

MHD Turbulence - An Introduction

See : S. Galtier (best)
Kulsrud
+ Literature

N.B. : MHD Turbulence is large, evolving subject.

Contents

- i.) Key features of K41
- ii.) Basic facts MHD Turbulence
- iii.) Kraichnan - Iroshnikov Phenomenology
 Cascade - as - Random Walk of Wave Packet
 [Weak Turbulence] - short wave packet transit time thru scale
- iv.) Goldreich - Sridhar Phenomenology
 (I-K with anisotropy)
 [Anisotropy strengthened by cascade,
 $k_{\parallel} \sim 1$]
- v.) 4/5 Law, again.

c.) Key Features of k_4

- self similarity and energy flux in k
i.e. nonlinear transfer

- dissipates energy; dissipation rate

indep. Re , as $Re \rightarrow \infty$, but $\nu \neq 0$

$$\langle \tilde{F} \tilde{U} \rangle = G = \nu \langle (\nabla u)^2 \rangle \approx \nu \langle \omega^2 \rangle$$

$$l \sim 1/\sqrt{\nu} \rightarrow \text{singularity}$$

- inertial range

$$l_d < l < l_0$$

rate $\rightarrow$ Galilean
invariance
 $\downarrow$

$$G = \frac{u(e)^2}{l(e)} = \frac{u(e)^3}{l} = [u(e)]^2 \frac{u(e)}{l}$$

$$= \frac{1}{l} [u(e)]^2$$

- energy balance equation.

- time rate of change energy $\rightarrow$ triads

$$E(e) \sim G^{2/3} l^{2/3} \quad E(k) \sim G^{2/3} k^{-5/3}$$

$$l_d \text{ from } \nu/l^2 \sim \frac{u(e)}{l} \sim \frac{G^{1/3}}{l^{2/3}} \quad \text{turbulent "flow is" rough}$$

- alternatively



$$\frac{dl}{dt} = v(dl)$$

$$dl^2 \sim \epsilon^2 t^3$$

→ Super-diffusive separation

- Some Things you Can Trust:

a.) Kármán - Howarth Egn. (Modulo useful)

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \nabla_i \cdot \left\langle \overbrace{u(x+l) - u(x)}^{u(x+l) - u(x)} \partial_i u \right\rangle$$

$$+ 2\nu \partial_{l_n, l_n}^2 \left\langle \frac{u_i u_i'}{2} \right\rangle$$

$$u_i = u(x)$$

$$u_i' = u(x')$$

$$x' = x + l$$

with external forcing:

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \nabla_i \cdot \left\langle (\partial_i u)^2 \right\rangle$$

$$+ 2\nu \partial_{l_n, l_n}^2 \left\langle \frac{u_i u_i'}{2} \right\rangle + \overline{F}$$

⇒ scale energetics balance

b.) 4/5 Law

Balancing external stirring with energy dissipation (must for steady state):

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{\tau}{4} \nabla_i \cdot \langle (u_i u_i')^2 \partial_i \rangle + 2 \tau \partial_{ln} \partial_{ln} \left\langle \frac{u_i u_i'}{2} \right\rangle + \epsilon$$

cr.

in inertial range:

$$\langle (u_i u_i')^2 \rangle = - \frac{4}{5} \epsilon l \quad \rightarrow \text{decay}$$

i.e. energy transfer at scale l (arbitrary in inertial range)

$$\boxed{\langle (u_i u_i')^2 \rangle = - \frac{4}{5} \epsilon l}$$

proportional (with 4/5 factor) to dissipation rate

k.) MHD (3D, Incompressible)

① - ^{often} wave strong, slowly varying external magnetic field

$$\rho (\partial_t \underline{v} + \underline{v} \cdot \nabla \underline{v}) = -\nabla p^* + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi} + \nu \nabla^2 \underline{v} + \tilde{f}$$

$$\partial_t \underline{B} + \underline{v} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{v} + \eta \nabla^2 \underline{B}$$

→ general
→ need not B_0

$$\nabla \cdot \underline{B} = 0$$

Now here can (for B_0) use reduced MHD: (Stenflo 76)

$$\begin{cases} \nabla \cdot \underline{v} = 0 \\ \nabla \cdot \underline{v} + \nabla_{\parallel} v_{\parallel} = 0 \end{cases} \stackrel{\text{en}}{=}$$

$$\rho (\partial_t \nabla_{\perp}^2 \phi + \underline{v}_{\perp} \cdot \nabla_{\perp} \nabla_{\perp}^2 \phi) = B_0 \partial_z \nabla^2 A + \nabla_{\perp} A \times \underline{B} \cdot \nabla \nabla^2 A + \tilde{f} + \nu \nabla^2 \nabla^2 \phi$$

