take the Laplace transform with respect to time

$$\tilde{E}(k,\bar{\omega}) = \int_{-\infty}^{\infty}\hat{E}(k,t)e^{i\bar{\omega}t}dt \qquad \hat{E}(k,t) = \frac{1}{2\pi}\int_{-\infty+i\mu}^{\infty+i\mu}\tilde{E}(k,\bar{\omega})e^{-i\bar{\omega}t}d\bar{\omega}$$

where $\bar{\omega} = \omega + i\gamma$ and $\gamma > 0$. Hence the equations take the form

$$-\hat{F}_1(v,t=0) - i(\bar{\omega} - kv)\tilde{F}_1(k,\bar{\omega}) + \frac{q}{m}\tilde{E}\frac{\partial F_0}{\partial v} = 0$$

Solving for $\tilde{F}_1$ and substituting into the equation for $\tilde{E}$ we obtain

$$\tilde{E}(k,t) = \frac{4\pi e}{k\varepsilon(k,\bar{\omega})}\int_{-\infty}^{\infty}\frac{\hat{F}_{1i}(v,0) - \hat{F}_{1e}(v,0)}{\bar{\omega} - kv}dv \qquad J(k,\bar{\omega}) \equiv \tilde{E}(k,\bar{\omega})\varepsilon(k,\bar{\omega})$$

where $\varepsilon(k,\bar{\omega})$ is the dielectric constant

$$\boxed{\varepsilon(k,\bar{\omega}) = 1 + \frac{4\pi e^2}{m_i k}\int\frac{\partial F_{0i}/\partial v}{\bar{\omega} - kv}dv + \frac{4\pi e^2}{m_e k}\int\frac{\partial F_{0e}/\partial v}{\bar{\omega} - kv}dv}$$

II.2 Ion-Acoustic Wave

Since we restrict ourselves to the electrostatic waves, the dielectric constant is $\varepsilon(k,\omega)$. This follows as by definition of the dielectric constant $\nabla \times B_1 = -i(\omega/c)\varepsilon \cdot E$ but we have $B_1 = 0$. Hence the dispersion relation for the electrostatic waves are obtained through $\varepsilon(k,\omega) = 0$. In particular it is given by

$$1 = -\frac{4\pi e^2}{m_i k}\int \frac{\partial F_{0i}/\partial v}{\bar{\omega} - kv}dv - \frac{4\pi e^2}{m_e k}\int \frac{\partial F_{0e}/\partial v}{\bar{\omega} - kv}dv$$

Suppose the equilibrium distribution functions of electrons and ions are Maxwellian.

$$F_{0i} = \frac{n_0}{\sqrt{2\pi}v_i}\exp(-v^2/2v_i^2) \qquad F_{0e} = \frac{n_0}{\sqrt{2\pi}v_e}\exp(-v^2/2v_e^2)$$

where $v_i^2 = kT_i/m_i$ and $v_e^2 = kT_e/m_e$ are ion and electron thermal velocities respectively and $n_0 = \int f_0 d^3v$.

Consider the phase velocities which satisfy $v_i \ll \omega/k \ll v_e$.

In the dispersion relation in the contribution of electron distribution we have $\omega - kv \approx -kv$.

$$-\frac{4\pi e^2}{m_e k}\int \frac{\partial F_{0e}/\partial v}{-kv}dv = -\frac{\omega_{pe}^2}{k^2 v_e^2}$$

In the dispersion relation in the contribution of ion distribution we have $\omega - kv \approx \omega$.

$$-\frac{4\pi e^2}{m_i k}\int \frac{\partial F_{0i}/\partial v}{\omega}dv = \frac{\omega_{pi}^2}{\omega^2}$$

Therefore the dispersion relation for waves in this limit is

$$1 = \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2 v_e^2} \implies \omega^2 = \frac{k^2 v_e^2 \omega_{pi}^2}{k^2 v_e^2 + \omega_{pe}^2} = \frac{m_e}{m_i}\frac{k^2 v_e^2}{1 + k^2\lambda_D^2} = \frac{k^2 c_s^2}{1 + k^2\lambda_D^2}$$

where $c_s = \sqrt{m_e/m_i}v_e$ is the effective sound speed and $\lambda_D = v_e/\omega_e$ is the Debye length. Waves in this regime are called Ion-Acoustic waves which correspond to longitudinal oscillations of the ions.

II.3 Landau Damping

Consider the expression we obtain for the Laplace transform of $\hat{E}$: $J(k,\bar{\omega}) \equiv \tilde{E}(k,\bar{\omega})\varepsilon(k,\bar{\omega})$. Taking the inverse Laplace form of the equation extending the integral to the contour closed from the lower half plane $\gamma \to -\infty$ and applying the Cauchy-Residue Theorem we obtain

$$\boxed{\hat{E}(k,t) = i \sum_{\varepsilon(k,\bar{\omega})=0} \frac{J(k,\bar{\omega})}{\partial\varepsilon/\partial\bar{\omega}} e^{-i\bar{\omega}t}}$$

Hence we observe that the evolution of the electrostatic wave induced by the small perturbations in the distribution function can be determined by the root of the dielectric constant and so from the equilibrium distributions. Denote the root $\bar{\omega}_1 = \omega_1 + i\gamma_1$ be the root of ε with largest γ. Hence for $t \gg 1$ the electric field will behave

$$\hat{E}(k,t) \approx e^{-i\omega_1+\gamma_1 t}$$

In particular, if $\gamma_1 < 0$ then this corresponds to the damping of the electrostatic wave. This is called the Landau damping. On the other hand if $\gamma_1 > 0$ this corresponds to exponential growth and is called the Landau growth.

