A Brief Overview of the Dynamo Theory

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Dynamo theory aims to figure out how the magnetic fields of celestial bodies are generated and maintained. Since the evolutions of magnetic field and velocity field are strongly coupled, it is difficult to formulate a fully self-consistent model. To find out under which conditions a magnetic field can be excited, a series of kinematic dynamo theory is constructed, in which the velocity field is prescribed artificially. A powerful approach, called mean-field dynamo theory, is introduced. The novelty of this approach is that fields are divided into a mean part and a fluctuating part by taking proper averaging operation. In this theory, the magnetic field is excited by α-effect, which requires the turbulence has no mirror-symmetry. A simple quenching mechanism of the α-effect is also elucidated in this paper.

1. Introduction: The History of Dynamo Theory

In 1919, Sir Joseph Larmor posed a famous question—i.e., "*How could rotating body such as the sun become a magnet?*" Therefore, the story of dynamo theory starts from people's interest in the origination of the magnetic field of the sun. One may ask why people were not motivated by the origination of the geomagnetic field? For a long time, it was believed that the interior of the earth was a permanent magnet, as shown in Fig. 1. As time went by, people realized that this is wrong. Because the core temperature of the earth is far beyond the Curie point, earth can by no means be ferromagnetic material. The long duration of the geomagnetic field and the cycle of the solar magnetic field ask for a theory to explain them. Larmor offered several suggestions, one of which provides the basis for dynamo theory: *solar magnetic field originates from the motion of electrically conducting fluid within the rotating body*.

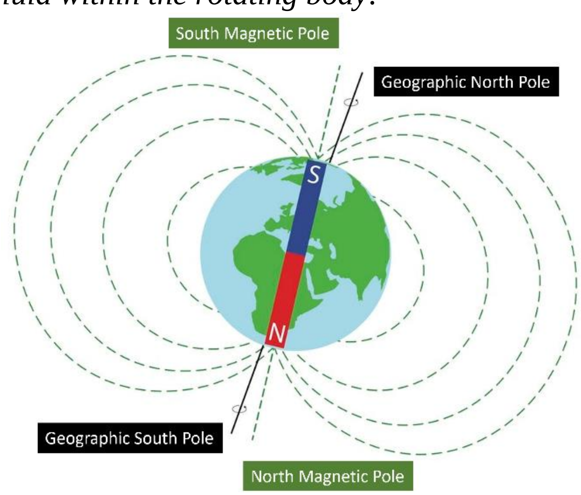

Fig. 1 The Earth's Magnetic Field

The rationality of this idea was first confirmed by Bullard's 'homopolar' disc dynamo in 1955 [1]. As shown in Fig. 2, a copper disc is rotating about a vertical axis with a constant angular velocity Ω. Its rim and axle are connected by a twisted conducting wire in which a current $I(t)$ flows. This current generates a magnetic flux across the disc, and, as the disc is rotating relative to the magnetic field, a motional electromotive force is generated. Since the disc and the wire form a closed circuit, we then have the following equation

$$L\frac{dI}{dt} + RI = \mathcal{E} = M\Omega I, \tag{1.1}$$

where M, L, and R are the mutual inductance, self-inductance, and resistance of the system. Clearly, if $\Omega > R/M$, the current will grow exponentially without limit, provided the growing magnetic field has no feedback on disc's rotation. This simple model reveals at least two insights: first, the system exhibits differential rotation. In this case the discontinuity of the rotation is concentrated at the rim of the disc. In a real dynamo system, the differential rotation can be distributed uniformly in space; second, there is no reflectional symmetry in this system. If the disc rotates in the opposite direction, the initial current can only decay exponentially. This property is of tremendous importance for the turbulent dynamo theory. As will be shown in Sec. 3, a nonvanishing electromotive force provided by turbulence requires a non-trivial turbulent helicity. There is, however, an essential difference between 'homopolar' disc dynamo and a real dynamo, i.e., the difference in topology. Bullard's model is delicately devised. The conducting wire is twisted and covered by insulating material. This kind of 'complex' geometric structure does not exist in the celestial bodies, in which the conducting fluid is confined to a single-connected region.

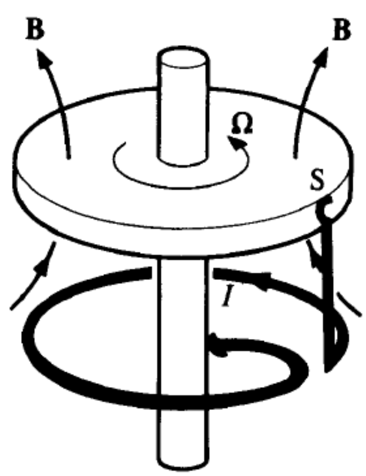

Fig. 2 The 'Homopolar' Disc Dynamo [2]

As the rotation and magnetic field of the Earth and the sun are approximately axisymmetric, it is natural for people to focus on models owning this symmetry. However, in 1934, Cowling proposed the first ‘anti-dynamo’ theorem, which concluded that *a steady axisymmetric magnetic field can not be maintained by axisymmetric motions* [3]. Obviously, it was necessary to consider non-axisymmetric, especially three-dimensional cases. In 1946, Elsasser summarized the dynamo problem as the interaction between a general prescribed axisymmetric velocity field and a general non-axisymmetric magnetic field in a conducting fluid contained within a rigid spherical boundary (medium outside the sphere is assumed to be insulate) [4]. Theories of this type are called *kinematic dynamo theory*, as the feedback of magnetic field on the velocity field is not included in these models. Since then, a series of kinematic dynamo theories appeared. In 1958, Herzenberg constructed the first kinematic dynamo model [5]. In his model, two smaller separate spherical regions rotating like rigid bodies are inside a spherical conducting medium. With special rates of rotations and a special arrangement of the axes of rotation, self-excitation occurs. And in 1973, Gailitis proved that oscillatory magnetic field is also possible, which might explain the periodic changes of solar magnetic field.

At that stage, when people wanted to go one step forward— i.e., including the interaction between magnetic field and velocity field, they found that a fully self-consistent dynamo theory (which is called *dynamic dynamo theory*) is too complex to be solved by a purely computational approach. A more advanced theoretical approach was urgently needed to mitigate the difficulty in numerical computation. In 1955, Parker adopted an averaging method to incorporate the non-axisymmetric upwellings in equations for *mean magnetic field* [6]. Although Parker’s idea was not mature enough, it is indeed a great breakthrough, and ignited a flame.

