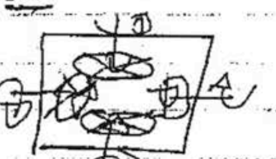
# Turbulence Theory

## — An Introduction

P. Diamond

i.e.

I.) Basics of Fluid Turbulence (3D)



) Characteristics of Fluid Turbulence:

"turbulence" vs "noise" → $\boxed{\text{energy flux}}$

- broad range of spatio-temporal scales excited — contrast TAM

Ref. U. Frisch "Turbulence - The Legacy of A. N. Kolmogorov"

- decay of large scale energy ↔ need input/stirring to maintain stationarity

- energy input dissipated as heat (to maintain stationarity) → viscosity → irreversibility

- irreversible mixing occurs ↔ i.e. passive tracer

- intermittency manifested
  i.e. spatial ↔ coherent structures (i.e. vortices) occur
  temporal ↔ bursts — probe trace

- self-similarity / scale-similarity:

  turbulence "looks the same" on all scales, except the very largest (stirring) and the very smallest (dissipation)
  Caveat: Intermittancy - memory of large scales on small.

iii) Navier Stokes Equation – {Describes Fluid

$$\underbrace{\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}}_{\substack{\text{advection/}\\ \text{straining}\\ \to\ \text{nonlinearity}}} = \underbrace{-\frac{\underline{\nabla} p}{\rho}}_{\text{pressure}} + \underbrace{\nu \nabla^2 \underline{V}}_{\substack{\text{viscous}\\ \text{diffusion of}\\ \text{momentum}}}$$

$\rho = 1$ hereafter

$$\underline{\nabla}\cdot\underline{V} = 0 \qquad \text{incompressibility}$$

Note: Pressure determined from incompressibility

i.e.

$$\nabla\cdot\left[\cancel{\frac{\partial \underline{V}}{\partial t}} + \underline{V}\cdot\underline{\nabla}\underline{V}\right] = -\nabla^2 p + \nu\nabla^2\cancel{(\underline{\nabla}\cdot\underline{V})}$$

$$\nabla^2 p = -\nabla[\underline{V}\cdot\underline{\nabla}\,\underline{V}]$$

$$p = -\nabla^{-2}[\nabla\cdot\underline{V}\cdot\underline{\nabla}\,\underline{V}]$$

$$= \frac{1}{4\pi}\int \frac{d^3x'\,\nabla_{x'}[\underline{V}(x')\cdot\nabla_{x'}\underline{V}(x')]}{|\underline{x}-\underline{x}'|}$$

More generally, can eliminate $p$

$$\partial_t V_i + (\delta_{i\ell} - \partial_{i\ell}\nabla^{-2})\,\partial_j(V_j V_\ell) = \nu\nabla^2 V_i$$

Key Parameter: Reynolds #

$$\boxed{Re = |\underline{V}\cdot\nabla\underline{V}| / |\nu\nabla^2\underline{V}| \sim \frac{V(L)\,L}{\nu}}$$

$\sim \dfrac{\text{nonlinearity}}{\text{collisional diffusion}}$

measure of strength of NL.

- Re usually referenced to largest scale

$L = L_{max}$
$V(L)$ = large scale velocity

- Re always referenced to a particular scale

$L_{max}$, $\lambda = \left[\langle(\partial_i V_j)^2\rangle / \langle V_i^2\rangle\right]^{-1/2}$, $\lambda_{dissp.}$,
(Taylor Scale) $(Re=1)$

- $Re \gg 1$ in turbulent { pipe flow, atmosphere, etc

$Re \sim 10^6 - 10^8$, etc.

i.e. planetary boundary layer : $l_{out} \sim 1\,km \sim 10^5\,cm$
$l_{diss} \sim .1\,cm$

$\Rightarrow$ 6 decades!

- Re : measure of ratio of inertial mixing of momentum to collisional mixing ...

(4)

iii) Experimental 'Laws' of Fully Developed Turbulence

▷ Much / <u>most</u> of turbulence theory <u>is empirically</u> motivated. Experimental info / results preceeded sophisticated theoretical analyses ...

The experimental facts:

1.) <u>2/3 Law</u> (Mundane)

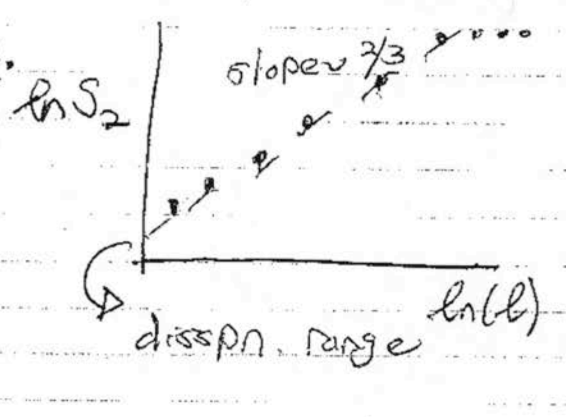
In a turbulent flow with $Re \gg 1$, $\langle \delta V(\ell)^2 \rangle$ (mean square velocity increment between two scales separated by distance $\ell$) scales as $\ell^{2/3}$.

<u>i.e.</u> $\delta V(\ell) = |\underline{V}(\underline{r}+\underline{\ell}) - \underline{V}(\underline{r})|$ ⇒ {a difference!

$S_2(\ell) = \langle \delta V(\ell)^2 \rangle \sim \ell^{2/3}$ → fundamental scaling relation

$S_2$: 2nd order structure function

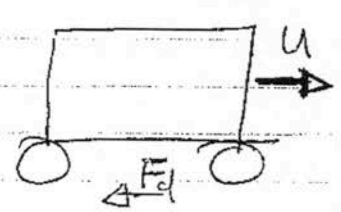
<u>i.e.</u>



B.) Law of Finite Energy Dissipation (Profound)

If, in an experiment on turbulent flow, all the control parameters are kept the same, except the viscosity, which is lowered as much a possible, the energy dissipation per unit mass $dE/dt$ behaves in a way consistent with a finite limit.