$\perp \rightarrow$ emphasize

$$\partial_t A + \underline{v}_{\perp} \cdot \nabla A = B_0 \partial_z \phi + \eta \nabla^2 A \quad \begin{cases} E_{\parallel} = -\nabla_{\parallel} \phi \\ -\partial_t v_{\parallel} - \nabla_{\parallel} \phi = -\eta v_{\parallel} \end{cases}$$

→ anisotropic turbulence, if B_0 strong. (2D + SAW in z)

→ wave component i.e. MHD turbulence is just of waves (Alfvenic) *

⇒ exact use methodology of wave turbulence

B_0 strong → wave turbulence.

trid coherence

- ① $\leadsto$ nonlinear transfer in $\perp$, i.e.
 $k_{\perp}$ (3D turbulence has coupling etc)
 $\leadsto$ what controls trid coherence?
 $\Rightarrow$ Alfvénic transfer (over scale) =

From HW:

- ② Nonlinear interaction exclusively via counter-propagating Elsasser populations

i.e. $\underline{Z}^{\pm} = \underline{v} \pm \underline{b}$ $\underline{b} \equiv \text{part. } \tilde{B}$
 2 MHD eqns $\Rightarrow$

$$\partial_t \underline{Z}^{\pm} \mp \underline{b}_0 \cdot \nabla \underline{Z}^{\pm} + \underbrace{\underline{Z}^{\mp} \cdot \nabla \underline{Z}^{\pm}}_{\text{interaction}} = -\nabla p^* + \nu_{\pm} \nabla^2 \underline{Z}^{\pm} + \nu \nabla^2 \underline{Z}^{\pm}$$

$$\nu_{\pm} = \frac{\nu \pm \eta}{2}$$

so NL coupling exclusively via $\underline{Z}^{\mp} \cdot \nabla \underline{Z}^{\pm}$

so NL transfer

$$\underline{u}^+ \xrightarrow{l} \underline{u}^-$$

$$T_{\text{int}} \sim l/v_A \Rightarrow \text{coherence time}$$

i.e. counter-propagating packets' interaction time limited to l/v_A .

$v_A \gg v_{\text{visc}}$

i.e. saves us from renormalization in absence of dispersion.

On reduced MHD:

$$RMHD =$$

2D MHD:

$$\partial_t \nabla_{\perp}^2 \phi + \nabla_{\perp} \phi \times \hat{z} \cdot \nabla_{\perp} \nabla_{\perp}^2 \phi = \nabla_{\perp} A \times \hat{z} \cdot \nabla_{\perp} \nabla_{\perp}^2 A$$

$$\partial_t A + \nabla_{\perp} \phi \times \hat{z} \cdot \nabla_{\perp} A = \mu \nabla_{\perp}^2 A$$

+
Shear AlFven Wave on $\hat{z}$

$$\partial_t \nabla_{\perp}^2 \phi = B_0 \partial_z \nabla_{\perp}^2 A$$

$$\partial_t A = B_0 \partial_z \phi$$

→ Nonlinearities are $\perp$ straining

→ excitations are AlFven wave packets

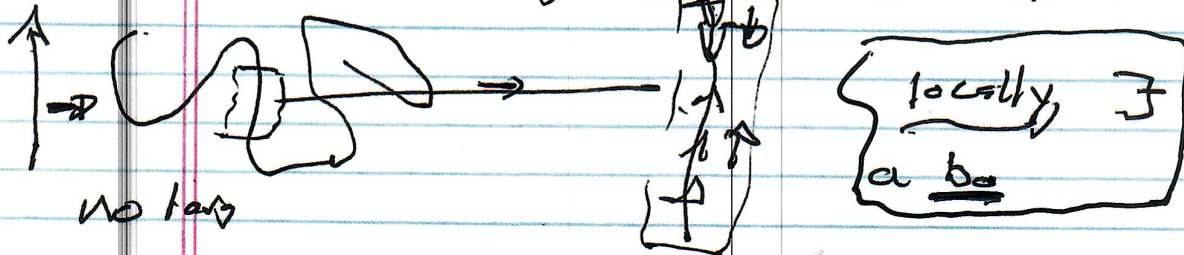
classic

so, Some Phenomenology:

(Iroshnikov, Kraichnan)
(64, 65)

→ nonlinear transfer - (2) isotropic turbulence

For weak mean $\underline{B}$, $b \sim b_{rms} > |\underline{B}|$



→ so, nonlinear scattering from "collisions" of counter-propagating packets



so (modulo 4/5!) - T.B.D.

$$\epsilon \sim \frac{\underline{Z}^2}{T_A l}$$

→ Alfvén wave cascade

→ const. energy throughput

$l \equiv \text{scale}$

universally
waves, =
counter E/S
populations
locality?

