An Analysis of Helicity and Knotted Plasma with Topological Invariants

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Abstract

Helicity is a measure of a field's self circulation or "knottedness", and is invariant for incompressible Ideal MHD. Within incompressible Ideal MHD the helicity takes the form of a collection of tangled vortex lines. In this paper we will construct the helicity of vortex loops in terms of topological invariants (Gauss Linking Number) and use that to then construct the helicity for knotted plasmas in terms of a new invariant (Calugareanu Invariant). The main interest in helicity is that it offers a lower bound on how the plasma can relax. We will reexamine this lower bound and find some knots whose helicity would suggest that the system can fully relax (whitehead link and borromean rings), but are non-trivial and can't. In response to this a method of finding the minimum magnetic energy of a knot based off its path instead of helicity is found. Finally, a discussion of numeric methods (PENCIL-code) resent uses in the topic and findings with regards to relaxation of these plasmas.

1 Tangled Vortex Lines

To gain some insight into properties of MHD helicity we will first examine how tangled vortex lines can be broken down into connected loops and their helicity expressed as topological invariants. Consider $\vec{A}$ is the vector field of a fluid who's curl forms of two closed vortex filaments that are unknotted, C_1 C_2, with strength, κ_1 & κ_2, the circulation around C_1,

$$K_1 = \oint_{C_1} \vec{A} \cdot d\vec{l} = \oint_{S_1} (\nabla \times \vec{A}) \cdot d\vec{S} \tag{1}$$

K_1 equal either 0 or $\pm\kappa_2$ depending on whether the loops are connected and their orientation. This can be generalized to any two knotted filaments by seeing that adding pairs of equal and opposite vorticity segments allows us to break C_2 into unknotted subloops around C_1.

$$K_1 = \oint_{C_1} \vec{A} \cdot d\vec{l} = \alpha_{12}\kappa_2 \tag{2}$$

Where $\alpha_{12} = \alpha_{21}$ is the winding number of the two loops or the number of times they wind around each other.

If instead of two loops connected to each other there was one loop that is knotted (linked to itself). The same method could be used, but now we count both parts of the connection since they are both parts of the original knot.

$$\oint_{C_1} \vec{A} \cdot d\vec{l} = \sum_i \oint_{C_i} \vec{A} \cdot d\vec{l} = 2\alpha_{11}\kappa_1 \tag{3}$$

Given that $C_1 = \sum_i C_i$. [1]

The most basic example of this method of inserting opposite and equal lines to break a closed integral into a group of them is the decomposition of the trefoil knot into two linked and unknotted loops.

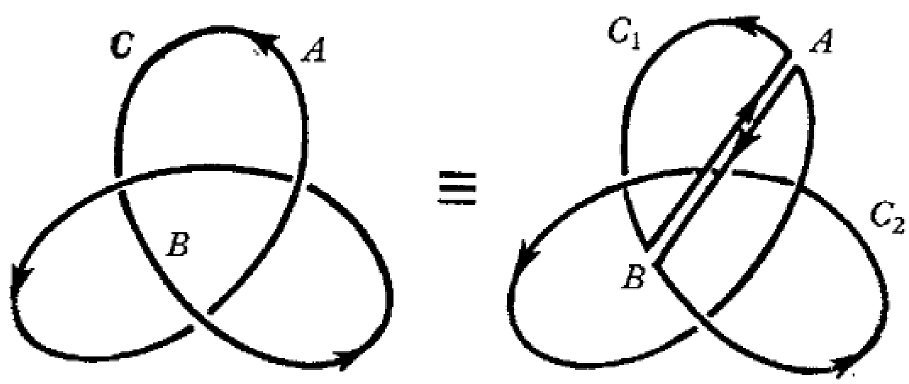

Figure 1: The trefoil knot, where the pair of lines can be inserted to separate a part of the tangle and turn the knot into two connected and unknotted loops.