- What means 'Energy Dissipation Rate'?



Consider a car, experiencing atmospheric drag

$$F_d = \frac{1}{2} C_D \rho S U^2$$

face surface area

i.e. 

$$P = \rho S (U\tau) U$$

→ momentum in air of slug:

$M = \rho S U \tau$ → mass

$V = U$

if assume air momentum completely transferred to car

$$\frac{dP_{car}}{dt} = F_d = \rho S U^2$$

$\frac{C_D}{2}(Re) \equiv$ drag coefficient (slowly varying function of $Re$, depends on shape, etc.)

$$\therefore \quad F_d = \frac{C_D}{2} \rho S U^2$$

Now, power dissipated by drag force

$$P_d = F_d U$$

$$\Rightarrow \quad P_d = \frac{C_D}{2} \rho S U^3$$

Energy dissipation rate (per volume) $\epsilon = P_d / \text{Mass}$

$$= \frac{C_D}{2} \frac{U^3}{L}$$

$$\text{also } NS \Rightarrow \partial_t \langle v^2 \rangle \sim -\langle \underline{v} \cdot \underline{\nabla}\, \underline{v}^2 \rangle \sim U^3/L$$

$\Rightarrow$ Why should we care?

Note energy budget:

$$\frac{\partial}{\partial t} v_i + v_j \partial_j v_i - \nu \nabla^2 v_i = -\partial_i p$$

$$\partial_t \frac{v_i^2}{2} + \partial_j \frac{v_j v_i^2}{2} - \nu v_i \nabla^2 v_i = -v_i \partial_i p$$

$\langle\ \rangle \equiv$ ensemble (fast space-time avg.)

$$\partial_t \left\langle \frac{v_i^2}{2} \right\rangle + \cancel{\left\langle \partial_j \frac{v_j v_i^2}{2} \right\rangle} - \nu \langle v_i \nabla^2 v_i \rangle = \cancel{\langle \partial_i v_i p \rangle}$$

surface terms $\to 0$; upon IBP

$\Rightarrow$
$$\partial_t \left\langle \frac{v_i^2}{2} \right\rangle = -\nu \langle |\nabla v_i|^2 \rangle$$

but $\epsilon \equiv -\partial_t \left\langle \frac{v_i^2}{2} \right\rangle$ ! (- dissp'n rate)

$\therefore$ $\boxed{\epsilon = \nu \langle |\nabla v_i|^2 \rangle}$ !

$\Rightarrow$ experiments suggest that $\epsilon \to$ finite as $\nu \to 0$ !!! $\leftrightarrow$ remarkable

$\leftrightarrow$ suggests that extremely large $\nabla \underline{V}$ forms as $\nu \to 0$, singular vortex sheets

$\Rightarrow$ singular velocity gradients formed in limit of weak viscosity !!

{Heart of turbulence problem is grappling with singularity (especially its degree) of velocity gradients.}

N.B.: $\begin{cases}\text{Dissipation Law} \\ \text{Singularity formation is at the heart of why turbulence is a "hard" problem.}\end{cases}$

Re: Dissipation Law:

$$\epsilon \sim \frac{u^3}{L} \sim u^2/(L/u)$$

$$\sim \frac{\text{K.E. per Mass}}{\text{Circulation Time}}$$

i.e. $\Rightarrow$ in 1 macro circulation time a finite fraction of (macro) kinetic energy is dissipated by viscosity.

$\Rightarrow$ dissipation time scale is $(L/u)$.

V.) Kolmogorov's Hypotheses and their Predictions / Implications. $\Rightarrow$ 'K41 Theory' of Turbulence

1: { In the limit of $Re \to \infty$, all possible symmetries of the Navier-Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundarys.

lose memory

What means?

- "small scales": $\ell \ll \ell_0$

$\ell_0$: integral scale → characteristic of production.

- symmetries

First, symmetries of Navier-Stokes Eqn.!!

a.) space translations $\underline{r} \to \underline{r} + \underline{\rho}$
(no explicit $\underline{r}$ dep.)

b.) time translation $t \to t + \tau$
(no $t$ dep.)

* c.) Galilean boosts
(no frame dep.)
$$\begin{cases} \underline{r} \to \underline{r} - \underline{u}t \\ \underline{v} \to \underline{v} + \underline{u} \end{cases} \qquad \underline{V} = \underline{u} + \underline{v}(\underline{r} - \underline{u}t, t)$$

i.e. $$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nu \nabla^2 \underline{v}$$

insert ⇒
$$\cancel{-\underline{u} \cdot \nabla \underline{v}} + \cancel{\underline{u} \cdot \nabla \underline{v}} + \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nu \nabla^2 \underline{v}$$

d.) Parity left-right $\underline{x} \to -\underline{x}$, $\underline{v} \to -\underline{v}$
(no preferred direction)

e.) Rotation
(no preferred direction)
$$\begin{cases} \underline{r} \to R\, \underline{r} \\ \underline{v} \to R\, \underline{v} \end{cases}$$

★ e.) Scaling (for $\nu \to 0$) $\Rightarrow$ critical: scale elimination, band passing

$$\underline{r}, \underline{v}, t \rightarrow \lambda \underline{r}, \lambda^h \underline{v}, \lambda^{1-h} t$$

i.e. $\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} = -\underline{\nabla} p$

$\underline{v} \rightarrow \lambda^a \underline{v}$
$t \rightarrow \lambda^b t$

$$\frac{\lambda^a \partial \underline{v}}{\lambda^b \partial t} + \frac{\lambda^{2a}}{\lambda} \underline{v} \cdot \underline{\nabla} \underline{v} = -\underline{\nabla} p$$