Several years later, this idea was recognized, and two approaches were developed. The first of these two masterpieces was “*nearly symmetric dynamo*” by Braginskii in 1964 [7]. He considered that the generation of magnetic field is a result of the simultaneous presence of an axisymmetric motion and some special non-axisymmetric motions. The second approach, named as “mean-field magnetohydrodynamics” or “turbulent dynamo theory”, by Steenbeck, Krause, and Rädler in 1966 [8], might be more general. They divide the velocity field and magnetic field into a mean part and a turbulent part. The key point is that turbulence can offer a mean electromotive force which has a component parallel to the prevailing local mean magnetic field. This is called “*α-effect*”. As will be discussed in Sec. 3, α-effect requires the turbulence to have a non-vanishing helicity, which echoes the lack of the reflectional symmetry of Bullard's 'Homopolar' disc dynamo. Unfortunately, even though the "mean-field magnetohydrodynamics" makes dynamic dynamo be manageable to some extent, the dynamic dynamo theory is still undeveloped compared with kinematic dynamo theory.

The following sections of this paper are arranged as follows. In Sec. 2, I will focus on the kinematic dynamo theory. The formulation of this theory, and some fundamental types of dynamos are introduced. Instead of listing the zoology of different kinematic dynamo theories, I will show the constraints on the construction of a kinematic dynamo. In Sec. 3, I will show the equations for mean fields and elaborate how the mean-field theory is closed. A subsection will be devoted to the illustration of the α-effect and its relation to turbulence helicity. In Sec. 4, I will talk a little about the dynamical dynamo theory, focusing on the α-quenching effect. Finally, a brief conclusion is given in Sec. 5.

2. Kinematic Dynamo Theory

One of the most obvious features of kinematic dynamo theory is that the velocity field is prescribed and doesn't evolve. Though it's not self-consistent, it provides insights that under which conditions the magnetic field of a celestial body can be excited and sustained. In addition, kinematic dynamo theory is more developed than dynamic dynamo theorem, and a lot of brilliant work have been done on this topic, so it plays a key role in dynamo theory.

2.1. Formulation of Kinematic Dynamo Theory

According to resistive MHD theory, the evolution of the magnetic field is governed by the induction equation

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \eta \nabla^2 \boldsymbol{B}, \tag{2.1}$$

where η is the resistivity. Resembling what Braginskii did, we separate $\boldsymbol{B}$ and $\boldsymbol{v}$ into axisymmetric and asymmetric parts by writing

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \widetilde{\boldsymbol{B}}, \quad \boldsymbol{v} = \overline{\boldsymbol{v}} + \widetilde{\boldsymbol{v}}. \tag{2.2}$$

More specifically, the axisymmetric parts of these two fields can be divided into zonal and meridional parts, i.e.

$$\begin{aligned} \overline{\boldsymbol{B}} &= B(s,z,t)\boldsymbol{e}_{\boldsymbol{\phi}} + \boldsymbol{B}_{\boldsymbol{M}}, \ \ \boldsymbol{B}_{\boldsymbol{M}} = \nabla \times \left[A(s,z,t)\boldsymbol{e}_{\boldsymbol{\phi}}\right] \\ \overline{\boldsymbol{v}} &= s\zeta(s,z,t)\boldsymbol{e}_{\boldsymbol{\phi}} + \boldsymbol{v}_{\boldsymbol{M}}, \ \ \boldsymbol{v}_{\boldsymbol{M}} = \nabla \times \left[\chi(s,z,t)\boldsymbol{e}_{\boldsymbol{\phi}}\right]' \end{aligned} \tag{2.3}$$

where ζ represents the differential rotation. Then, substituting Eq. (2.2) into Eq. (2.1), we can get

$$\begin{aligned} &\frac{\partial}{\partial t}\widetilde{\boldsymbol{B}} - \nabla \times \left(\overline{\boldsymbol{v}} \times \widetilde{\boldsymbol{B}}\right) - \nabla \times \left(\widetilde{\boldsymbol{v}} \times \widetilde{\boldsymbol{B}}\right) - \eta \nabla^2 \widetilde{\boldsymbol{B}} \\ &= -\frac{\partial}{\partial t}\overline{\boldsymbol{B}} + \nabla \times (\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}}) + \nabla \times (\widetilde{\boldsymbol{v}} \times \overline{\boldsymbol{B}}) + \eta \nabla^2 \overline{\boldsymbol{B}} \end{aligned}. \tag{2.4}$$

By taking the average over azimuthal angle ϕ, Eq. (2.4) can be divided into two equations:

$$\frac{\partial}{\partial t}\overline{\boldsymbol{B}} - \nabla \times (\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}}) - \eta \nabla^2 \overline{\boldsymbol{B}} = \nabla \times \overline{\boldsymbol{\mathcal{E}}}, \tag{2.5}$$

$$\frac{\partial}{\partial t}\widetilde{\boldsymbol{B}} - \nabla \times \left(\overline{\boldsymbol{v}} \times \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{\mathcal{E}}}\right) - \eta \nabla^2 \widetilde{\boldsymbol{B}} = \nabla \times (\widetilde{\boldsymbol{v}} \times \overline{\boldsymbol{B}}), \tag{2.6}$$

where $\boldsymbol{\mathcal{E}} = \widetilde{\boldsymbol{v}} \times \widetilde{\boldsymbol{B}}$ is the electromotive force created by asymmetric fields, and $\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{\mathcal{E}}} + \widetilde{\boldsymbol{\mathcal{E}}}$. In terms of the representation Eq. (2.3), Eq. (2.5) can be rewritten as

$$\frac{\partial}{\partial t}A + \frac{1}{s}\boldsymbol{v_M} \cdot \nabla(sA) = \eta \nabla^2 A + \overline{\boldsymbol{\mathcal{E}}}_\phi, \tag{2.7}$$

$$\frac{\partial}{\partial t}B + s\boldsymbol{v_M} \cdot \nabla\left(\frac{B}{s}\right) = s\boldsymbol{B_M} \cdot \nabla\zeta + \eta \nabla^2 B + (\nabla \times \overline{\boldsymbol{\mathcal{E}}})_\phi. \tag{2.8}$$

Here comes a subtle question, what is the expression for the mean electromotive force $\overline{\boldsymbol{\mathcal{E}}}$? In Eq. (2.4), the left-hand side is linear in $\widetilde{\boldsymbol{B}}$, and the right-hand side is linear in $\overline{\boldsymbol{B}}$. Therefore, $\widetilde{\boldsymbol{B}}$ is, in principle, a functional of $\overline{\boldsymbol{v}}$, $\widetilde{\boldsymbol{v}}$, and $\overline{\boldsymbol{B}}$, especially, a functional proportional to $\overline{\boldsymbol{B}}$. Consequently, $\overline{\boldsymbol{\mathcal{E}}}$ is proved to be a functional of $\overline{\boldsymbol{v}}$, $\widetilde{\boldsymbol{v}}$, and $\overline{\boldsymbol{B}}$ which proves to be linear in $\overline{\boldsymbol{B}}$. Therefore, it is fair to guess that $\overline{\boldsymbol{\mathcal{E}}}$ can be approximated by $\overline{\boldsymbol{B}}$ and its first-order derivative—i.e., in isotropic case

$$\overline{\boldsymbol{\mathcal{E}}} = \alpha\overline{\boldsymbol{B}} - \beta \nabla \times \overline{\boldsymbol{B}}. \tag{2.9}$$

This approximation can be better justified in turbulence dynamo theory. Note that $\boldsymbol{\mathcal{E}}$ is vector while $\boldsymbol{B}$ is a pseudo-vector, this implies that α is a pseudo-scalar. Therefore, for a system with mirror symmetry, α can only be equal to zero. In other words, a non-vanishing α requires a system that has no reflectional symmetry, which is consistent with the result of the 'homopolar' disc dynamo.