One might see that the strength of a filament κ_i is the curl of $\vec{A}$ along the filament C_i, so we can say $\kappa_i d\vec{l} = (\nabla \times \vec{A})dV$ where dv is the volume containing

dl.

$$\kappa_i K_i = \oint_{c_i} \vec{A} \cdot \kappa_i d\vec{l} = \int_{v_i} \vec{A} \cdot (\nabla \times \vec{A}) dV \tag{4}$$

Therefor

$$H_M = \int_V \vec{A} \cdot (\nabla \times \vec{A}) dV = \sum_{i,j} \alpha_{ij} \kappa_i \kappa_j \tag{5}$$

This integral is invariant under the Euler equations of ideal fluid flow when $\vec{A} \cdot \vec{n} = 0$, so too is the description of helicity in terms of winding numbers and filament strength. [1]

It is important to note that by substatutin the geneal expression,

$$\vec{A}(\vec{r}) = -\frac{1}{4\pi} \int \frac{(\vec{r} - \vec{r'}) \times (\nabla \times \vec{A}(r'))}{|\vec{r} - \vec{r'}|^3} dV$$

in for H_M and re-expression as line integrals we get

$$\alpha_{ij} = \alpha_{ji} = \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\vec{r} - \vec{r'}) \cdot [d\vec{l_i} \times d\vec{l_j}]}{|\vec{r} - \vec{r'}|^3} \tag{6}$$

which is Gauss's integral definition for the linking number of two loops.

One can see that this discussion of tangled vortex line is not the same as the magnetic helicity in MDH, the helicity in MDH is in magnetic Fields not discrete vortex lines of the B field. Regardless, this has still offers us insight into how to think and describe these knots. This insight is very clear with regards to tangled flux tubes, which we will construct a description for using knotted vortex lines. [1]

2 Knotted Flux Tubes

To construct the helicity for knotted vortex tubes we treat them as a collection of vortex lines. The easiest example of this is the two linked, untwisted, and unknotted flux tubes, where their vortex lines are unlinked with lines from the same tube. They therefor are connected in the same way as the tubes, so they have helicity $H_{M1_i} = \pm\alpha_{12}\kappa_{1_i}\phi_2$, since the sum of the vortex line strength passing through line 1_i is the flux of tube 2. Summing this helicity over all the vortex lines in tube 1 gives us the helicity of tube 1, and this argument can be mirrored for the perspective of tube 2. Therefor, the total helicity of the tubes is

$$H_M = \pm 2\alpha_{12}\phi_1\phi_2 \tag{7}$$

Where ϕ_1 & ϕ_2 are the fluxes of the tubes.

Once again we are interested in the helicity of a knot (self linkage). Now we will examine the helicity of a knotted tube with flux ϕ and form C (the innermost part of the knot is given by curve C). The helicity of the flux tube is

$$H = h\phi^2 \tag{8}$$

where h is the Calugareanu invariant. The Calugareanu invariant is sum of the writhe (W_r), average torsion (T), and rotation (N) of the flux tube ie. $h = W_r + T + N$. N is an integer since the vortex loops are continues, writhe is the same expression for α_{ij} when $i = j$

$$W_r = \frac{1}{4\pi} \oint_C \oint_C \frac{(\vec{r} - \vec{r'}) \cdot (d\vec{l} \times d\vec{l'})}{|\vec{r} - \vec{r'}|^3}$$

and the torsion is a function of position on C, $\tau(l)$, so

$$T = \frac{1}{2\pi} \oint_C \tau(l) dl$$

It should be noted that the W_r, T, and N are still subject to change under certain conditions so long as h is conserved.[2] Figure 2 is a diagram of showing the change in T in relation to N.

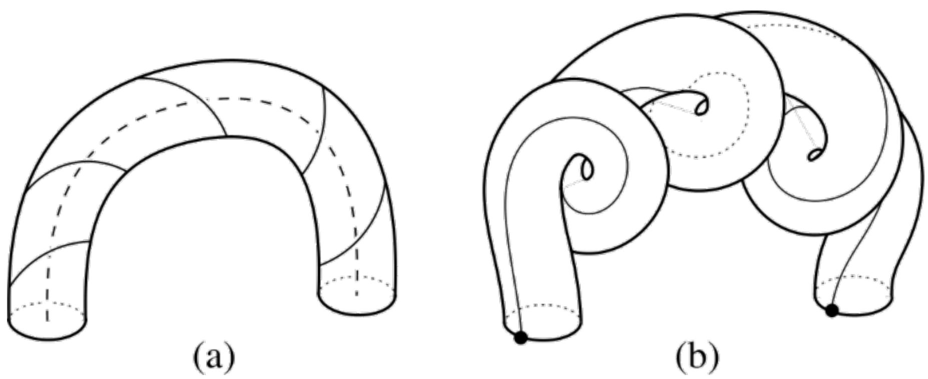

Figure 2: In diagram (a) we see that the vortex filament (solid line) twists around the center of the tube 5 times, but in (b) it only twists 2 times. This is due to the center of the tube in (b) making these three loops due to its torsion.