Now Alfvén transit time on scale l , is b_0

$$T_A = l / b_0$$

Far transfer:

- transfer by wave scattering $\Rightarrow$ random walk of amplitude/intens
- define τ_{Tr} by:

$$\langle \delta Z^2(\tau_{Tr}) \rangle \sim \langle Z^2 \rangle$$

τ_{Tr} sufficient #
kicks to make

i.e. randomization of scattering in amplitude

$$dZ_{rms} \sim Z$$

or

$$Z_e(t + \tau_A) = Z_e(t) + \tau_A \partial_t Z_e$$

$$\partial_t dZ_e \sim \frac{Z_e^2}{\rho}$$

kick in 1 Alfvén transit time =

$$dZ_e \approx \tau_A \frac{Z_e^2}{\rho} \rightarrow \text{kick}$$

in τ_A

uncor.
d.f.f.a.

$$\partial_t X \sim F$$

$$\langle X^2 \rangle \sim \frac{F^2}{\tau_A} t$$

accumulated kicks

$$D t$$

$\Rightarrow$

$$\langle \delta Z_e^2 \rangle = \left(\tau_A \frac{Z_e^2}{\rho} \right)^2 t$$

$$= \left(\frac{Z_e^2}{\rho} \right) \tau_A t$$

thus:

$$\langle \delta Z_e^2 \rangle \sim \langle Z_e^2 \rangle$$

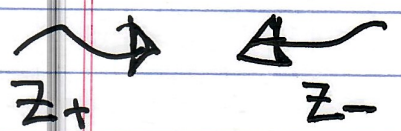
$$\Rightarrow Z_e^2 \approx \tau_A^2 \frac{(Z_e^2)^2}{\rho^2} \frac{\tau_{Tr}}{\tau_A}$$

defines τ_{Tr}

$$\frac{1}{\tau_{Tr}} \approx \frac{Z_e^2}{\rho^2} \tau_A$$

determiner
transfer rate

→ Cascade as Random Walk


 , Average time $\tau_A \sim l/b_0$

$\epsilon \sim \frac{Z^2}{\tau_{tr}}$
 $\tau_{tr} \rightarrow$ transfer time {many kicks here, as opposed to streaming}

- Now,
- interaction produces δZ
 - $\langle (\delta Z)^2 \rangle \sim \langle Z^2 \rangle$ defines transfer time
 - akin $\langle (\delta \theta)^2 \rangle \sim \langle \theta^2 \rangle \Rightarrow$ deflection.

How determine $\langle \delta Z^2 \rangle$

Recall, from Eikonal equations:

$\partial_t \delta Z_{\pm} \sim Z_{\pm} \cdot \underbrace{\nabla}_{\substack{\text{random scattering} \\ \text{event}}} Z_{\mp} \sim \frac{Z^2}{l}$

Then treat as random walk

$$m \frac{dx}{dt} + \beta v = \tilde{F}$$

$$\frac{dx}{dt} = \frac{\tilde{F}}{\beta}$$

$$dx = \int \tilde{F} / \beta dt$$

$$\langle dx^2 \rangle \approx 2 \frac{|\tilde{F}|^2}{\beta^2} \tau_{ac} t$$

$\downarrow$
 random force autocorrelation time

Then, $\downarrow$
coherence time interaction

$$\langle dx^2 \rangle \sim \tau_A \left[\left(\frac{Z(e)^2}{\ell} \right) \right]^2 t$$

$$\langle dx^2 \rangle \sim \langle Z^2 \rangle \sim \tau_A \left(\frac{Z(e)^2}{\ell} \right)^2 \tau_{Tr.}$$

$$1/\tau_{Tr} \approx \tau_A \frac{Z(e)^2}{\ell^2}$$

$$1/T_{Tr} \approx \frac{\rho}{b_0} \frac{Z(e)^2}{l^2} \approx \frac{Z(e)^2}{b_0 b}$$

$$1/T_{Tr} \sim \frac{Z(e)^2}{l^2} \tau_A \sim \frac{Z(e)}{\rho} \underbrace{\tau_A}_{< 1} \frac{Z(e)}{b}$$

$\ll 1$

< 1

reduced transfer rate,
due short τ_A .