$\hookrightarrow$ from $\underline{\nabla} \cdot \underline{v} = 0$

$\therefore \quad \lambda^{2a-1} = \lambda^{a-b}$

$\Rightarrow \lambda^{-(a-1)} = \lambda^b \qquad b = 1-a$

$\therefore$ scalings $\underline{r} \rightarrow \lambda \underline{r}$, $\underline{v} \rightarrow \lambda^h \underline{v}$, $t \rightarrow \lambda^{1-h} t$

Now, turbulence onset $\Rightarrow$ symmetry breaking!

i.e. ① KH : shear breaks $\left\{ \begin{array}{l} \text{translational} \\ \text{rotational} \end{array} \right.$ invariance

②  rigid body boundary flow, etc

③ flushing toilet $\left\{ \begin{array}{l} \text{space} \\ \text{time} \end{array} \right.$

etc.

However, fully developed turbulence tends to
* restore symmetry, except near boundarys,
on small scale. b.c.'s $\leftrightarrow$ boundary conditions

i.e. if $\delta v(\underline{r}, \ell) = \underline{v}(r+\underline{\ell}) - v(\underline{r})$
$\Rightarrow$

$$\delta v(\underline{r}+\underline{\rho}) = \delta v(\underline{r})$$

similarly: isotropy, parity ...
(facilitates scaling approach)

H2 For $Re \to \infty$ turbulence, at small scales and
* away from boundaries, the flow is self-similar at small scales
i.e. possesses a unique scaling exponent $h$ s/t

$$\delta v(\underline{r}, \lambda \underline{\ell}) \to \lambda^h \delta \underline{v}(\underline{r}, \underline{\ell})$$

($\to$ addresses 2/3 Law)

H3 With assumptions similar to H1, the
* turbulent flow has a finite, nonvanishing mean rate of dissipation $\epsilon$ per unit mass.

$Re \to \infty \Rightarrow \nu \to 0$ with $v_0 = v_{rms}$, $\ell_0$ fixed

$$\epsilon = v_0^3 / \ell_0$$

Alternative (not necessary): Kolmogorov's 'Second Universality Assumption': In the limit of infinite Reynolds number, all the small-scale statistical properties are uniquely and universally determined by the scale $\ell$ and the mean energy dissipation rate $\epsilon$.
(First: $-\frac{4}{5}\epsilon\ell = \langle u^3 \rangle$)

i.e. Implications:

$-\langle \delta V(\ell)^2 \rangle = S_2$ ?

$S_2 \sim L^2/T^2$, dimensionally

Now $\epsilon \sim L^2/T^3$

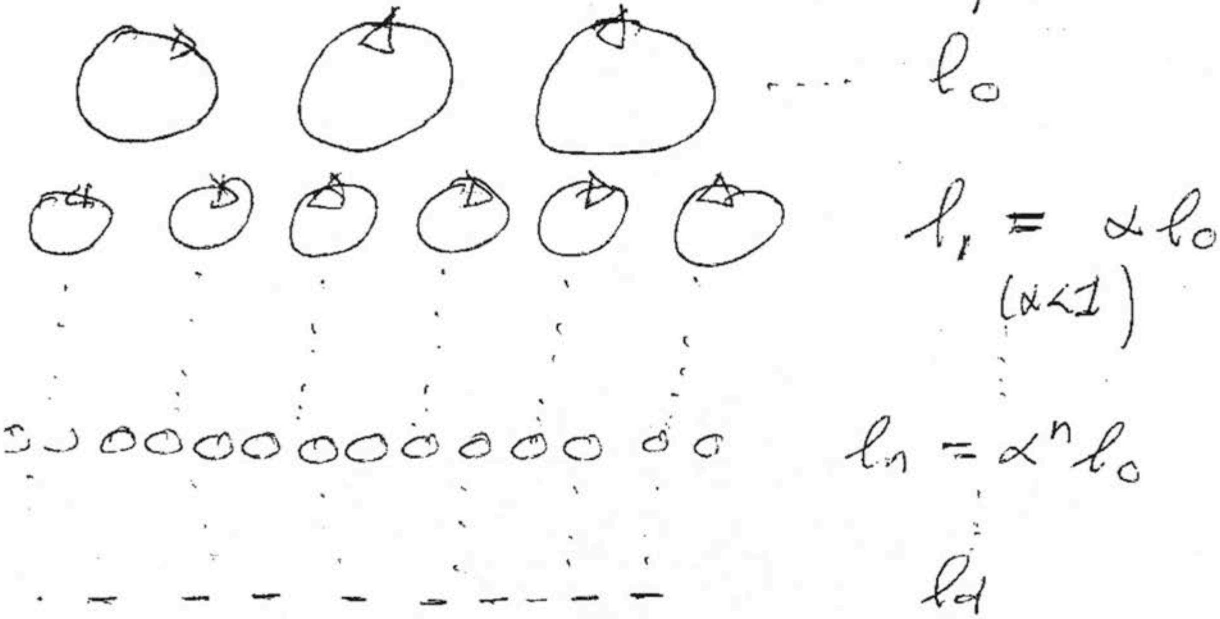
$\Rightarrow \langle \delta V(\ell)^2 \rangle \sim \epsilon^{2/3} \ell^{2/3} \Rightarrow$ recovers 2/3 Law.

also, implies $h = 1/3 \Leftrightarrow$ scaling exponent, etc.

$\underline{H1}$, $\underline{H2}$, $\underline{H3}$ (+2$^{nd}$ Universality Assumption) $\Rightarrow$ K41 phenomenology.