Plugging Eq. (2.9) into Eq. (2.7) and Eq. (2.8). we get

$$\frac{\partial}{\partial t}A + \frac{1}{s}\boldsymbol{v_M} \cdot \nabla(sA) = \eta_T \nabla^2 A + \alpha B, \tag{2.10}$$

$$\frac{\partial}{\partial t}B + s\boldsymbol{v_M} \cdot \nabla\left(\frac{B}{s}\right) = \eta_T \Delta B + [\nabla\zeta \times \nabla(sA)]_\phi - \left[\alpha \nabla^2 A + \frac{1}{s}\nabla\alpha \cdot \nabla(sA)\right], \tag{2.11}$$

where $\eta_T = \eta + \beta$ is the total diffusivity. The boundary conditions of this system are

$$[A] = [\partial A/\partial n] = [B] = 0 \ \ on\ S. \tag{2.12}$$

Two differential equations Eq. (2.10), Eq. (2.11), and along with boundary conditions Eq. (2.12), formulate the basic kinematic dynamo theory.

2.2. α^2-dynamo and $\alpha\omega$-dynamo

An important observation of Eq. (2.10) and Eq. (2.11) is that there are two parameters: α and ζ. In Eq. (2.9), the first term on the right-hand side means there is an electromotive force parallel to the local magnetic field, which is not predicted in classical electrodynamics, and has been known as "*α-effect*" (the choice of the letter α is just a coincidence in history). As defined by Eq. (2.3), ξ represents the differential zonal rotation which called "*ω-effect*" (again, the choice of the letter ω is due to historical reasons). Then we can define two dimensionless numbers

$$R_\alpha = \frac{\alpha_0 L}{\eta_T}, \quad R_\omega = \frac{\zeta_0 L^2}{\eta_T}. \tag{2.13}$$

According to the ratio of R_α to R_ω, there are two extreme cases: i) $R_\omega \ll R_\alpha$ (*α^2-dynamo*); ii) $R_\omega \gg R_\alpha$ (*$\alpha\omega$-dynamo*). These two limiting cases will be discussed separately.

i. α^2-dynamo

In this case, α-effect appears in both Eq. (2.10) and Eq. (2.11), which means α-effect can generate not only A from B, but also B from A. When R_α reach a critical value $R_{\alpha c}$, α^2-dynamo can maintain a magnetic field. Here we only analyze the free modes of the α^2-dynamo.

For simplicity, we can assume $\boldsymbol{v_M} = \zeta = 0$, α and β are uniform and constant. Then Eq. (2.5) can be rewritten as

$$\frac{\partial}{\partial t}\overline{\boldsymbol{B}} - \eta_T \nabla^2 \overline{\boldsymbol{B}} = \alpha \nabla \times \overline{\boldsymbol{B}}. \tag{2.14}$$

For a 'force-free' field,

$$\nabla \times \overline{\boldsymbol{B}} = k\overline{\boldsymbol{B}}, \tag{2.15}$$

$$\nabla^2 \overline{\boldsymbol{B}} = -\nabla \times \nabla \times \overline{\boldsymbol{B}} = -k^2 \overline{\boldsymbol{B}}, \tag{2.16}$$

where K is a constant. So, if $\overline{\boldsymbol{B}}(\boldsymbol{x}, 0) = \widehat{\boldsymbol{B}}(\boldsymbol{x})$, and $\overline{\boldsymbol{B}}(\boldsymbol{x}, t) = \widehat{\boldsymbol{B}}(\boldsymbol{x})e^{pt}$, the growth rate p satisfies

$$p = \alpha k - \eta_T k^2. \tag{2.17}$$

Hence the mean magnetic field will grow exponentially if $0 < k < k_c = \alpha/\eta_T$, and the maximum growth rate is $p_m = \alpha^2/4\eta_T$.

ii. $\alpha\omega$-dynamo

In this case, the α-effect in Eq. (2.11) can be neglected, which means only A is generated from α-effect, and B is generated from A. The ω-effect is illustrated in Fig. 3. For a rotating conducting sphere, if there is a meridional magnetic field threading though it, due to the flux-freezing law, field lines inside the sphere will rotate with the sphere and be twisted. A toroidal field is thus generated. To maintain a magnetic field, the product of these two numbers, called the dynamo number, $D = R_\alpha R_\omega$, must be not less than a critical value, D_c. Again, here we only discuss the free modes of the $\alpha\omega$-dynamo.

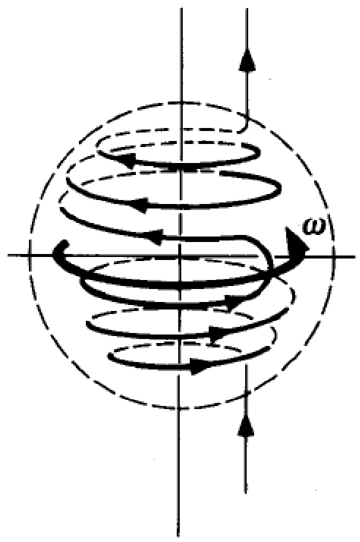

Fig. 3 Generation of a Toroidal Field Due to the ω-effect [2]

In Cartesian coordinate, the $\alpha\omega$-dynamo is described by

$$\frac{\partial}{\partial t}A + \boldsymbol{v_M} \cdot \nabla A = \alpha B + \eta_T \nabla^2 A, \tag{2.18}$$

$$\frac{\partial}{\partial t}B + \boldsymbol{v_M} \cdot \nabla B = \boldsymbol{B_M} \cdot \nabla v + \eta_T \nabla^2 B, \tag{2.19}$$

where $\overline{\boldsymbol{B}} = B\boldsymbol{e_y} + \nabla \times (A\boldsymbol{e_y})$, $\overline{\boldsymbol{v}} = v\boldsymbol{e_y} + \boldsymbol{v_M}$. Over regions of limited extent in which $\boldsymbol{v_M}$, α, and ∇v can be treated as constant, these equations have local solutions of the form

$$(A, B) = (\hat{A}, \hat{B}) \exp(pt + i\boldsymbol{k} \cdot \boldsymbol{x}), \qquad \boldsymbol{k} = (k_x, \boldsymbol{0}, \boldsymbol{k_y}). \tag{2.20}$$

Then the growth rate p satisfies

$$\tilde{p}^2 = (p + \eta_T k^2 + i\boldsymbol{v_M} \cdot \boldsymbol{k})^2 = 2i\gamma = -i\alpha(\boldsymbol{k} \times \nabla v)_y. \tag{2.21}$$

To simplify the analysis, we can assume $\boldsymbol{v_M} = 0$. Then we can conclude that it is possible to maintain a magnetic field when $|\gamma| > \eta_T^2 k^2$.

