Lecture 12b – Notions of Selective Decay

ii.) 4/5 Law – See Lecture I.

iii.) Cascades and Relaxation

$\Rightarrow$ Selective Decay

Recall: { Taylor Relaxation { 3D
2D – "Taylor in Flatland"

Argued: $\int d^3x\, B^2/8\pi$ minimized
subject to constraint of
$\int d^3x\, A\cdot B$ conserved.

$\Rightarrow$ $J_\parallel = J\cdot B/B^2 \to$ const.

(2D $J/A \to$ const.).

Arguements heuristic. [Power counting (k)
stoch. fields
⋮

Now, – dissipation at small scale
η, ν

– expect energy transfer to small scale

- Inverse {cascade / transfer} of magnetic helicity would set up "selective Decay" scenario

i.e. magnetic energy scattered to small scale and dissipated
⇒ relaxation

magnetic helicity inverse cascades
⇒ avoids dissipation. Constraint; as survives.

c.f. {Frisch (75), Pouguet, et. al. (76) (posted)
see also: Montgomery}

- Why, where from?

→ Primarily: Statistical Mechanics

→ c.f.: Frisch 75, though not transparent.

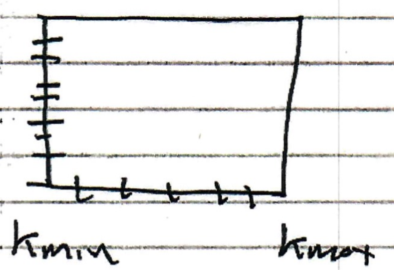
easier → "Taylor in Flatland" Problem.

Recall: Relaxation $\begin{cases} \text{minimizes } \langle B^2 \rangle \\ \text{conserving } \langle A^2 \rangle \end{cases}$.

Does this follow from Selective Decay?

⇒ Explore Absolute Equilibrium.

i.e.



finite box

- remove forcing, dissipation etc.
- input excitation.

For 2D MHD (ignoring cross helicity):

have $\quad A \rightarrow X_i$

↳ Modal amplitude

⇒

$$E_M = \sum_{i=1}^{N} k_i^2 X_i^2$$

$$H = \sum_{i=1}^{N} X_i^2 \qquad - \langle A^2 \rangle$$

$$\phi \to y_i$$

$$E_k = \sum_{i=1}^{N} k_i^2 y_i^2 .$$

Now, $H \to \alpha$

$E = E_{pot} + E_k \to \beta$ } conserved

conserved, so PdF of this closed system is given by micro-canonical ensemble/distribution:

$$P(x,y) = \underset{\substack{\downarrow \\ \text{norm}}}{c} \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) x_i^2 + \beta k_i^2 y_i^2\right]$$

and can integrate out y_i (KE) part, so:

$$\boxed{P(x) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) x_i^2\right]}$$

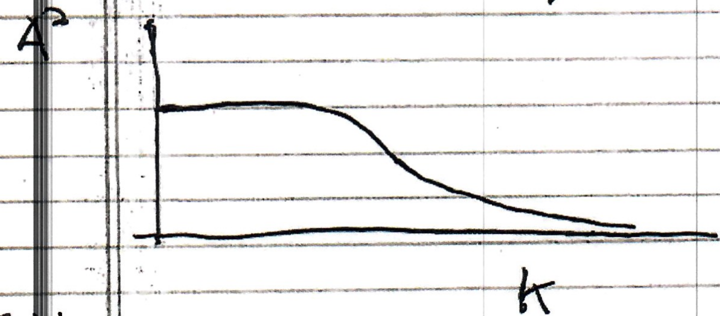
then:

$$\langle A^2(k) \rangle = \int dx_i \, x_i^2 \, P(x_i)$$

$$= 1/(\alpha + \beta k^2)$$

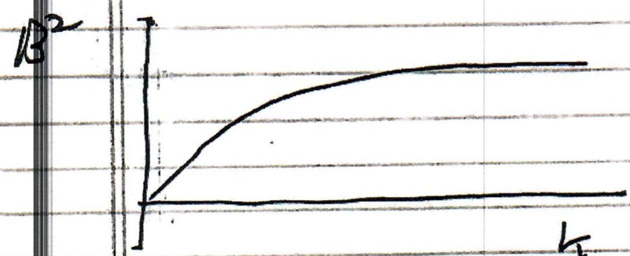
$$\langle B^2(k) \rangle = \left[k^2 / (\alpha + \beta k^2)\right]$$

⊗ observe immediately:



"A^2 wants remain at large scale"

$k_{min} \to 0$



"B^2 approaches equipartition"

$\Rightarrow$ A^2 distribution most populated at large scales. Decays at small

$\Rightarrow$ B^2 distribution most populated at smaller. Approaches equipartition at small scale.

$\therefore$ - suggests A^2 populates large scales, B^2 approaches equipartition.

- suggestive of inverse cascade of A^2, along with forward cascade of energy.

— supports Selective Decay Hypothesis as foundation for "Taylor in Flatland".

— similar story for Magnetic Helicity, though more laborious.

N.B. For 2D Fluid:

$$E = \int d^2x \, (\nabla \phi)^2 \quad \text{— energy}$$

$$\Omega = \int d^2x \, (\nabla^2 \phi)^2 \quad \text{— enstrophy}$$

$$\Omega_i = k_i^2 E_i$$

$$\underline{v} \rightarrow \underline{X}_i$$

$$P(X) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2 \right]$$

so

$$E(k) = \langle v^2(k) \rangle = 1/(\alpha + \beta k^2)$$

$$\Omega(k) = k^2/(\alpha + \beta k^2)$$

similar suggestion of dual cascade and minimum enstrophy state.