K41 Phenomenology

Picture : (Richardson) Cascade / Eddy Mitosis



Key Idea : ① → Flux of energy in 'scale space',
from $l_0$ (integral scale) to $l_d$,
(dissipation scale)

② → energy flux is self-similar

ⓐ symmetry restoration.

Flux ↔ ⓑ energy dissipation → finite limit as $\nu \to 0$ (i.e. endpoint re-adjustment)

self-similarity ↔ 2/3 Law $\quad S_2 = C\,(l/l_0)^{2/3}$

$l_0 \to \alpha l_0$ etc. $\quad , \; C \to C\alpha^2$

Ingredients in K41 Phenomenology:

→ $\ell$ : scale paramer : eddy scale

→ $v(\ell)$ : $\tilde{v}(\ell) \sim \left( \langle \delta v_{\parallel}(\ell)^2 \rangle \right)^{1/2}$

eddy velocity

$\delta v_{\parallel} \sim \left( \underline{v}(\underline{r}+\underline{\ell}) - v(\underline{r}) \right) \cdot \frac{\underline{\ell}}{\ell}$

≡ longitudinal velocity increment

→ $V_0$ : rms velocity fluctuation (large scale dominates)

$V(\ell_0) \sim V_0$

→ $\tau(\ell)$ : eddy lifetime / turn-over rate

characteristic rate of transfer thru scale $\ell$.

self-similarity : energy thru-put rate is scale invariant

⇒

$$\underset{\text{dissipation rate}}{\epsilon} = \underset{\text{scale } \ell \text{ energy}}{v(\ell)^2} / \underset{\text{scale } \ell \text{ life-time}}{\tau(\ell)}$$

energy balance / thru-put eqn.

, ow, $\tau(\ell)$ ?

compare $\tau(\ell)$ with $\tau_{tw}(\ell)$



$\tau(\ell) \rightarrow$ 'lifetime' of structure of scale $\ell$

$\rightarrow$ i.e. time for structure to be distorted out of existence'

scales $\ell' \gg \ell$ :

$\rightarrow$ advect eddy $\rightarrow$ apply Galilean boost, but dont affect life-time.

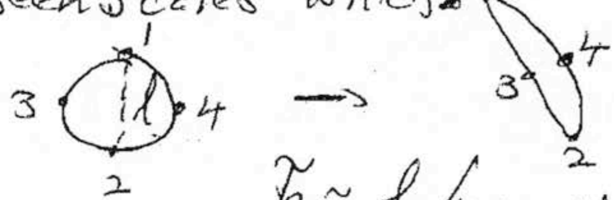
irrelevant/? $\rightarrow$ symmetry under random Galilean transformations

$\rightarrow$ would also violate symmetry restoration.

scales $\ell' \ll \ell$ :

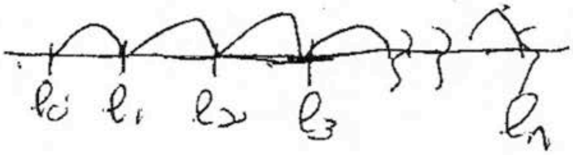
$\rightarrow$ irrelevant as very little energy/shear in such eddys/scales

∴ seek scales which:

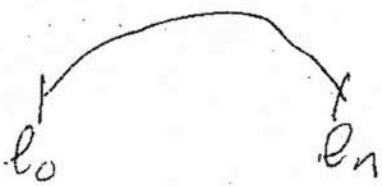



$$\tau_\ell \sim \ell / |v_1 - v_2|$$

$\rightarrow \tau(\ell) \sim \dfrac{\ell}{v(\ell)}$ : cascade local in scale space.




not



$$\Rightarrow \quad \epsilon = \frac{v(\ell)^3}{\ell}$$

$$\Rightarrow v(\ell) \sim (\epsilon \ell)^{1/3} \quad : \text{K41 scaling relation}$$

$$\boxed{v(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3}}$$

– verifies "2/3 Law"

– for spectrum:

if $E(k) = |\tilde{v}(k)|^2$

s/t $E = \int dk \, E(k)$ {i.e. absorbs density of states

then $v(\ell) = \int_{k_{\ell_{n-1}}}^{k_{\ell_n}} dk \, E(k)$

$$v(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3} = \epsilon^{2/3} k_\ell^{-2/3}$$

$$\Rightarrow \boxed{E(k) = \epsilon^{2/3} k^{-5/3}} \quad : \quad \underline{\text{Kolmogorov Spectrum}}$$

at $\ell_0$:

$$v_0 \sim \epsilon^{1/3} \ell_0 \quad \Rightarrow \quad \frac{v_0^3}{\ell_0} = \epsilon.$$

For dissipation scale:

$l_d$ occurs in $l$-space when cascade terminated
<u>i.e.</u> viscosity asserts itself $\rightarrow Re(l) \rightarrow 1$

$$1/\tau(l) \sim 1/\tau_d = \nu/l^2$$

$$\Rightarrow \quad \epsilon^{1/3} l^{-2/3} = \frac{\nu}{l^2}$$

$$l^{4/3} = \nu/\epsilon^{1/3} \quad \Rightarrow \boxed{l_d = \nu^{3/4}/\epsilon^{1/4}}$$

$\boxed{l_d \equiv \eta \text{, in Frisch}}$

Recall: $\quad \epsilon = \nu\langle(\nabla \tilde{v})^2\rangle$

$\Rightarrow \nu \rightarrow 0 \Rightarrow \langle(\nabla \tilde{v})^2\rangle$ divergent

$$\langle(\nabla \tilde{v})^2\rangle = \int_{k_0}^{k_d} dk\, k^2 \epsilon^{2/3} k^{-5/3}$$

$$= \int_{k_0}^{k_d} dk\, k^{1/3} \epsilon^{2/3}$$

$$\simeq k_d^{4/3} \epsilon^{2/3}$$

$$= \frac{\epsilon}{\nu}^{1/3} \epsilon^{2/3} = \epsilon/\nu$$

$\hookrightarrow \langle(\nabla \tilde{v})^2\rangle$ divergent as $\nu \rightarrow 0$.