2.3. Fast Dynamo and Slow Dynamo [9]

Besides α^2-dynamo and $\alpha\omega$-dynamo, there is another way to classify dynamos—i.e., fast dynamo and slow dynamo.

For ideal MHD, the magnetic field obeys Alfven's theorem. In terms of the initial field $\boldsymbol{B}(\boldsymbol{X},0)$, the expression for $\boldsymbol{B}(\boldsymbol{x},\boldsymbol{t})$ is

$$B_i(\boldsymbol{x},t) = B_j(\boldsymbol{X},t)\partial x_i/\partial X_j, \tag{2.22}$$

where $\boldsymbol{x}(\boldsymbol{X},t)$ is the position of the fluid element initially at $\boldsymbol{x}(\boldsymbol{X},0) = \boldsymbol{X}$. Since field lines are frozen into the fluid element, the growth rate of the magnetic field is determined by "deformation gradient matrix" $\partial x_i/\partial X_j$. If one of the eigenvalues of this matrix grows exponentially, characterized by a positive Lyapunov exponent,

$$\lambda_L(\boldsymbol{X}) = \lim_{t\to\infty}(t^{-1}\ln\Lambda) > 0, \tag{2.23}$$

the magnetic field will also grow exponentially on the advective time scale, $T_{adv} = \lambda_L^{-1}$. Dynamos in which magnetic field grows exponentially are classified as "*fast dynamo*". Otherwise, if $\lambda_L(\boldsymbol{X}) = 0$, for example, $\Lambda(t) \propto t$, this kind of dynamos is classified as "*slow dynamo*".

The mechanism behind fast dynamo can be illustrated by *stretch-twist-fold picture* (see Fig. 4) by Vainshtein and Zeldovich in 1972 [10]. Suppose a closed flux tube of magnetic flux $\Phi_0 = \int_S \boldsymbol{B}\cdot\boldsymbol{n}dS$, where S is a cross section of the tube. If we stretch this tube to double its length, as the plasma is incompressible, the cross section of this tube will reduce by a half. Due to the flux-freezing law, the flux across the cross section is conserved, which means the magnitude of the magnetic field also doubles. Then the tube is twisted in a way shown in Fig. 4 and is folded so that the number of the loops becomes two. Finally, these two loops merge with each other to make this process irreversible. Repeating this process for n times, the field inside the loop will be amplified by a factor of 2^n. Several points are worth addressing. First, shear is needed in this process. Without shear, flux tube cannot be twisted, and then the field in the folded loop would cancel rather than add coherently. Second, the last step, i.e., merging, can be done at any time, and dynamo growth is not limited by this step. This is why the characteristic time scale of fast dynamo is the convection time instead of the resistive diffusion time. However, if the merging rate is slow, as we can expect, twisting field lines will cost more and more energy, the exponential grow becomes hard to be kept.

In 1989, Vishik argued that, when there is no stretching anywhere in a smooth fluid flow (mathematically, $\lambda_L = 0$ everywhere), the resulting dynamo is slow dynamo [11]. Another simple

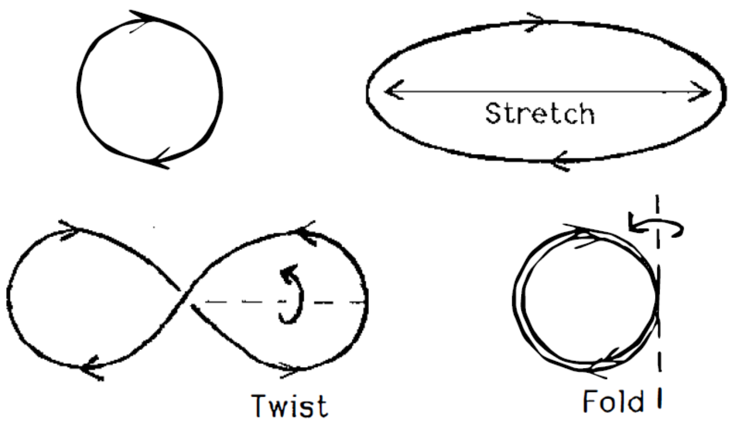

Fig. 5 Flux of Tube Doubles by the Stretch-Twist-Fold Mechanism [9]

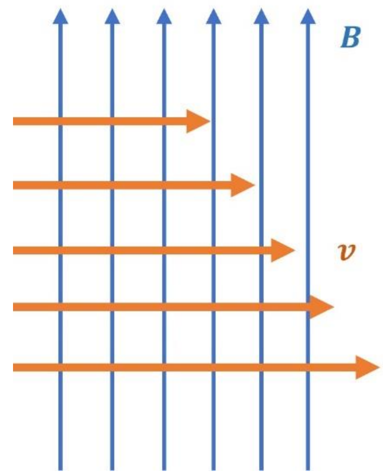

Fig. 4 An Illustration of a Slow Dynamo

picture can illustrate a slow dynamo. Consider a uniform magnetic field. When there is a shear flow transverse to the field, field lines will be stretched perpendicular to the magnetic field (see Fig. 5). In this condition, Λ is proportional to t.

2.4. Necessary Conditions for Dynamo

In the development of dynamo theory, several necessary conditions for dynamos are found. The most remarkable one must be Cowling's *"anti-dynamo" theorem*, as mentioned in Sec. 1. According to Eq. (2.7) and Eq. (2.8), if magnetic field and velocity field are both axisymmetric, i.e., $\widetilde{\boldsymbol{v}} = \widetilde{\boldsymbol{B}} = 0$, then we have (assuming there is no meridional convection)

$$\frac{\partial}{\partial t}A - \eta\Delta A = 0, \tag{2.24}$$

$$\frac{\partial}{\partial t}B - \eta\Delta B = s\boldsymbol{B_M} \cdot \nabla\zeta. \tag{2.25}$$

Obviously, in Eq. (24), as there is no source for A, A will decay exponentially. In Eq. (2.25), the only source for B is $\boldsymbol{B_M}$. However, after A decays to zero, B will lose its source and decay exponentially, either. Later, this theorem is further developed, and leads to a conclusion that a steady axisymmetric magnetic field can never be maintained by dynamo action. Nevertheless, a non-axisymmetric magnetic field *can* be maintained by a steady axisymmetric velocity field. This is proved by Gailitis in 1970 [12].