Obvious Analogy

Brownian Motion

$\vec{F} \rightarrow$ thermal fluct

$\tau_{AC} \rightarrow$ autocorrelation
time random force

$$\langle \delta x^2 \rangle \sim |\vec{F}|^2 \tau_{AC} t$$

$$\sim D t$$

diffusion time

Alfven Wave Cascade

$$\underline{Z}_+ \cdot \underline{Z}_- \sim \frac{Z^2}{\rho}$$

scattering

$\tau_A \rightarrow$ Alfven packet
passage time

$$\langle \delta z^2 \rangle \sim \left(\frac{Z(e)^2}{\rho} \right) \tau_A t$$

$$\tau_{Tr} \text{ from } \langle \delta z^2 \rangle \sim \langle z^2 \rangle$$

$$\frac{1}{\gamma_{Tr}} \approx \frac{Z_e^2}{l^2} \frac{l}{b_0}$$

$$\approx \frac{Z_e^2}{l b_0}$$

Thus:

$$\epsilon \approx \frac{Z_e^2}{\gamma_{Tr}} \approx \frac{Z_e^2 Z_e^2}{l b_0}$$

const. rate $k = k_{max}$

$$Z_e^3 \sim \sqrt{b_0 \epsilon} l^{1/2}$$

$$Z_e \sim (b_0 \epsilon)^{1/4} l^{1/4}$$

$$E(k) \approx \sqrt{b_0} k^{-3/2}$$

$$\rightarrow \text{Kraichnan spectrum}$$

N.B. — as work with Z
 $E_M \sim E_k$

(equal Elsasser regulations)

$$P_M \sim 1$$

$$\frac{\gamma}{l^2} \approx \frac{\gamma}{\gamma_{Tr}}$$

$\Rightarrow$ dissipation scale altered.

$$1/\gamma_{Tr} \approx \frac{Z_e^2}{l^2} \frac{l}{b_0} \approx \frac{\gamma_A}{\gamma_{Tedy}} \frac{1}{\gamma_{Tedy}}$$

Reduction factor

compute $k41$: $1/T_{Tr} \sim \frac{1}{T_{ddy}}$

Transfer rate reduced by factor T_A/T_{ddy} !

Also, recall for weak wave turbulence:

$$CONS = \sum_{k'} |V|^2 N_{k'} N_k \tilde{T}_C \rightarrow \text{general structure of collision integral for weak interaction}$$

(tried) coherence time $\rightarrow$ interaction

~~so~~ $\frac{1}{T_{Tr}} \approx \sum_{k'} |V|^2 N_{k'} \tilde{T}_C \rightarrow \text{scattering rate}$

but $N_k \rightarrow \frac{2\pi^2}{\ell^2}$ (intensity) (assumes local transfer)

$$|V|^2 \sim |\nabla^2| \sim \frac{1}{\ell^2} \rightarrow \frac{1}{\ell^2}$$

$$\tilde{T}_C \sim T_A \rightarrow \text{tried coherence}$$

$\downarrow$
Alfvenic packet transit time

$$1/T_{Tr} \sim \frac{2\pi^2}{\ell^2} T_A \Leftrightarrow \text{W.T.T.}$$

I-K is fundamentally wave scattering process!

Obvious analogy:

Transition rate $\rightarrow$ Fermi Golden Rule

$$\frac{1}{T} \sum_F |\langle F|V|i\rangle|^2 \delta(E_F - E_i - \hbar\omega_{fi}) \rho(F)$$

$\downarrow$ rate out of i $\downarrow$ coupling / Matrix element $\downarrow$ selection rule $\downarrow$ occupation ρ

for waves: $\delta(E_F - E_i - \hbar\omega_{fi}) \rightarrow \delta(\omega_f - \omega_i - \omega_{fi})$

$\sim T \omega$

$$|\langle F|V|i\rangle|^2 \sim \left(\frac{Z(e)}{r}\right)^2$$

$$\rho_i \sim \text{~~density~~ } \#$$

$$\Gamma \sim \frac{Z(e)^2}{T \omega} \quad \text{is rate balance}$$

→ 'triad' → 2 AWs + cell

$$\underline{z}_1 = \underline{k}_1 + \underline{k}_1'$$

$$\omega_2 = \omega_1 + \omega_1'$$

→ akin NLD

$$\phi \sim k_{\perp} V_A - k_{\perp} V_A$$

→ Alfvén Waves non-dispersive, but coherence time controlled by packet propagation.