› Counting Degrees of Freedom

How big is the inertial range:

$$\underset{\#\,\ell's}{\overset{\uparrow}{n}} \sim \frac{\ell_0}{\ell_d} \sim \frac{\ell_0}{(\nu^3/\epsilon)^{1/4}}$$

$$\sim \frac{\ell_0}{\nu^{3/4}} \left(V_0^3/\ell_0\right)^{1/4} \sim \left(\frac{V_0 \ell_0}{\nu}\right)^{3/4}$$

$$\sim R_e^{3/4}$$

∴ number of degrees of freedom for 3D turbulence is:

$$\boxed{N \sim R_e^{9/4}}$$

: would be (minimum) # grid points to resolve range of scales in numerical simulation

Now, i.e. atmospheric boundary layer:

$\ell_0 \sim 1\,Km$

$\ell_d \sim 1\,mm$

$$n \sim 10^6 \Rightarrow \begin{cases} N \sim 10^{18} \\ Re \sim 10^8 - 10^9 \end{cases} !$$

⇒ subgrid scale modelling ...

N.B.: Sometimes able to exploit reduced degrees of freedom models, i.e. when some class of scales slaved to others.

Exercises:

⇒ Consider passive scalar, with concentration $C$:

$$\frac{\partial C}{\partial t} + \underline{V} \cdot \underline{\nabla} C - \mathcal{K} \nabla^2 C = \tilde{f}_c$$

$C$ dissipation rate in K41 turbulence $\equiv \alpha$

i.e. $\alpha = \tilde{C}_0^2 \dfrac{V_0}{l_0}$

⇒ a.) Calculate K41 spectrum for $C$.

b.) Discuss What if $\mathcal{K} \begin{smallmatrix} \ll \\ \gg \end{smallmatrix} \nu$ ?

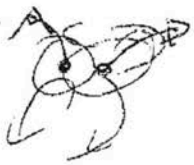
⇒ Consider incompressible turbulence with $M \equiv \dfrac{V_0}{C_s} \ll 1$.

Show: $\dfrac{l_d}{l_{mfp}} \sim M^{-1} R_e^{1/4}$

⇒ validity of continuum hydrodynamics gets better at high $Re$.

· Particle Separation / Richardson Law

Consider 2 particles (test) in K41 turbulence. Rate of separation?



→ larger eddys advect both

→ smaller eddys do ⊙ nothing

⇒ divergence controlled by eddys of scale $\ell \sim |\underline{x}_1 - x_2|$.

∴ if $\lambda \equiv |\underline{x}_1 - \underline{x}_2|$

$$\Rightarrow \quad \frac{d\lambda}{dt} = v(\lambda) = \epsilon^{1/3}\lambda^{1/3}$$

$$\lambda^{2/3} = \epsilon^{1/3} t$$

$$\therefore \quad \boxed{\lambda \sim \epsilon^{1/2} t^{3/2}}$$ Richardson's 3/2 Law

N.B.: Non-diffusive!

$$\lambda^2 \sim \epsilon t^3 \Rightarrow \boxed{\tau_{sep.} \sim \lambda^{2/3}/\epsilon^{1/3}}$$

N.B.:

→ process is self-accelerating ⇒ large eddys move faster

· non-diffusive.

→ [Momentum] Flux Driven Turbulence

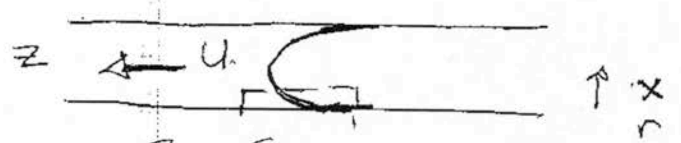
› *Turbulent Pipe Flow*

(cf. Landau, Lifshitz "Fluid Mechanics")

*Till now* → homogeneous flow in a periodic box
→ cascade in scale space (*Kolmogorov*)

*Now* → inhomogeneous flow in a pipe
→ momentum transport in a turbulent boundry layer (*Prandtl*)

Consider turbulent pipe flow:




→ core

→ logarithmic profile (inertial sublayer) (virtually all high Re experiments → "profile consistency")

→ linear profile (viscous sublayer)

Common features of pipe flow:

- linear → logarithmic $U(x)$ profile

- logarithmic profile persists over a broad range of Re

$$\left( Re = 2Ua/\nu \right)$$

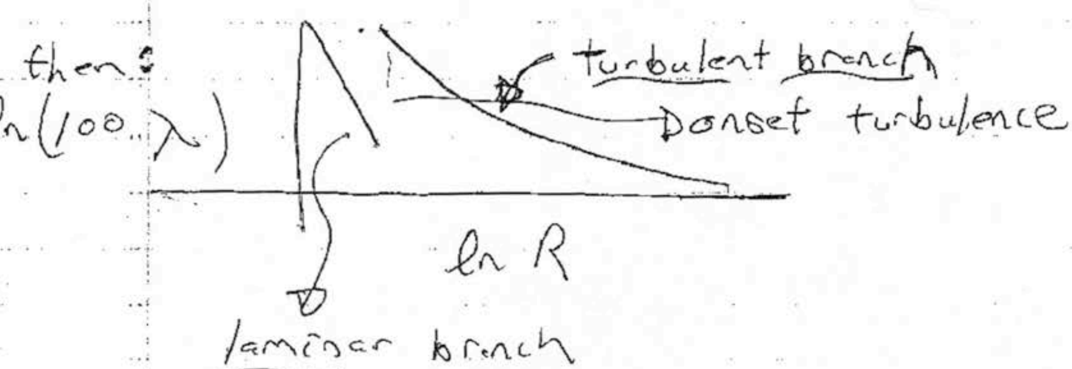
∴ logarithmic profile "universal" (Prandtl "Law of the Wall")

- resistance increases with increasing Re, discontinuously

$\lambda = \dfrac{2a\,\Delta p/l}{\frac{1}{2}\rho U^2}$ → pressure drop/length

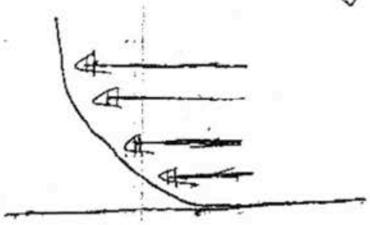
$\frac{1}{2}\rho U^2$ → mean flow energy

then:




- turbulent resistance curve universal.