Consider k sufficiently small such that $v_i, v_e \ll |\omega/k|$ then we have the following expansion

$$\frac{1}{\omega - kv} \approx \frac{1}{\omega}\left(1 + \frac{kv}{\omega} + \frac{k^2v^2}{\omega^2} + \frac{k^3v^3}{\omega^3}\right)$$

Substituting this expansion into the dispersion relation given by $\varepsilon(k,\omega)$ and noting that the equilibrium distributions are Maxwellian we obtain the following dispersion relation

$$\omega^2 \approx \omega_{pe}^2 + 3kv_e^2$$

the term $3kv_e^2$ is called the Bohm-Gross correction. Now we consider the plasma oscillation with resonant particles in this limit. To the first order interaction we have $\bar{\omega} = \omega_0 + \omega_1$ with $\omega_0 = \omega_{pe}$. Decomposing the dielectric constant to real and imaginary part the root equation takes the form

$$\varepsilon_1(\omega_0 + \omega_1) + i\varepsilon_2(\omega_0) = 0$$

The imaginary part can be obtained by integrating over ω/k with a semi-circle contour

$$i\varepsilon_2 = -\pi i \frac{4\pi e^2}{k^2 m_e} \frac{\partial F_{0e}}{\partial v}\Big|_{\omega/k} - \pi i \frac{4\pi e^2}{k^2 m_i} \frac{\partial F_{0i}}{\partial v}\Big|_{\omega/k}$$

The real part of the dielectric constant from the above expansion is given by

$$\varepsilon_1(\omega_0 + \omega_1) \approx 1 - \frac{\omega_{pe}^2 + 3kv_e^2}{(\omega_0 + \omega_1)^2} \qquad \frac{\partial\varepsilon_1}{\partial\omega} = \frac{2}{\omega_0}$$

Hence substituting inside the root equation and noting that $\varepsilon_1(\omega_0) = 0$ we have

$$\omega_1 = -\frac{i\varepsilon_2(\omega_0)}{\partial\varepsilon_1/\partial\omega_0}$$

Hence the growth rate is given by

$$\gamma_1 = \frac{\omega_1}{i} = \frac{\pi\omega_0}{2n_0k^2}\left[\omega_{pe}^2\frac{\partial F_{0e}}{\partial v} + \omega_{pi}^2\frac{\partial F_{0i}}{\partial v}\right]$$

Ignoring the contribution of the ion distribution and noting that the equilibrium electron distribution is Maxwellian $\partial F_{0e}/\partial v = -(v/ve^2)F_{0e}$ the growth rate can be written as

$$\boxed{\frac{\gamma_1}{\omega_0} = -\sqrt{\frac{\pi}{8}}\frac{1}{k^3\lambda_D^3}\exp(-1/2k^2\lambda_D^2)}$$

Note that the growth rate strictly negative hence the waves are damped. However this damping rate γ_1 is small compare to ω_1 if $k\lambda_D \ll 1$.

II.4 Current Driven Ion-Acoustic Instability

Now we take account the addition of electron drift velocity to the ion-acoustic wave. Set our frame such that the ion distribution function is Maxwellian and the electron is a shifted Maxwellian.

$$\partial F_{0i}/\partial v = -(v/vi^2)F_{0i} \qquad \partial F_{0e}/\partial v = -((v-v_d)/ve^2)F_{0e}$$

Consider phase velocities in the limit $v_i \ll \omega/k$ and $|\omega/k - v_d| \ll v_e$. In this case we have $\omega_0^2 = \frac{k^2c_s^2}{1+k^2\lambda_D^2}$. Decomposing the dielectric constant to real and imaginary parts again we have obtain

$$\partial\varepsilon_1/\partial\omega|_{\omega_0} = 2\omega_{pi}^2/\omega_0^3$$

instead and the imaginary part stays the same. Hence the corresponding growth rate is

$$\frac{\gamma_1}{\omega_0} = \frac{\pi\omega_0^2}{2n_0k^2}\left(\frac{m_i}{m_e}\frac{\partial F_{0e}}{\partial v} + \frac{\partial F_{0i}}{\partial v}\right)\Big|_{\omega_0/k}$$

