Lecture 11b

Flux Expulsion / Homogenization

- the problem
- 3 balance → $\tau \simeq \Omega^{-1} R_m^{1/3}$ (with b)

- Physics: Shear Dispersion $1/\tau \sim \left(\frac{D V'^2}{\rho_0^2} \right)^{1/3}$

- Systematics: Shearing coordinates

  - eddy tilting: $k_x = k_{x0} - k_y V' t$
  → tilt aligns with shear
  - decay rate → shear dispersion.

- Expulsion → Homogenization

A

$\omega \to$ const relaxation

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{v}$$

$dE = T\,dS - p\,dV$

$dH = T\,dS + V\,dp$

$\underline{v} \cdot \nabla \underline{v} = -\nabla \frac{\underline{v}^2}{2} - \underline{v} \times \underline{\omega}$

$\frac{dS}{dt} = 0 \quad \Rightarrow \frac{\nabla p}{\rho} = -\nabla H$

$$\frac{\partial \underline{v}}{\partial t} = \nabla \times \underline{v} \times \underline{\omega}$$

$\nabla \cdot \underline{v} = 0$

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{v} \cdot \nabla \underline{\omega} = \cancel{\underline{\omega} \cdot \nabla \underline{v}}$$

2D

$$\boxed{\begin{array}{l} \dfrac{d\underline{\omega}}{dt} = \nu \nabla^2 \underline{\omega} \\ \dfrac{\partial \omega}{\partial t} + \underline{v} \cdot \underline{\nabla} \omega = \nu \nabla^2 \omega \end{array}}$$

$\omega = \nabla^2 \phi$

$\underline{v} = \underline{\nabla} \phi \times \hat{z}$

analogy → charge.

→ P-B Proof

$\nu = 0$
$\nu \to 0$ sing. limit
global

→ Key Points.
- simple room
- $\nabla \cdot$ flux dissip
- closed, bounding stream line

N.B. : Similar story

$$\frac{\partial \omega}{\partial t} + \underline{v} \cdot \underline{\nabla} \omega = \nu \nabla^2 \omega$$

Vort
Homog

$$\frac{\partial}{\partial t} c + \underline{v} \cdot \underline{\nabla} c = D \nabla^2 c$$

Passive
Scalar
Homogen.

$$\frac{\partial \Omega}{\partial t} + \underline{v} \cdot \nabla \Omega = \nu \nabla^2 \nabla^2 \phi$$

PV

$$\Omega = \nabla^2 \phi - \phi - \ln n_0(r)$$

HM

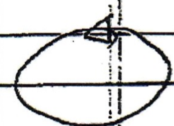
$$\frac{\partial A}{\partial t} + \underline{v} \cdot \underline{\nabla} A = \eta \nabla^2 A$$

Flux
Expulsion.

→ analog of "flux expulsion"

E) Prandtl - Batchelor Theorem

* G. Batchelor, JFM 1 177 (1956) (posted)
P.B. Rhines and W.R. Young, JFM 122, 347 '82 (posted)
JFM 133 133 '83
J. Pedlosky, "Ocean Circulation Theory"
Springer 1996, esp. 3.8
see also



Prandtl - Batchelor Theorem

Thm: Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline C_0. Then, if diffusive dissipation,

i.e. $\partial_t \omega + \nabla\phi\times\hat{z}\cdot\nabla\omega = \nabla\cdot(\nu\nabla\omega)$

then vorticity → uniform (homogenization), as $t\to\infty$, within C_0.

N.B.: finite ν ⇒ radically different final state

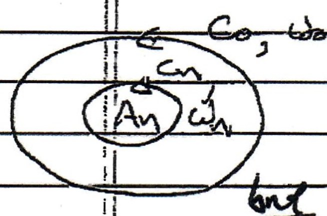
no comment on "how long"?

$t \to \infty$ before $\nu \to 0$ 6.