∘ What is going on ?



no slip boundary condition

$U = U(x) \xrightarrow[x \to 0]{} 0$

$\therefore\ U = U(x) \Rightarrow \left\{ \begin{array}{l} \text{momentum flux} \\ \text{to wall} \end{array} \right\}$

→ momentum flux to wall ⇒ stress on the wall

→ wall stress must balance pressure drop, for steady flow

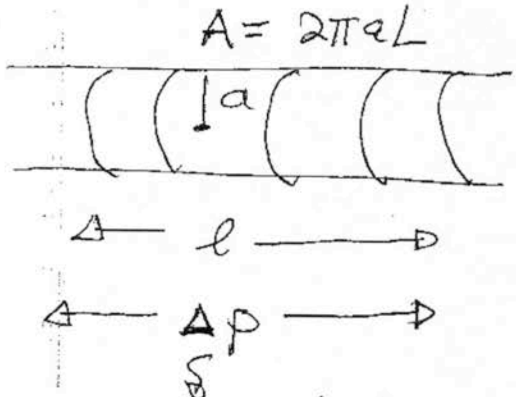
so  wall stress : $\rho U_*^2$  i.e.

$U_* \equiv$ friction velocity

$$\underset{F \sim SA}{\underline{v}\,\underline{\nabla} v} - \nu \nabla^2 \underline{v} = -\nabla p \sim \underset{F \sim pA}{\frac{\Delta p}{\ell}}$$

$$2\pi\, V_r^2 \ell R \sim \Delta P \pi R^2$$

$\therefore$ $$\rho U_*^2 2\pi a\, \ell = \Delta p\, \pi a^2$$




Force on wall ≅ $\rho U_*^2 A_{wall}$

(Pressure Drop) $A_{Flow}$ = Force on Fluid

ition anty ⇒ $$\rho U_*^2 (2\pi a \ell) = (\Delta p)\, \pi a^2$$

$$U_* = \left[ (\Delta p / 2\rho) \left( \frac{a}{2\ell} \right) \right]^{1/2}$$

Friction Velocity

$u_*$ ≡ friction velocity
≡ "typical" velocity of turbulence in turbulent pipe

Deriving the inertial sublayer profile:

i) dimensional reasoning

in pipe flow inertial sublayer, have

3 dimensional parameters $\rho$, $\tau$, $x$

$\rho$: density; $\tau$: wall stress, $u_*$; $x$: distance from wall

Key Point: ~~Assumption~~ Assumption of scale invariance

on scale $l_{vs} = \frac{\nu}{u_*} < x < a$.

→ universality of logarithmic profile motivates scale invariance assumption

now, seek velocity gradient $du/dx$,

$\frac{du}{dx}$ : $u_*$, $x$, $\rho$

so simplest form for $dU/dx$ is:

$$\frac{dU}{dx} = \frac{U_*}{x}$$

$$\Rightarrow \boxed{\begin{aligned} U &= \frac{U_*}{\kappa} \ln(x/x_0) \\ &= \frac{U_*}{\kappa} \ln x + \text{const.} \end{aligned}}$$

→ logarithmic profile (consequence of scale invariance in pipe flow)

→ $\kappa \sim .4$ universal constant → von-Karman constant

→ $x_0 \leftrightarrow$ width of viscous sublayer $\sim \nu/U_*$

ii.) Physical Reasoning

stationary flow ⇒

momentum flux to wall = pressure drop

$$\therefore \quad \underbrace{\langle \tilde{v}_x \tilde{v}_z \rangle}_{\text{Reynolds stress}} = u_*^2$$

$$\rho \langle \tilde{v}_x \tilde{v}_z \rangle = \Gamma_p$$

$\hookrightarrow$ momentum flux

$$\Gamma_p / \rho = u_*^2$$

Now, to calculate $\langle \tilde{v}_x \tilde{v}_z \rangle$:

→ take velocity fluctuation as generated by <u>mixing</u> of $u(x)$, so

$$\Rightarrow \quad \tilde{v}_z \sim \underbrace{\ell}_{\text{"mixing length"}} \frac{\partial u}{\partial x}$$

analogous to Chapman-Enskog expansion, i.e.

$\ell \Leftrightarrow \ell_{mfp}$.
$\tilde{v}_x \Leftrightarrow v_{Th}$.

here, scale invariance $\Leftrightarrow$ $l \sim x$

mixing length set by distance from wall

so

$$\langle \tilde{V}_x \tilde{V}_z \rangle = \langle V_x l \rangle \frac{\partial U}{\partial x}$$

$$\cong U_* x \frac{\partial U}{\partial x}$$

$$\nu_T = U_* x$$ $\rightarrow$ "eddy viscosity" / "turbulent viscosity" — Key concept

$\Rightarrow$ rate of turbulent transport of momentum

then momentum balance $\Rightarrow$

$$U_* x \frac{\partial U}{\partial x} = U_*^2$$

$\Rightarrow$

$$U = \frac{U_*}{\kappa} \ln(x/x_0)$$ $\rightarrow$ Logarithmic Profile

$\rightarrow$ Law of the Wall

Some comments:

→ as in K41, clear phenomenology critical to guiding the approximations ↔ scale invariance

"Mixing length theory always works ... provided you know the mixing length..."
— P.D.