Denote $\alpha = (1+k^2\lambda_D^2)$. Substituting the equilibrium distribution F_{0e} and F_{0i} we obtain

$$\begin{aligned}\frac{\gamma_1}{\omega_0} &= \sqrt{\frac{\pi}{8}}\alpha^3c_s^3\left[\frac{m_i}{m_e}\left(\frac{v_d}{\alpha c_s}-1\right)\frac{\exp(-(\alpha c_s - v_d)^2/2v_e^2)}{v_e^3} - \frac{\exp(-(\alpha c_s)^2/2v_i^2)}{v_i^3}\right]\\ &= \alpha^3\sqrt{\frac{\pi}{8}}\alpha^3c_s^3\left[\sqrt{\frac{m_i}{m_e}}\left(\frac{v_d}{\alpha c_s}-1\right) - \left(\frac{T_e}{T_i}\right)^{3/2}\exp(-(\alpha c_s)^2/2v_i^2)\right]\end{aligned}$$

In the limit $k\lambda_D \ll$ we have $\alpha \to 1$ hence the growth rate becomes

$$\boxed{\frac{\gamma_1}{\omega_0} \approx \sqrt{\frac{\pi}{8}}c_s^3\left[\sqrt{\frac{m_i}{m_e}}\left(\frac{v_d}{c_s}-1\right) - \left(\frac{T_e}{T_i}\right)^{3/2}\exp(-T_e/-2T_i)\right]}$$

In particular the destabilization of the ion acoustic wave occurs when the electron drift velocity satisfies

$$v_d > c_s\left[1+\sqrt{\frac{m_i}{m_e}}\exp(-T_e/-2T_i)\right]$$

we note that this limit is valid when $T_e \gg T_i$ and thus the exponential factor reduces the threshold to $v_d > c_s$ as T_e/T_i is increased.

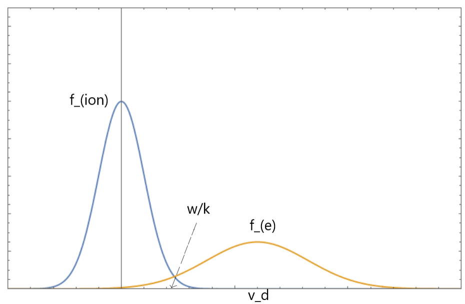

Figure 2: Current Driven Ion-Acoustic Instability: F_{0i} (blue) and F_{0e} (orange)

III Energy Spectrum and Anomalous Resistivity

The anomalous resistivity is due to the momentum transfer of the electrons. In order to calculate the anomalous resistivity we calculate the energy spectrum of ion-acoustic waves in the steady state. After obtaining the wave amplitudes over the spectrum we can calculate the momentum transfer rate.

The effective resistivity is obtain through the Ohm's law which takes the form

$$\frac{d\mathscr{P}_e}{dt} = -n_0 e E^{eff}$$

The wave-particle interactions dominates the momentum loss of the electrons. This process corresponds to linear Landau damping and the momentum loss of the electrons are derived from the conservation of momentum. In particular the loss of the momentum of the non-resonant particles transfer to the resonant particles, ion-acoustic waves. The expression for the momentum loss of the particles are derived in Galeev Sagdeev (1969) to be

$$\frac{d\mathscr{P}_e}{dt} = -\pi\frac{e^2}{m}\int\frac{dk}{k}\hat{I}(k,t)\frac{\partial F_{0e}}{\partial dv}\Big|_{\omega/k}$$

where $\hat{I}(k)$ is the intensity of the electric field per unit wave number given by $\hat{I}(k) = |\hat{E}(k)|^2/L$ and L is the length scale of the system. From the definition of resistivity it follows that

$$\eta^{eff}(j) = \frac{c^2}{4\pi\sigma^{eff}} = \frac{E^{eff}(j)c^2}{4\pi j} = \frac{c^2}{4\pi n_0^2 e^2 v_d}\left|\frac{d\mathscr{P}_e}{dt}\right|$$

Plugging in the momentum transfer rate of the electrons due to the wave-particles interactions we obtain the anomalous resistivity to be [3].

$$\eta^{eff}(v_d) = \frac{c^2}{\omega_{pi}}\frac{c_s}{4v_d}\int \frac{dk}{k\lambda_D}\frac{\hat{I}(k,t)}{n_0 m_e v_e^2}\frac{v_e^2}{n_0}\frac{\partial F_{0e}}{\partial v}\Big|_{\omega/k}$$

Note the dependence of the anomalous resistivity to the electron drift velocity. In particular for the regime $T_e \gg T_i$ the threshold for exciting ion-acoustic instability hence anomalous resistivity is $v_d > c_s$ while for $T_e \approx T_i$ the threshold becomes much higher. Either case as $v_d \approx v_e$ the anomalous resistivity will flatten out.

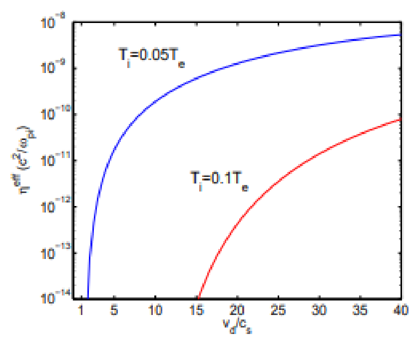

Figure 3: The Anomalous resistivity η^{eff} as a function of the electron drift velocity v_d. The figure is taken from reference [3]

IV References

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