For spherical models with constant electrical conductivity, in 1964, Braginskii generalized Cowling's theorem by including non-steady magnetic fields. He proved that any field which is symmetric with respect to a given axis is bound to decay if $\boldsymbol{v}$ is solenoidal ($\nabla \cdot \boldsymbol{v} = 0$) and symmetric with respect to the same axis. Actually, this statement retains its validity even if the velocity field is not symmetric.

As Cowling's theorem provides a condition for the magnetic field, Elsasser (in 1946), and Bullard and Gellman (in 1954), and Moffatt (in 1978) concluded that a magnetic field cannot be maintained by solenoidal motions without radial components if electrical conductivity is a constant. A lower bound for the magnitude of the radial motion necessary for dynamo action has been given by Busse in 1975.

Last but not least, Childress gave a requirement of the magnitude of the velocity field. In particular, for a spherical model with constant electrical conductivity and solenoidal motions, the magnetic Reynolds number should satisfy

$$R_M = \frac{U_{max}R}{\eta} \geq \pi, \tag{2.26}$$

where U_{max} is the maximum relative velocity inside the sphere and R is the radius.

3. Mean-Field Dynamo Theory

In both the core of the Earth and the convection zone of the sun, the motions of conducting fluids have been observed to be highly turbulent. In addition, the solar magnetic field has a wide range of spatial and temporal scales. However, there is no doubt that the magnetic fields of the Earth and the sun have large-scale structures. These facts imply that the emergence of self-excited dynamos is closely connected to turbulence, and there is a separation of scales between the large-scale structures and small-scale structures (e.g., turbulent motions, irregular fields). Therefore, it is possible to divide the magnetic field and velocity field into a mean field and a fluctuating field by adopting proper averaging operation. This is known as mean-field magnetohydrodynamics, which is concerned with the evolution and behavior of mean electromagnetic and hydrodynamic fields in a turbulent conducting medium.

In both Sec. 1 and Sec. 2, there is a common feature in technical dynamos and the kinematic dynamo theory—i.e., systems lack mirror-symmetry. *The lack of reflectional symmetry is the origin of α-effect.* But one question still remains unsolved: where does the Eq, (2.9), i.e., the expression for the mean electromotive force, come from? Up to now, this equation appears as a magic. This question can be better answered in mean-field dynamo theory.

3.1. Equations for the Mean Fields

Before constructing the equations for the mean field, we need to clarify one question: what is the definition of the averaging operation here? This question is of great subtlety. Theoretically, the averaging operation we use is *ensemble average (statistical average)*. For example, as the cycle of the solar magnetic field is 22 years, each complete cycle can be treated as a sample of the ensemble. Then we just need to take the average over a large number of different cycles to get the ensemble average. However, in reality, it is common to admit *space or time average*.

Assume the characteristic length and time scales of the mean field are $\bar{\lambda}$ and $\bar{\tau}$, and those of the fluctuations are λ_{cor} and τ_{cor}. We denote the scale of the averaging range by λ_{av} and τ_{av}. In order to separate large and small scales, we must require inequalities

$$\bar{\lambda} \gg \lambda_{av} \gg \lambda_{cor}, \tag{3.1}$$

$$\bar{\tau} \gg \tau_{av} \gg \tau_{cor}. \tag{3.2}$$

to be fulfilled.

The effect of the averaging is to divide a field into two parts. Let $\boldsymbol{F}$ be a irregular field. Its corresponding mean field, $\overline{\boldsymbol{F}}$, is defined as the expectation value of $\boldsymbol{F}$ in an ensemble of many identical systems. $\widetilde{\boldsymbol{F}}$ is defined as the residual part of $\boldsymbol{F}$, i.e., $\widetilde{\boldsymbol{F}} = \boldsymbol{F} - \overline{\boldsymbol{F}}$. Then we have the following Reynolds relations:

$$\begin{aligned} \boldsymbol{F} = \overline{\boldsymbol{F}} + \widetilde{\boldsymbol{F}}, \quad \overline{\overline{\boldsymbol{F}}} = \overline{\boldsymbol{F}}, \quad \overline{\widetilde{\boldsymbol{F}}} = 0, \\ \overline{\boldsymbol{F} + \boldsymbol{G}} = \overline{\boldsymbol{F}} + \overline{\boldsymbol{G}}, \quad \overline{\overline{\boldsymbol{F}}\overline{\boldsymbol{G}}} = \overline{\boldsymbol{F}}\overline{\boldsymbol{G}}, \quad \overline{\overline{\boldsymbol{F}}\widetilde{\boldsymbol{G}}} = 0 \end{aligned}. \tag{3.3}$$

The mean-field equations are derived as follows. We start with basic resistive MHD equations:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \times \boldsymbol{H} = \boldsymbol{j}, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{3.4}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}). \tag{3.5}$$

By combining Eq. (3.4) and Eq. (3.5), we can get the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \tag{3.6}$$

Taking the average of Eq. (3.4) and Eq. (3.5), we obtain

$$\nabla \times \overline{\boldsymbol{E}} = -\frac{\partial \overline{\boldsymbol{B}}}{\partial t}, \qquad \nabla \times \overline{\boldsymbol{H}} = \bar{\boldsymbol{j}}, \qquad \nabla \cdot \overline{\boldsymbol{B}} = 0, \tag{3.7}$$

$$\overline{\boldsymbol{B}} = \mu \overline{\boldsymbol{H}}, \qquad \bar{\boldsymbol{j}} = \sigma\left(\overline{\boldsymbol{E}} + \overline{\boldsymbol{u}} \times \overline{\boldsymbol{B}} + \overline{\widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{B}}}\right). \tag{3.8}$$

Compared with Eq. (3.5), one extra term appears in Eq. (3.8), i.e., $\overline{\widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{B}}}$, which gives an additional electromotive force in Ohm's law for the mean field. It is named as "*turbulent electromotive force*", and in the remaining part of the paper we will use the notation

$$\mathfrak{E} = \overline{\widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{B}}}. \tag{3.9}$$

Clearly, to determine $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{H}}$, $\overline{\boldsymbol{E}}$, and $\overline{\boldsymbol{J}}$, we need to study the $\mathfrak{E}$ in detail.