- $\nabla\phi \times \hat{z} \cdot \nabla\omega = \nabla \cdot \nu \nabla \omega$

for stationarity [note $t \to \infty$ before $\nu \to 0$]

- choose arbitrary closed C_n within C_0.
Here C_n a streamline



n.b. - assume simply connected region, i.e. no holes
- stationarity ⇒ ω constant along streamlines
- C_0 → specified on 1 bndry

∴ $\omega \to \omega_0$ on C_0 (ultimately C_0 sets b.c.)
$\omega \to \omega_n$ on C_n

If A_n is area enclosed by C_n,

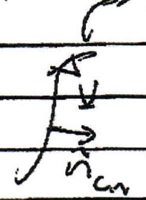
$$\int_{A_n} d^2x \, \underline{v} \cdot \nabla \omega = \int_{A_n} d^2x \, \nabla \cdot (\nu \nabla \omega)$$

but

$$\int_{A_n} d^2x \, \underline{v} \cdot \nabla \omega = \int d^2x \, \nabla \cdot [\underline{v}\omega]$$
$$= \oint_{C_n} dl \, \hat{n}_{C_n} \cdot (\underline{v}\omega)$$

normal

now



so $$\int_{C_n} dl\,(\hat{n}_{cn}\cdot \underline{V})\,\omega = 0$$

as $\underline{V}$ is along streamline

$$0 = \int_{A_n} d^2x\;\underline{\nabla}\cdot(\nu\,\underline{\nabla}\omega)$$

$$= \nu\int_{C_n} dl\;\hat{n}_{cn}\cdot\underline{\nabla}\omega$$

now in stationary state must have $\omega \rightarrow$ const along streamline

∴ $$\omega = \omega(\phi)$$

so $$\omega_{cn} = \omega(\phi_n)$$

$$0 = \nu\int_{C_n} dl\;\hat{n}_{cn}\cdot\underline{\nabla}\phi_n\,\frac{\delta\omega}{\delta\phi_n}$$

$$0 = \nu\,\frac{\delta\omega}{\delta\phi_n}\int_{C_n} dl\;\hat{n}_{cn}\cdot\underline{\nabla}\phi_n$$

but $\Gamma = \int d\underline{\ell}\cdot\underline{v}$

$= \int d\underline{\ell}\cdot(\nabla\phi\times\hat{z})$

$= \int(\hat{z}\times\vec{n})\cdot(\nabla\phi\times\hat{z})$

$= -\int d\ell\,(\nabla\phi\cdot\vec{n}) = -\int d\ell\,(\nabla\phi\cdot\vec{n})$

∴

$0 = \nu\,\dfrac{d\omega}{d\phi_n}\,\Gamma_n$

∴

$d\omega/d\phi_n = 0$

C bnd. bndy stline

but ϕ_n arbitrary $\Rightarrow$ $\boxed{d\omega/d\phi = 0,\ \text{all } \phi}$

arbitrary $\Rightarrow$ no variation from line to line

$\Rightarrow$ $\boxed{\omega \text{ homogenized}}$

so, expect $\nabla\omega$ larger at bounding contour C_0.

$\nabla\omega \neq 0$, within $\Rightarrow$ $\nabla\omega$ held at boundary

Some Comments:

⇒ Homogenization theory looks 'magical' → caveat emptor!

i.e.

* 1.) note assumptions of:

$t \to \infty$ ⇒ time asymptopia

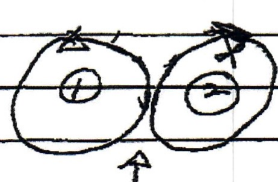
$q = q(\phi)$ ⇒ concentric streamlines

how long to achieve configuration?!

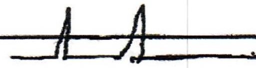
2.) simply connected domain ⇒ annulus? (Z.F.!)

3.) single structure → expulsion from neighbors and possible interaction not addressed

i.e. what happens if? → 'interference of boundary layers'?



or



⇒ straining interaction

⇒ ② 'strips' ① etc.