One assumption we make in this approach to knotted flux tubes in MHD is that the vortex lines in the tubes are all of the same rotation N. In reality the vortex lines can rap around C a different number of times, but taking this into account required more advanced methods.

3 Magnetic Relations

The magnetic energy has a lower bound placed in it by the field's helicity since it cannot infinitely contract field lines trapped by the knots topology. From this we have the expression,

$$E_M \geq |H_M|/l_0 = |h|\phi^2/l_0 \tag{9}$$

where l_0 is the initial length scale of the knot. An issue with this expression can be see, the field's ability to relax is based on its topology, but our expression is based off its Calugareanu invariant, these are not always equivalent. For example, both the Whitehead link and Borromean Rings (figure 3) have $H_M = 0$ but non-trivial topology that stops them from relaxing to $E_M = 0$. [2]

Whitehead link   Borromean rings

Figure 3: A Whitehead link and Borromean Rings. both have $h = 0$ since for each right-handed circulation of each other there is a corresponding left-handed one, but they they still not topologically trivial, so their minimum magnetic energy is greater than 0.

The question becomes how to express this minimum energy in a way that takes into account any non-trivial topology. We start with a dimensional expression

$$E_{M_{min}} = m(h)\phi^2 V^{-1/3} \tag{10}$$

and see that the minimum energy occurs when there is no more dissipation of kinetic energy due to viscosity since dissipation is the mechanism by which it relaxes. This mean the fluid is in magnetostatic equilibrium, so

$$\nabla p = (\nabla \times \vec{B}) \times \vec{B}$$

This gives us that the minimum energy of the knot is one that has the same topology as the initial knot, and is stable since its the minimum energy. For flux tubes of circular uniform cross section and no kinks, $m(h)$ can be computed and a graph of them for $T_{2,n}$ is included below. If there were kinks and non-uniform cross sections the minimum energy would go down, by an undetermined, though likely small for small h, amount. [2]

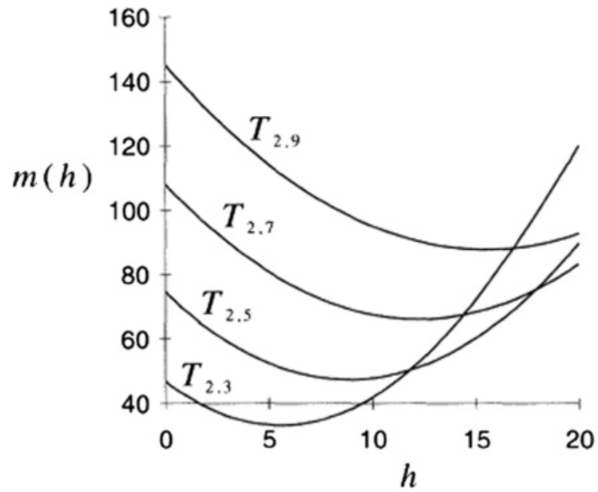

Figure 4: graphs of $m(h)$. The minimum of $m(h)$ is gives the minimum magnetic energy through equation 10.

4 Numerical Methods and Simulation

More recently numeric methods have been used to show that when incompressible MHD relaxes the minimum energy caused by helicity adds stability. It was found that as the plasmas relax their initial helicities organize to form nested toroidal surfaces. The method used to find this is the PENCIL-code, a solver favored in astrophysics for its versatility, which was chose since it can solve for the vector potential, letting the condition $\nabla \cdot \vec{B} = 0$ be maintained. Using this method it was found that various non-trivial topologies all time evolved into resulted in the same shape with a different distribution of energies. Remarkably, this relaxation still conserves helicity, would potentially offer stability to the magnetic fields of fusion plasmas . [3]

References

[1] H. K. Moffatt. The degree of knottedness of tangled vortex lines. *Journal of Fluid Mechanics*, 35(1):117–129, 1969.

[2] H. Keith Moffatt. Helicity and singular structures in fluid dynamics. *Proceedings of the National Academy of Sciences*, 111(10):3663–3670, 2014.

[3] Christopher Berg Smiet. *Knots in plasma.* PhD thesis, Leiden University, 2017.