Recall Eq. (2.4) and our discussion in Sec. 2.1, $\mathfrak{E}$ is a functional of $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{u}}$, and $\widetilde{\boldsymbol{u}}$. It is the feature of turbulence that the correlation time and correlation length of quantities like $\widetilde{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{B}}$ are very small. Therefore, to determine the value of $\mathfrak{E}$ at a specific space-time point, we just need to know $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{u}}$, and $\widetilde{\boldsymbol{u}}$ in a small neighborhood of the point considered. As mentioned in Sec. 2.1, $\mathfrak{E}$ is a linear homogeneous functional of $\overline{\boldsymbol{B}}$. Since $\overline{\boldsymbol{B}}$ varies slowly in space and the neighborhood we are considering is quite small, it is reasonable to say that $\mathfrak{E}$ can be approximated by $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{B}}$'s first order derivative. Moreover, we can further restrict our choices for the expression for $\mathfrak{E}$. We notice that $\mathfrak{E}$ is a vector, so the only vectors available to the construction of $\mathfrak{E}$ is: $\overline{\boldsymbol{B}}$, $\nabla \times \overline{\boldsymbol{B}}$, $\overline{\boldsymbol{B}} \times (\nabla \times \overline{\boldsymbol{B}})$, $(\overline{\boldsymbol{B}} \cdot \nabla)\overline{\boldsymbol{B}}$, etc. Since $\mathfrak{E}$ is linear in $\overline{\boldsymbol{B}}$, the only possible choice is

$$\mathfrak{E} = \overline{\widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{B}}} = \alpha\overline{\boldsymbol{B}} - \beta\nabla \times \overline{\boldsymbol{B}}, \tag{3.10}$$

which is almost the same as Eq. (2.9). Substituting Eq. (2.9) into Eq. (3.8), in the case of $\boldsymbol{u} = 0$, the Ohm's law is rewritten as

$$\overline{\boldsymbol{J}} = \sigma_T(\overline{\boldsymbol{E}} + \alpha\overline{\boldsymbol{B}}), \tag{3.11}$$

where σ_T, the turbulent conductivity, is given by

$$\sigma_T = \frac{\sigma}{1 + \mu\sigma\beta}. \tag{3.12}$$

To pin the value of α and β down, we need to utilize linearized Eq. (2.6)

$$\frac{\partial}{\partial t}\widetilde{\boldsymbol{B}} - \eta\nabla^2\widetilde{\boldsymbol{B}} = \nabla \times (\widetilde{\boldsymbol{v}} \times \overline{\boldsymbol{B}}). \tag{3.13}$$

Here we assume that magnitudes of turbulent quantities are small, which allows us to discard all the higher order terms. As we shall see in Sec. 3.2, this approximation is called "*second order correlation approximation*".

In addition to second order correlation approximation, we only focus on two limiting cases:

i. High conductivity limit

In this case, $\eta \to 0$, so we have a first-order partial differential equation

$$\frac{\partial}{\partial t}\widetilde{\boldsymbol{B}} = \nabla \times (\widetilde{\boldsymbol{v}} \times \overline{\boldsymbol{B}}), \nabla \cdot \widetilde{\boldsymbol{B}} = 0 \tag{3.14}$$

Integrating Eq. (3.14) over time from t_0 to t, we get

$$\widetilde{\boldsymbol{B}}(\boldsymbol{x}, t) = \int_{-\infty}^{t} \nabla \times \left(\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t') \times \overline{\boldsymbol{B}}(\boldsymbol{x}, t')\right) dt' \tag{3.15}$$

The initial condition $\widetilde{\boldsymbol{B}}(\boldsymbol{x}, t_0)$ are omitted due to fact that the turbulent system has a very short memory. Plugging Eq. (3.15) into Eq. (3.9), the turbulent electromotive force is

$$\mathfrak{E}(\boldsymbol{x}, t) = \int_0^\infty \overline{\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t) \times \nabla \times \left(\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t-\tau) \times \overline{\boldsymbol{B}}(\boldsymbol{x}, t)\right)} d\tau. \tag{3.16}$$

After long boring algebra, we can obtain

$$\alpha = -\frac{1}{3}\int_0^\infty \overline{\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t) \cdot \nabla \times \widetilde{\boldsymbol{u}}(\boldsymbol{x}, t-\tau)} d\tau = -\frac{1}{3}\overline{\widetilde{\boldsymbol{u}} \times \nabla \times \widetilde{\boldsymbol{u}}}\tau_{cor}, \tag{3.17}$$

$$\beta = \frac{1}{3}\int_0^\infty \overline{\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t) \cdot \widetilde{\boldsymbol{u}}(\boldsymbol{x}, t-\tau)} d\tau = \frac{1}{3}\tilde{u}^2\tau_{cor}. \tag{3.18}$$

In Eq. (3.17), the correlation $\overline{\widetilde{\boldsymbol{u}} \times \nabla \times \widetilde{\boldsymbol{u}}}$ is always defined as *turbulence helicity* h. Once again, we can see α-effect is directly related to the lack of reflectional symmetry of turbulence. If a turbulence prefers left-handed screw, α has a negative sign, otherwise α is positive.

ii. Low conductivity limit

In this case, Eq. (3.13) is still a second-order partial differential equation, which is

$$-\eta\nabla^2\widetilde{\boldsymbol{B}} = \nabla \times (\widetilde{\boldsymbol{v}} \times \overline{\boldsymbol{B}}). \tag{3.19}$$

Eq. (3.19) is a Poisson equation, we can get $\widetilde{\boldsymbol{B}}$ by using the Green's function. So, the solution of Eq. (3.19) is

$$\widetilde{\boldsymbol{B}}(\boldsymbol{x}, t) = \frac{1}{4\pi\eta}\int \frac{\nabla \times \left(\widetilde{\boldsymbol{u}}(\boldsymbol{x}', t) \times \overline{\boldsymbol{B}}(\boldsymbol{x}', \boldsymbol{t})\right)}{|\boldsymbol{x}' - \boldsymbol{x}|} d\boldsymbol{x}' \ . \tag{3.20}$$

Substituting Eq. (3.20) into Eq. (3.9), we get

$$\mathfrak{E}(\boldsymbol{x}, t) = \frac{1}{4\pi\eta}\int \overline{\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t) \times \left[\nabla \times \left(\widetilde{\boldsymbol{u}}(\boldsymbol{x} + \boldsymbol{\xi}, t) \times \overline{\boldsymbol{B}}(\boldsymbol{x} + \boldsymbol{\xi}, t)\right)\right]} \frac{d\xi}{\xi}. \tag{3.21}$$

And

$$\alpha = -\frac{1}{3\eta}\overline{\boldsymbol{a_1} \cdot \nabla \times \boldsymbol{a_1}}, \tag{3.22}$$

$$\beta = \frac{1}{3\eta}\left(\overline{\boldsymbol{a}_1^2} - \overline{\varphi^2}\right), \tag{3.23}$$

where $\widetilde{\boldsymbol{u}} = \nabla \times \boldsymbol{a_1} - \nabla\varphi$.