⇒ prevents negative spectra etc.
(coherence time not large).

→ Strong Bo. (explicit)

Some simple observations:

→ turbulence clearly anisotropic

→ nonlinear transfer in k_z .

So consider: $k_z \rightarrow 1/l_z$
 $k_{||}$

$k_{\perp} > k_{||}$ so

$$E \sim \frac{Z(l_z)^2 Z(l_z)^2}{l_z^2 (k_{||} V_A)}$$

$|A(k_{||})| V_A \rightarrow$
coherence
time

fairly $|A(k_{||})| V_A \sim |k_{||} V_A|$

i.e. $1/\tau \sim \frac{V_A}{l_{||}}$

So

$$Z(l_z)^2 \approx (E |k_{||} V_A|)^{1/2} l_z$$

⇒

$$E(k_{\perp}) \approx (E |k_{||} V_A|)^{1/2} / k_{\perp}^2$$

so, accounting for Density of states in $k_{||}$:

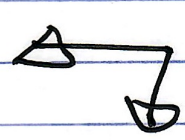
$$E(k_{\perp}, k_{||}) \sim [eV_A]^{1/2} / k_{||}^{1/2} k_{\perp}^2$$

- sharp! Weak wave turbulence spectrum
- why: interaction restricted!
- strongly anisotropic
- $k_{||} \sim \text{frozen}$.

But — is this consistent?

$$Z(l_{\perp}) \sim \delta B(l_{\perp}) \sim (e |k_{||} V_A|)^{1/4} l_{\perp}^{1/2}$$

Now, $\omega \sim k_{||} V_A$

$$\nabla_{||} \approx \partial_z + \frac{\delta B_z}{B_0} \cdot \nabla_{\perp}$$


Magnetic nonlinearities
 ↓
 AW scattering

$$\begin{cases} \nabla_{||} \phi \\ \nabla_{||} T_{||} \end{cases}$$

So natural to define

$$k_{\text{ubo}} \# \sim \frac{\delta B_{\perp} \cdot \sigma_{\perp}}{\underline{B}_0} / \sigma_z$$

$$\sim \frac{NL}{L}$$

$$\sim \frac{l_{\text{ec}}}{\Delta_{\perp}} \frac{\delta B_{\perp}}{\underline{B}_0} \rightarrow \frac{1}{|k_{\text{u}}| l_{\perp}} \frac{\delta B_{\perp}}{\underline{B}_0}$$

Analogy: $k_{\text{u}} \sim \frac{\nabla T_e}{\Delta_{\perp}}$

$k_{\text{u}} \rightarrow$ # kicks in correlation length

- $\rightarrow k_{\text{u}} < 1 \rightarrow$ diffusive
- $\rightarrow k_{\text{u}} \geq 1 \rightarrow$ coherent.

Now, $\frac{\delta B_{\perp}}{l_{\perp}} \sim (\leq k_{\text{u}} V_A)^{1/4} / l_{\perp}^{1/2}$

so

$$k_{\text{u}} \sim \frac{V_A^{1/4}}{|k_{\text{u}}|^{3/4}} / l_{\perp}^{1/2}$$

→ Now observe:

$$k_u \sim \frac{E^{1/4} v_A^{1/4}}{|k_{||}|^{3/4}} \frac{1}{l_\perp^{1/2}}$$

k_u increases as $R_\perp$ decreases

→ WTT, expect different picture → need

$$k_u < 1.$$

⇒ How high can k_u go and still retain physics of Alfvén wave cascade?

⇒ Critical Balance Conjecture

– Goldreich-Sridhar (1995)
(cf Kadomtsev-Pogutse 1978)

In MHD inertial range in strong field, turbulence sits at $k_u \sim 1$.
state of critical balance

i.e. $\sigma \underline{B}_\perp \cdot \underline{D}_\perp \sim B_0 D_\parallel$

$$\frac{Z(l_\perp)}{l_\perp} \sim k_{\perp} v_A$$

transit time sets
bound on
interaction strength.