→ why a single value of velocity, i.e. $u_*$?

Consistent with mixing length hypothesis, velocity fluctuations generated by mixing of mean flow gradient, i.e.

⇒

$$\tilde{V} \sim \ell \frac{\partial u}{\partial x} \sim x \frac{\partial u}{\partial x}$$

$$\sim \cancel{x} \frac{u_*}{\cancel{x}}$$

absence of preferred scale.

consistent. ∴ Assumption consistent with:

- logarithmic profile
- scale invariance.

⇒ viscous sublayer / cut-off of inertial layer ?

∴ when : $\nu_T < \nu$ { molecular viscosity dominates mixing

⇒ $u_* x \lesssim \nu$

$$x \lesssim \nu / u_* \equiv x_0$$

viscous sublayer scale.

In viscous sublayer, flow linear :

$$\nu \frac{\partial u}{\partial x} = u_*^2$$

$$\therefore \quad u = \frac{u_*^2}{\nu} x$$

⇒ note effect of turbulence is to:

- flatten profile – { higher transport at fixed wall stress
- reduce central velocity
- limit $Q$ (quality factor)

– matching, for const:

$$x_0 = \nu / u_* \quad \text{so}$$

$$u = \frac{u_*}{\kappa} \ln\left(\frac{u_* y}{\nu}\right)$$

Note: Flow in viscous sublayer is turbulent, but mixing there affected by dissipation range scales ⇒ linear profile

Now – turbulent dissipation ?.

Consider NSE:

$$\frac{\partial \hat{\underline{V}}}{\partial t} + \hat{\underline{V}} \cdot \nabla \hat{\underline{V}} + \langle V_z \rangle \frac{\partial}{\partial z} \hat{\underline{V}} + \hat{V}_x \frac{\partial}{\partial x} \langle V_z \rangle$$

$$= -\nabla \hat{p} + \nu \nabla^2 \hat{\underline{V}}$$

$\hat{\underline{V}} \cdot$ and avg ⇒

odd

$$\frac{\partial}{\partial t} \langle \hat{V}^2 \rangle + \langle \hat{\underline{V}} \cdot \hat{\underline{V}} \cdot \nabla \hat{\underline{V}} \rangle + \langle V_z \rangle \cancel{\langle \hat{\underline{V}} \cdot \frac{\partial}{\partial z} \hat{\underline{V}} \rangle}$$

$$+ \langle \hat{V}_x \hat{V}_z \rangle \frac{\partial}{\partial x} \langle V_z \rangle = -\cancel{\langle \hat{\underline{V}} \cdot \nabla \hat{p} \rangle} - \nu \langle (\nabla \hat{V})^2 \rangle$$

i.b.p

obviously: $\underbrace{\nu \langle (\nabla \tilde{v})^2 \rangle}_{\text{small scale dissipation}} = \overbrace{\nu_T \left(\frac{\partial U}{\partial x}\right)^2}^{\text{input} \to \text{mean flow mixing}}$

and

$$\epsilon = (U_* x) \left(\frac{U_*}{x}\right)^2 \qquad \text{(ignoring } R\text{)}$$
$$= \frac{U_*^3}{x}$$

$\to$ sets dissipation rate.

i.e. $\epsilon = \frac{V_0^3}{\ell}$ $\qquad V_0 \leftrightarrow U_*$, $\ell \leftrightarrow x$

$\to$ $\epsilon$ finite as $\nu \to 0$ (i.e. viscous sublayer gradient diverges then)

Additional References:

- S.B. Pope, "Turbulent Flows"
- H. Tennekes and J. Lumley, "A First Course in Turbulence"

For net energy budget:

$$\partial_t \mathcal{E} = \underbrace{-\langle \hat{V}_x \hat{V}_z \rangle \frac{\partial \langle V_z \rangle}{\partial x}}_{\text{input to fluctuations by relaxation of mean shear flow (Reynolds work)}} \underbrace{- \nu \langle (\nabla \tilde{V})^2 \rangle}_{\text{dissipation of fluctuation energy by viscosity}}$$

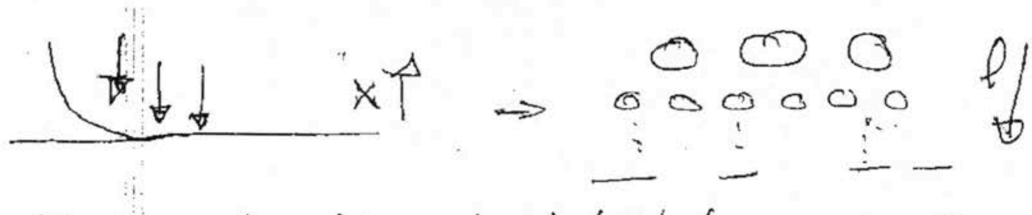
∴ Can define:

$$\underbrace{\epsilon}_{\text{turbulent dissipation rate}} = \langle \hat{V}_x \hat{V}_z \rangle \frac{\partial U}{\partial x}$$

and using mixing length theory:

$$\langle \hat{V}_x \hat{V}_z \rangle = U_* x \frac{\partial U}{\partial x}$$

$$\Rightarrow \epsilon = (U_* x)\left(\frac{\partial U}{\partial x}\right)^2 = \underbrace{\nu_T}_{\text{rate of "heating" by turbulent relaxation of mean flow.}} \left(\frac{\partial U}{\partial x}\right)^2$$