3.2. Closure of the Mean-Field Dynamo Theory

We can define the two-point-two-time correlation tensor of the velocity field as

$$Q_{ik}(\boldsymbol{x}, \boldsymbol{\xi}, t, \tau) = \overline{\tilde{u}_i(\boldsymbol{x}, t)\tilde{u}_j(\boldsymbol{x} + \boldsymbol{\xi}, t + \tau)}, \tag{3.24}$$

the mixed two-point-two-time correlation tensor of the velocity field and the magnetic field as

$$P_{ik}(\boldsymbol{x},\boldsymbol{\xi},t,\tau) = \overline{\tilde{u}_\iota(\boldsymbol{x},t)\tilde{B}_k(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)}, \tag{3.25}$$

and the two-point-two-time correlation tensor of the magnetic field as

$$B_{ik} = \overline{\tilde{B}_\iota(\boldsymbol{x},t)\tilde{B}_k(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)}. \tag{3.26}$$

In this light, we can define correlation-tensor of higher rank, i.e.,

$$Q_{ijk}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta},t,\tau,\sigma) = \overline{\tilde{u}_\iota(\boldsymbol{x},t)\tilde{u}_\jmath(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)\tilde{u}_k(\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\eta},t+\tau+\sigma)}, \tag{3.27}$$

$$P_{ijk}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta},t,\tau,\sigma) = \overline{\tilde{u}_\iota(\boldsymbol{x},t)\tilde{u}_\jmath(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)\tilde{B}_k(\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\eta},t+\tau+\sigma)}, \tag{3.28}$$

$$\begin{aligned} &Q_{ijkl}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta},t,\tau,\sigma,\rho) \\ &= \overline{\tilde{u}_\iota(\boldsymbol{x},t)\tilde{u}_\jmath(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)\tilde{u}_k(\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\eta},t+\tau+\sigma)}, \\ &\overline{\tilde{u}_k(\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\eta}+\boldsymbol{\zeta},t+\tau+\sigma+\rho)} \end{aligned} \tag{3.29}$$

$$\begin{aligned} &P_{ijkl}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta},t,\tau,\sigma,\rho) \\ &= \overline{\tilde{u}_\iota(\boldsymbol{x},t)\tilde{u}_\jmath(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)\tilde{u}_k(\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\eta},t+\tau+\sigma)}. \\ &\overline{\tilde{B}_k(\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\eta}+\boldsymbol{\zeta},t+\tau+\sigma+\rho)} \end{aligned} \tag{3.30}$$

And

$$\begin{gathered} p_{ij}(\boldsymbol{x},t) = P_{ij}(\boldsymbol{x},0,t,0), \qquad p_{ijk}(\boldsymbol{x},\boldsymbol{\xi},t,\tau) = P_{ijk}(\boldsymbol{x},\boldsymbol{\xi},0,t,\tau,0), \\ p_{ijkl}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta},t,\tau,\sigma) = P_{ijkl}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{\eta},0,t,\tau,\sigma,0). \end{gathered} \tag{3.31}$$

To make equations more compact, we introduce the operators

$$D_{jn} = \left(\frac{\partial}{\partial t} - \eta\Delta\right)\delta_{jn} - \epsilon_{jkl}\epsilon_{lmn}\frac{\partial}{\partial x_k}\bar{u}_m, \tag{3.32}$$

$$D_{jmn} = \epsilon_{jkl}\epsilon_{lmn}\frac{\partial}{\partial x_k}. \tag{3.33}$$

N.B., when we apply D_{jn} or D_{jmn} to any correlation, the differentiation is to be carried out with respect to the last set of space or time coordinates.

Now the average of Eq. (3.6) can be written as

$$D_{jn}\bar{B}_n = D_{jmn}p_{mn}. \tag{3.34}$$

If we multiply Eq. (3.34) by $\tilde{u}_m(\boldsymbol{x},t)$, and take the average again, we get

$$D_{jn}P_{mn} = D_{jnp}\left(Q_{mn}\bar{B}_p\right) + D_{jnp}p_{mnp}. \tag{3.35}$$

From Eq. (3.34) and Eq. (3.35) we can see, each correlation tensor is always coupled with higher rank correlations, which means these correlations constitute a hierarchy of equations. Therefore, we have to figure out a way to truncate this hierarchy.

If we only keep the turbulent velocity $\tilde{u}$ up to the second order, Eq. (3.35) becomes

$$D_{jn}P_{mn} = D_{jnp}(Q_{mn}\bar{B}_p). \tag{3.36}$$

In this way, the mean-field equations are closed. This approximation is called "*second order correlation approximation*". Omitting p_{mnp} in Eq. (3.35) is equivalent to omitting the term $\tilde{\boldsymbol{u}} \times \tilde{\boldsymbol{B}} - \overline{\tilde{\boldsymbol{u}} \times \tilde{\boldsymbol{B}}}$ in Eq. (3.13). Because if we multiply this term by $\tilde{\boldsymbol{u}}$ and take the average, $\overline{\tilde{\boldsymbol{u}} \times \overline{\tilde{\boldsymbol{u}} \times \tilde{\boldsymbol{B}}}}$ vanishes and $\overline{\tilde{\boldsymbol{u}} \times \tilde{\boldsymbol{u}} \times \tilde{\boldsymbol{B}}}$ is just p_{mnp}.

3.3. Illustration of the α-effect

To help reader understand α-effect better, we can use Fig. 6 to illustrate how α-effect work. In the high-conductivity limit, flux ropes are frozen into the fluid elements. So, if a flux rope meets a right-handed vortex, as shown in Fig. 6., it will be twisted. According to the Ampere's law, this field can produce a current anti-parallel to the unperturbed magnetic field. In contrast, if a flux rope meets a left-handed vortex, a current parallel to the unperturbed magnetic field will be produced. In order for the effect to be macroscopic, turbulence must have a preference in "chirality", i.e., a non-vanishing turbulence helicity.

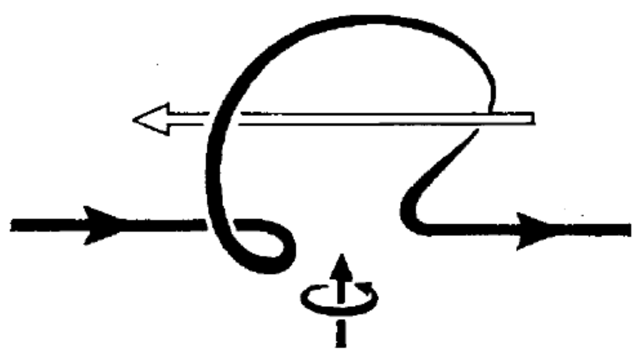

Fig. 6 When a Flux Rope meets a Right-handed Vortex, an Anti-parallel Current Is Produced. [13]

4. Dynamical Dynamo Theory: α-Quenching

The dynamos appearing in Sec. 2 all grow exponentially, which is not physical. The resultant magnetic field must react to the velocity field so that the system can reach equilibrium. To describe this process, we need to construct a dynamical dynamo theory, which is less developed compared with the kinematic dynamo theory. In this section I will simply introduce how the magnetic field can react to the velocity field through α-quenching [14,15].