→ Is this story true?

⇒ What does dynamics tell us?
∴ Consider interactions in 2D MHD.

Observe:

– Reduced MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = B_0 \partial_z \phi + \eta \nabla_\perp^2 \psi$$

– 2D MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = \eta \nabla_\perp^2 \psi$$

so; with strong $\underline{B}_0$:

$\langle A \cdot B \rangle \rightarrow \langle \psi \rangle \underline{B}_0$

so mean $\langle \psi \rangle$ in 2D ⊖ captures magnetic helicity dynamics in strongly magnetized system.

For $\langle A^2 \rangle_k$ transfer, consider closure of $\partial_t \langle A^2 \rangle$ equation, much akin to wave kinetics, though closure required.

See: Diamond, Hughes, Kim (posted).

Can write (see DHK):

$$\frac{1}{2}\left[\partial_t \langle A^2\rangle_{\underline{k}} + T(k)\right] = -\Gamma_A(k)\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\underline{k}}$$

triplet

$$\langle \nabla\cdot\langle \underline{V} A^2\rangle\rangle_{\underline{k}}$$

flux

$$\Gamma_A = \left[\tau_c^{\phi}(k)\langle V^2\rangle_{\underline{k}} - \tau_c^{A}(k)\langle B^2\rangle_{\underline{k}}\right]$$

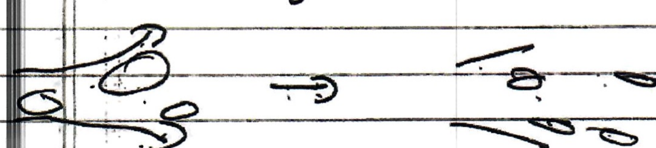
$$T_{\underline{k}} = \sum_{\underline{k}'} (\underline{k}\cdot\underline{k}'\times\hat{z})^2\, \Theta_{k,k',k''} \overset{(1)}{\left\{\langle\phi^2\rangle_{\underline{k}'}\right.} \overset{(2)}{\left. - \frac{(k'^2-k^2)}{|k+\underline{k}'|^2}\langle A^2\rangle_{\underline{k}'}\right\}}\{A^2_{\underline{k}}\}$$

$$\overset{(3)}{-\sum_{\substack{\underline{k}=\underline{p}+\underline{q}\\ \underline{p},\underline{q}}} (\underline{p}\cdot\underline{q}\times\hat{z})^2\, \Theta_{\underline{k},\underline{p},\underline{q}}\langle A^2\rangle_{\underline{p}}\langle\phi^2\rangle_{\underline{q}}}$$

(1), (3) → coherent damping, incoherent emission

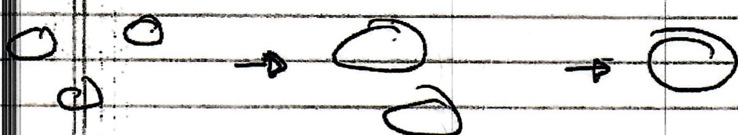
→ akin to scattering of passive scalar, → small scale/chop-up.

→ conserve $\langle\psi^2\rangle$ upon $\sum_k$ together.



(2) → coherent damping/growth - from back reaction ($J \times B$) into Ohm's Law:
→ reshuffle $\langle A^2 \rangle$ to larger scale. sign k'^2 vs k^2 !

→ $\sum_k \langle A^2 \rangle$ conserves independently.



→ correspondence to condensation aka wires (currents) attracting.

(1) + (2) → net effective resistivity sign.
→ see Γ_A, too. - 'negative resistivity'
- Alfvenized state

⇒ $E_K > E_M$ ⇒ $\langle A^2 \rangle_k$ shuffled to smaller scale.

$E_M < E_K$ ⇒ $\langle A^2 \rangle_k$ transfered to larger scale.

<u>and</u> transfer need not be local.

⇒ In dynamics, $\langle A^2\rangle$; $\langle A\cdot B\rangle$ evolution is complex.

⇒ N.B. Recall flux expulsion:

$\frac{V_{A0}^2}{\tilde{V}^2} R_m < 1 \rightarrow$ A passive, B expelled

$> 1 \Rightarrow J\times B$ disrupts vortex, expulsion stops

⇒ $B_0^2 \lessgtr \rho_0\langle\tilde{V}^2\rangle/R_m$

but $\langle\tilde{B}^2\rangle \gg B_0^2$, upon stretching, weak B_0 is sufficient!

Zeldovich:

$$\frac{\partial A}{\partial t} + \underline{v}\cdot\underline{\nabla} A = -v_r \frac{\partial\langle A\rangle}{\partial x} + \eta\nabla^2 A$$

$\times A$ and avg. ⇒

$$\eta\langle\tilde{B}^2\rangle = \langle\tilde{v}_r\tilde{A}\rangle\frac{\partial\langle A\rangle}{\partial x}$$

$$\langle\tilde{B}^2\rangle = \frac{\eta_T}{\eta}B_0^2$$

$$= \frac{\tilde{v}\,\tilde{v}\,\tau_c}{\eta}B_0^2 \cong R_m B_0^2 \quad \checkmark.$$

so, crudely:

$$\langle \tilde{B}^2 \rangle / R_m \lessgtr \langle \tilde{V}^2 \rangle / R_m$$

$\Rightarrow$ Questions still open!

$\therefore$ Taylor conjecture remains a conjecture!