+) Key Assumptions:

⇒ { closed, bounding streamline
{ viscous dissipation

i.e. can envision:

→ exact streamline, molecular viscosity

or

→ coarse-grained streamline, eddy viscosity

⇒ correspond to homogenization of

→ ~~total vorticity~~

→ mean/coarse-grained vorticity

⇒ time scales different

→ $\boxed{\dfrac{\tau_{circulation}}{\tau_{diffusion}} \ll 1 \rightarrow Re \gg 1}$ ← L →

then
- to establish concentric circulation lines
- diffuse across to homogenize → but slow!!

$$\frac{\tau_c}{\tau_d} = \frac{1}{(V/L)} \frac{D}{L^2} \ll 1 \Rightarrow \frac{D}{VL} \ll 1$$

i.e. $Re \gg 1$

or equivalently $\frac{VL}{D} \gg 1$

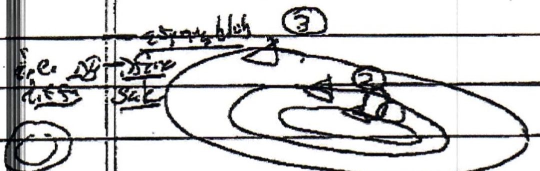
i.e. $(Re)_{\substack{eff \\ cell}} \gg 1$ $Pe \gg 1$

related: – essential idea is that Ω constant along streamlines established on fast ($\sim \tau_c$) scale
few

– dissipation homogenizes on slower ($\sim \tau_D$) time scale
(but this is slow ...)

→ what are the time scales? – { how resolve slow time scale problem

– useful to consider differentially rotating sheared flow within closed pattern



$V_{①} \neq V_{②} \neq V_{③}$

~~need blob with finite~~ l_y

→ what is the mixing time scale? $\boxed{\text{shear dispersion}}$

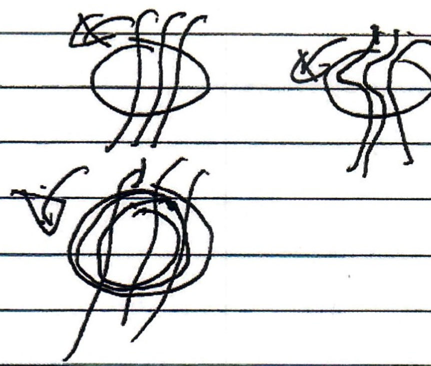
① finer, finer scale ...

② key: synergism between { shear, diffusion !

c.f. { H. Biglari, P.H. Diamond, P.W. Terry
Phys Fluids (B2), 1, 1990
~~(first noted by G.I. Taylor)~~

→ $\boxed{\text{When does kinematic expulsion end? (MGH)}}$

~ qualitatively, when magnetic tension grows to point of competition with vortex evolution.



Now,
$$b\,\ell = B_0 L_0$$

$$\frac{\eta}{\ell^2} b = \frac{B_0 U_0}{L_0}$$

so
$$\eta \frac{b}{\left(\frac{B_0 L_0}{b}\right)^2} = \frac{B_0 U_0}{L_0}$$

$$\Rightarrow \boxed{b \sim B_0 \left(\frac{U_0 L_0}{\eta}\right)^{1/3}} \rightarrow \frac{b}{B_0} \sim \frac{L_0}{\ell} \sim R_m^{1/3}$$

For tension;

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = -\frac{|B|^2}{r_c}\hat{n} + \underbrace{\frac{d}{ds}\left(\frac{|B|^2}{2}\right)\hat{t}}_{\text{stretched field}}$$

$$|\underline{B}\cdot\underline{\nabla}\,\underline{B}| \approx \frac{b^2}{L_0} \qquad \begin{cases} r_c \sim L_0 \\ \frac{d}{ds} \sim L_0^{-1} \end{cases}$$

PHYSICAL REVIEW FLUIDS 2, 113701 (2017)

Vortex disruption by magnetohydrodynamic feedback

J. Mak,* S. D. Griffiths, and D. W. Hughes
Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom
(Received 10 September 2016; published 22 November 2017)