The linearized equation of motion for conducting fluid is

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{u}} = \boldsymbol{B_0} \cdot \nabla\tilde{\boldsymbol{b}} + \tilde{\boldsymbol{b}} \cdot \nabla\boldsymbol{B_0} - \nabla\tilde{p}_{tot}, \tag{4.1}$$

where the fluid is incompressible, i.e., $\nabla \cdot \boldsymbol{v} = 0$. We can further assume $\boldsymbol{B_0}$ is constant, so Eq. (4.1) can be rewritten as

$$\frac{\partial}{\partial t}\widetilde{\boldsymbol{u}} = \boldsymbol{B_0} \cdot \nabla\widetilde{\boldsymbol{b}}. \tag{4.2}$$

N.B. $\nabla\tilde{p}_{tot}$ vanishes as we are considering incompressible fluid. Then we have

$$\frac{\partial}{\partial t}\left(\widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{b}}\right) = -\widetilde{\boldsymbol{b}} \times (\boldsymbol{B_0} \cdot \nabla)\widetilde{\boldsymbol{b}} + \widetilde{\boldsymbol{u}} \times (\boldsymbol{B_0} \cdot \nabla)\widetilde{\boldsymbol{u}}. \tag{4.3}$$

In Eq. (4.3), induction equation is used to deal with $\partial_t\widetilde{\boldsymbol{b}}$. Then integrating Eq. (4.3) over t and taking the ensemble average, we get

$$\boldsymbol{\mathfrak{E}} = -\overline{\int \widetilde{\boldsymbol{b}} \times (\boldsymbol{B_0} \cdot \nabla)\widetilde{\boldsymbol{b}}\, dt} + \overline{\int \widetilde{\boldsymbol{u}} \times (\boldsymbol{B_0} \cdot \nabla)\widetilde{\boldsymbol{u}} dt}\,. \tag{4.4}$$

In isotropic cases, Eq. (4.4) reduces to

$$\boldsymbol{\mathfrak{E}} = -\frac{1}{3}\tau_{cor}\left(\overline{\widetilde{\boldsymbol{u}} \cdot \nabla \times \widetilde{\boldsymbol{u}}} - \overline{\widetilde{\boldsymbol{b}} \cdot \nabla \times \widetilde{\boldsymbol{b}}}\right)\boldsymbol{B_0}. \tag{4.5}$$

Therefore, in this condition

$$\alpha = -\frac{1}{3}\tau_{cor}\left(\overline{\widetilde{\boldsymbol{u}} \cdot \nabla \times \widetilde{\boldsymbol{u}}} - \overline{\widetilde{\boldsymbol{b}} \cdot \nabla \times \widetilde{\boldsymbol{b}}}\right) = \alpha_u + \alpha_b. \tag{4.6}$$

Compare Eq. (4.6) with Eq. (3.17), there is an extra term, which indicates α-effect can be reduced by correlation $\overline{\widetilde{\boldsymbol{b}} \cdot \nabla \times \widetilde{\boldsymbol{b}}}$.

For resistive MHD, the vector potential $\widetilde{\boldsymbol{a}}$ is governed by

$$\frac{\partial}{\partial t}\widetilde{\boldsymbol{a}} = \widetilde{\boldsymbol{u}} \times \boldsymbol{B_0} + \widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{b}} - \overline{\widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{b}}} - \eta\nabla \times \widetilde{\boldsymbol{b}}. \tag{4.7}$$

Multiplying Eq. (4.7) by $\widetilde{\boldsymbol{b}}$, and taking the statistical average, in Coulomb gauge, we get

$$\frac{\partial}{\partial t}\overline{\widetilde{\boldsymbol{a}} \cdot \widetilde{\boldsymbol{b}}} = 2\overline{(\widetilde{\boldsymbol{u}} \times \boldsymbol{B_0}) \cdot \widetilde{\boldsymbol{b}}} - 2\eta k_f^2\overline{\widetilde{\boldsymbol{a}} \cdot \widetilde{\boldsymbol{b}}}, \tag{4.8}$$

where $\overline{\widetilde{\boldsymbol{a}} \cdot \widetilde{\boldsymbol{b}}}$ is magnetic helicity, and k_f^2 is the characteristic wave vector of $\widetilde{\boldsymbol{a}}$. If the system starts with a state at which $\alpha_b = 0$, $\alpha_0 = \alpha_u > 0$, as $\overline{\nabla \times \widetilde{\boldsymbol{b}} \cdot \widetilde{\boldsymbol{b}}} \approx k_f^2\overline{\widetilde{\boldsymbol{a}} \cdot \widetilde{\boldsymbol{b}}}$, clearly a negative α_b will be generated, which effectively reduce the α-effect.

In stationary state, Eq. (4.8) gives us

$$-\boldsymbol{B_0} \cdot \overline{\widetilde{\boldsymbol{u}} \times \widetilde{\boldsymbol{b}}} = \eta\, \overline{\widetilde{\boldsymbol{b}} \cdot \nabla \times \widetilde{\boldsymbol{b}}}. \tag{4.9}$$

Plugging Eq. (3.10) into Eq. (4.9), we get

$$\overline{\widetilde{\boldsymbol{b}} \cdot \nabla \times \widetilde{\boldsymbol{b}}} = -\frac{\alpha}{\eta}B_0^2 + \frac{\beta}{\eta}\boldsymbol{B_0} \cdot \nabla \times \boldsymbol{B_0}. \tag{4.10}$$

Since $\overline{\tilde{\boldsymbol{b}} \cdot \nabla \times \tilde{\boldsymbol{b}}} \propto \alpha - \alpha_0$, we conclude that

$$\alpha = \frac{(\alpha_0 + \beta_0 \boldsymbol{R} \cdot \nabla \times \boldsymbol{R})}{1 + R^2}, \tag{4.11}$$

where $\boldsymbol{R}$ is just the large-scale magnetic field $\boldsymbol{B_0}$ renormalized by a factor $(\rho V^2/R_m)^{1/2}$.

5. Conclusions

In this paper, I first review the development of dynamo theory, which originates from people's interest in the solar magnetic field. One of greatest breakthrough in this area mean-field dynamo theory. In Sec. 2, I formulate the kinematic dynamo theory in a way similar to Braginskii's idea, i.e., dividing the fields into an axisymmetric part and a asymmetric part. Then I introduce two ordinary classifications of dynamo theories, and necessary conditions for the construction of dynamos. In Sec. 3, the derivations and closure of the mean-field dynamo theory are shown in great detail. The relation between the α-effect and the turbulence helicity is also discussed. A nontrivial α-effect requires a non-vanishing turbulence helicity. In Sec. 4, a simple α-qunching mechanism is derived by including the momentum equation of conducting fluids.

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