Problem Set IV: due Friday, 12/3

1) This problem asks you to explore the Current Convective Instability (CCI) in a homogeneous medium and its sheared field relative, the Rippling Instability.

a) Consider first a current carrying plasma in a straight magnetic field $\underline{B} = B_0\hat{z}$ - i.e. ignore the poloidal field, etc. Noting that the resistivity η is a function of temperature (ala' Spitzer - c.f. Kulsrud 8.7), calculate the electrostatic resistive instability growth rate, assuming T evolves according to:

$$\frac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T - \chi_{\|} \partial_z^2 T - \chi_{\perp} \nabla_{\perp}^2 T = 0$$

and the electrostatic Ohm's Law is just

$$-\partial_z \phi = \frac{1}{\eta}\frac{d\eta}{dT}\tilde{T} J_0.$$

b) *Thoroughly* discuss the physics of this simple instability, i.e.

- what is the free energy source?
- what is the mechanism?
- what are the dampings and how do they restrict the unstable spectrum?
- how does spectral asymmetry enter?
- what is the cell structure?

c) Now, consider the instability in a *sheared* magnetic field, $\underline{B} = B_o(\hat{z} + \frac{x}{L_s}\hat{y})$

i.) What difficulties enter the analysis?

ii.) Resolve the difficulty by considering coupled evolution of vorticity, Ohm's Law (in electrostatic limit but with temperature fluctuations) and electron temperature. Compute the growth rate in the limit $\chi_{\|}, \chi_{\perp} \to 0$. Compute the mode width. Discuss how asymmetry enters here. Explain why.

d) Noting that $\chi_{\|} >> \chi_{\perp}$ (why? - see Kulsrud 8.7), estimate when parallel thermal conduction becomes an important damping effect. Can $\chi_{\|}$ alone ever absolutely stabilize the rippling mode?

2) *Taylor in Flatland*

Taylor awakes one morning, and finds himself in Flatland, a 2D world. Seeking to relax, he sets about reformulating his theory for that planar universe.

a) Write down the visco-resistive 2D MHD equations, and show that *three* quadratic quantities are conserved, as $\eta \to 0$, $\nu \to 0$.

b) Which of these is the most likely to constrain magnetic relaxation? Argue that

i.) the local version of this quantity is conserved for an 'flux circle', as $\eta \to 0$,

ii.) the global version is the most "rugged", for finite η.

c) Formulate a 2D Taylor Hypothesis - i.e. that magnetic energy is minimized while the quantity you identified from b.) ii.) is conserved. What equation describes this state? Show that the solution is force-free. What quantity is constant in Flatland? Hence, what is the endstate of Taylor relaxation in 2D?

d) <u>Optional</u>: Consider the possibility that $\nu >> \eta$ in Flatland. Derive the mean field evolution equation for mean magnetic potential. Discuss!

e) *Optional - Extra Credit* - Describe the visit of the Terrifying Torus to Flatland. How would 2D Taylor perceive this apparition?

N.B. You may find it useful to consult *Flatland*, by E. Abbott.

3) Reformulate the Sweet–Parker Reconnection problem for weak collisionality. Assume a uniform, strong guide field $B_0\hat{z}$ orthogonal to the plane of reconnection.

What can be said about the reconnection speed? [Note: This is an open-ended problem that asks you to synthesize the stories of the current-driven ion–acoustic instability and the resulting scattering of momentum with the S–P problem. You may find it useful to consult relevant parts of Kulsrud, Chapter 14.]

4) a) Show that for incompressible MHD in two dimensions, the basic equations can be written as:

$$(\partial_t + \underline{v}\cdot\nabla)\nabla^2\phi = (B\cdot\nabla)\nabla^2 A + \nu\nabla^2\nabla^2\phi + \tilde{f}$$

$$(\partial_t + \underline{v}\cdot\nabla)A = \eta\nabla^2 A.$$

Here ν is viscosity, η is resistivity, $\underline{v} = \underline{\nabla}\phi \times \hat{z}$ and $\underline{B} = \underline{\nabla}A \times \hat{z}$. $\tilde{f}$ is a random force. Take $P = P(\rho)$.

b) Take $\underline{B} = B_0\hat{x}$ to be a weak in-plane magnetic field. Calculate the real frequency and damping for Alfven waves.

c) Taking $\underline{B} = B_0\hat{x}$ and $\langle \tilde{V}_y \tilde{A} \rangle = -\eta_T \partial A_0/\partial y$ as a definition of turbulent resistivity η_T. Show that at stationarity

$$\eta_T = \eta\langle \tilde{B}^2 \rangle / B_0^2,$$

assuming the system has periodic boundary conditions. Discuss your result and its implications. This is a famous result, referred to as the Zeldovich Theorem, after Ya.B. Zeldovich.

5) Kulsrud; Chapter 11, Problem 1. Ignore the last paragraph.