$$\frac{\tau_A}{\tau_{tr}} \rightarrow 1 \quad \text{but not } \gg 1.$$

$$\tau_{ac} \sim |k_{\perp} v_A|^{-1}$$

$$\rightarrow \left[(k_{\perp} v_A)^2 + \left(\frac{Z(l_\perp)}{l_\perp} \right)^2 \right]^{-1/2}$$

$$\epsilon \sim \frac{Z(l_\perp)^3}{l_\perp^2} \frac{Z(l_\perp)^2}{|k_{\perp} v_A|}$$

$$\rightarrow \frac{Z(l_\perp)^3}{l_\perp^2} \frac{Z(l_\perp)^2}{\left[(k_{\perp} v_A)^2 + \frac{Z(l_\perp)^2}{l_\perp^2} \right]^{1/2}}$$

$$\sim \frac{Z(l_\perp)^2}{l_\perp^2} \frac{Z(l_\perp)^3}{\frac{Z(l_\perp)}{l_\perp} \left[(k_{\perp} v_A)^2 \frac{l_\perp^2}{Z(l_\perp)^2} + 1 \right]^{1/2}}$$

$$k_{\perp} \sim 1 \Rightarrow$$

$$\epsilon \sim \frac{Z(l_{\perp})^2}{l_{\perp}^2} \frac{Z(l_{\perp})^2}{Z(l_{\perp})/l_{\perp}}$$

$$\sim \frac{Z(l_{\perp})^3}{l_{\perp}}$$

→ he is back!

$$Z(l_{\perp}) \sim (\epsilon l_{\perp})^{1/3}$$

$$E(k_{\perp}, k_{\parallel}) \sim \epsilon^{2/3} k_{\perp}^{-5/3}$$

GJ
Goldreich -
Sridhar spectrum

→ like $k_{\perp}^{-1}$
but different
Physics.

→ fits data
(simulation)

and → pronounced anisotropy

$$k_{\parallel} \sim 1, \quad B \cdot \nabla \sim \underline{B}_{\perp} \cdot \nabla_{\perp}$$

$$B \cdot k_{\parallel} \sim \epsilon^{1/3} \frac{l_{\perp}^{1/3}}{l_{\perp}}$$

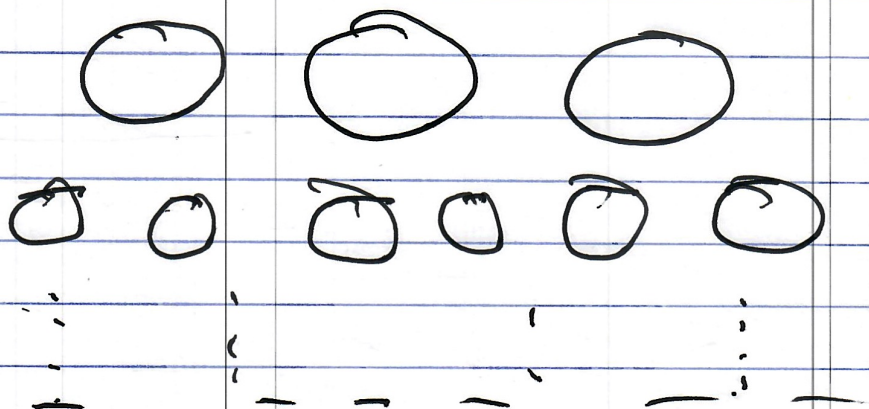
$$k_{\parallel} \sim \frac{\epsilon^{1/3}}{B} k_{\perp}^{2/3}$$

→ specifies
 $k_{\perp}$ vs $k_{\parallel}$
"relation."

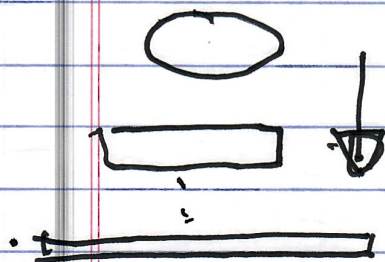
→ Goldreich - Sridhar
"cone" in k -space.

→ cascade is anisotropic,

ie 4/5



G-S 45:



anisotropy more pronounced as progress thru inertial range

→ Ongoing story
Magnetic helicity ?

→ NB: analogue of 4/5 Law
(Pouquet, Politano)

$$-4/3 \epsilon^{\pm} l = \langle (\delta z_l^{\pm} \cdot \delta z_p^{\mp}) \delta z_e^{\mp} \rangle$$

(need 2 streams for dissipation!)

or

$$-4/3 \epsilon^T l = \langle (\delta u_p \cdot \delta u_p + \delta b_p \cdot \delta b_p) \delta u_e \rangle$$

energy
injection

$$-2 \langle (\delta u_e \cdot \delta b_p) \delta b_p \rangle$$