→ Now, interesting to tabulate comparison between Pipe Flow and K41 Problem

| Pipe Flow (Prandtl) | K41 (Kolmogorov) |
|---|---|
| scales: $a$, $x$, $\nu/u_*$ | $l_0$, $l_n$, $l_d$ |
| invariance: $X \to$ real space | $l \to$ scale space |
| inertial sublayer<br>viscous sublayer | inertial range<br>dissipation range |
| alance: $u_*^2 = \nu_T \frac{\partial u}{\partial x}$ | $\epsilon = \frac{v(l)^2}{\tau(l)}$ |
| ynamics: eddy viscosity<br>$\nu_T = u_* X$ | turn-over rate<br>$1/\tau(l) = \frac{v(l)}{l}$ |
| sult: $u = \frac{u_*}{k} \ln(x)$ | $v(l) = \epsilon^{1/3} l^{1/3}$ |
| universal profile | universal spectral scaling |
| issipation: $\nu = \nu_T$<br>$x_0 = \nu/u_*$ | $v(l)/l = \nu/l^2$<br>$l_d = \nu^{3/4}/\epsilon^{1/4}$. |

→ Practical Issues

Resistance Law → Pipe Flows.

have: $\frac{\nu}{u_*} < x \le \underset{\text{radius}}{a}$

can push to $x \sim a$, with logarithmic accuracy

$$U \cong \frac{u_*}{k}\ln\left(\frac{u_* a}{\nu}\right)$$

but
$$v_* = u_* = \left(\frac{a}{\ell}\,\frac{\Delta P}{2\rho}\right)^{1/2}$$

⇒ can re-write:

$$U = \left(\frac{a\,\Delta P}{2\rho\,\ell k^2}\right)^{1/2} \ln\left(a\left(\frac{a\,\Delta P}{2\rho\ell}\right)^{1/2}\Big/\nu\right)$$

Convenient to define:

$$\lambda = \frac{2a\,\overset{\text{drive}}{\Delta P/\ell}}{\underbrace{\tfrac{1}{2}\rho U^2}_{\text{flow KE}}} \rightarrow \text{friction factor / resistance coefficient}$$

$\Rightarrow$ taking $R_e = 2aU/\nu$

can re-write friction law as:

$$1/\sqrt{\lambda} = .88 \ln\left(R_e\sqrt{\lambda}\right) - .85$$

phenom.

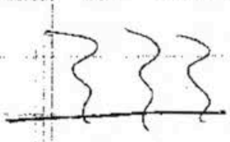
$$R_e = 2aU/\nu$$

$$\lambda = \frac{2a\,\Delta P/\ell}{1/2\,\rho U^2}$$

— good fit to pipe flow data.

Problems:

1 a.) A very strong explosion, with energy released $\Delta E$, creates a spherical blast wave in an atmosphere of pressure $P$, density $\rho$. Use dimensional analysis to derive the radius of the blast front as a function of time, i.e. $r(t)$? When does this scaling fail?

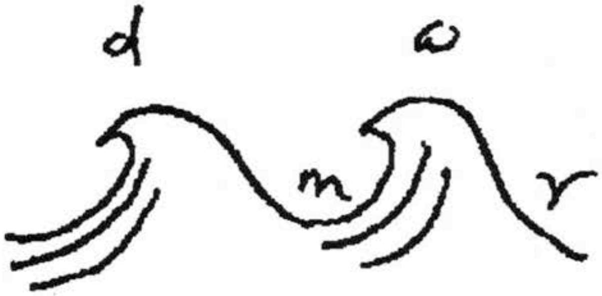
b.) 

A hot surface produces thermal convection above it. Assuming the convection is turbulent, use scaling arguments to calculate the temperature profile above the plate, assuming the hot plate drives a surface heat flux $Q$. (See Chapter 5, Landau).

# Why Wave "Kinetics"?

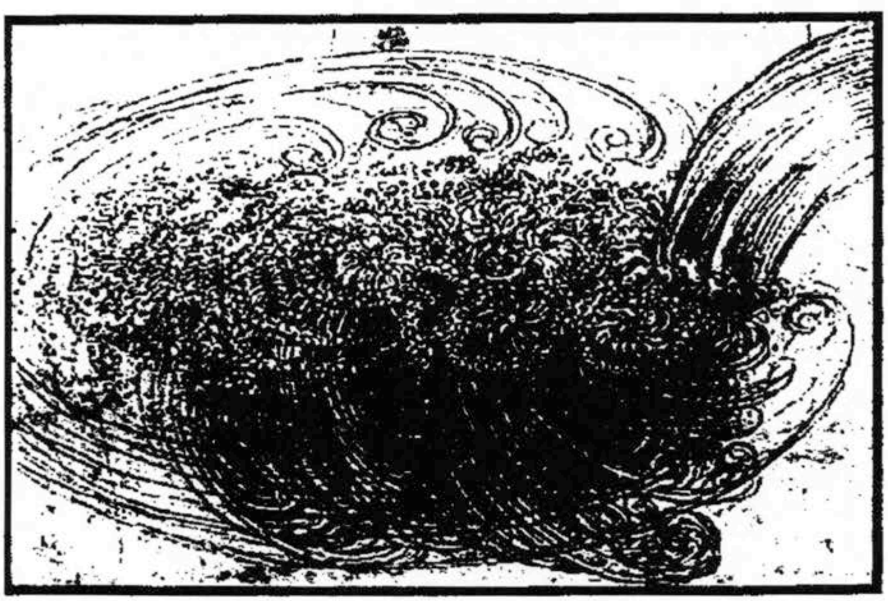
" A wave is never found alone, but is mingled with as many other waves as there are uneven places in the object where the said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions"

– Leonardo da Vinci
Codice Atlantico, c.1500.




[illegible]

[Leonardo
on waves...]



From Asian art.....

**The great wave at Kanagawa** Hokusai