In an electrically conducting fluid, vortices stretch out a weak, large-scale magnetic field to form strong current sheets on their edges. Associated with these current sheets are magnetic stresses, which are subsequently released through reconnection, leading to vortex disruption, and possibly even destruction. This disruption phenomenon is investigated here in the context of two-dimensional, homogeneous, incompressible magnetohydrodynamics. We derive a simple order of magnitude estimate for the magnetic stresses—and thus the degree of disruption—that depends on the strength of the background magnetic field (measured by the parameter M, a ratio between the Alfvén speed and a typical flow speed) and on the magnetic diffusivity (measured by the magnetic Reynolds number Rm). The resulting estimate suggests that significant disruption occurs when $M^2\mathrm{Rm} = O(1)$. To test our prediction, we analyze direct numerical simulations of vortices generated by the breakup of unstable shear flows with an initially weak background magnetic field. Using the Okubo-Weiss vortex coherence criterion, we introduce a vortex disruption measure, and show that it is consistent with our predicted scaling, for vortices generated by instabilities of both a shear layer and a jet.

DOI: 10.1103/PhysRevFluids.2.113701

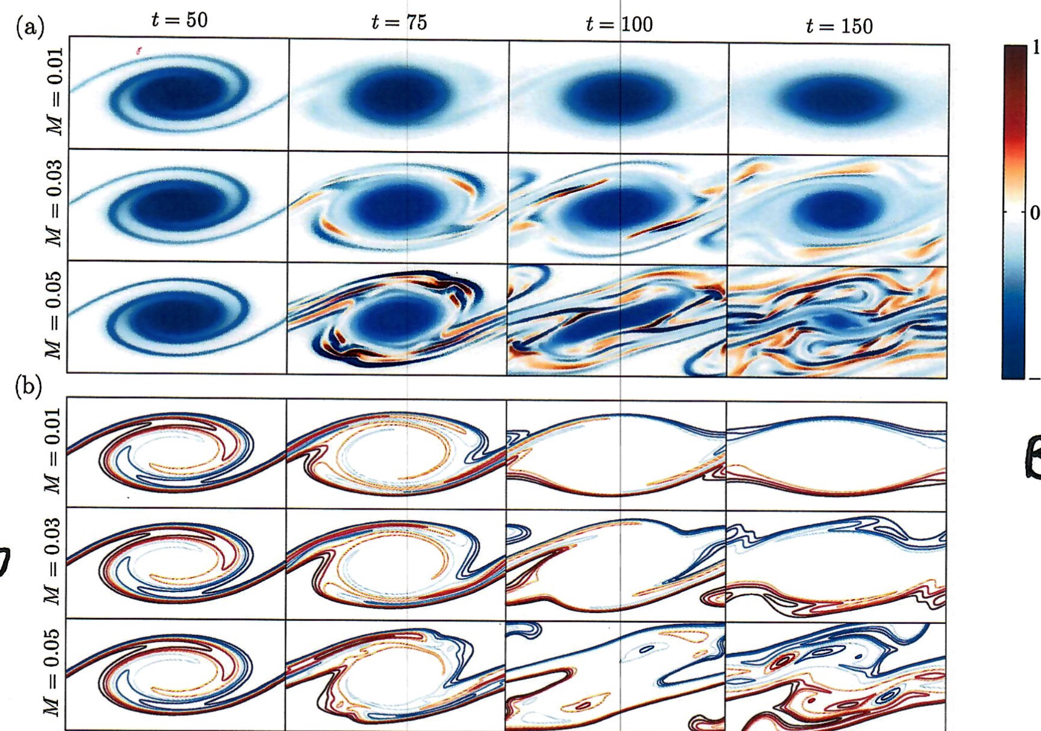
I. INTRODUCTION

The interaction of vortices with a magnetic field is a fundamental process in astrophysical magnetohydrodynamics (MHD). Such vortices could be generated, for example, by convection [1,2] or by the breakup of unstable shear flows [3–6]. In the absence of magnetic fields, vortices can be coherent, long-lived structures, particularly in two-dimensional or quasi-two-dimensional systems [e.g., [7]]. However, in the presence of a background magnetic field, various studies have shown how vortices can be disrupted, by which we mean either a reduction in strength or spatial coherence, or completely destroyed [8–16]. Here we show explicitly how this disruption depends on both the field strength and on the magnetic Reynolds number Rm.

Astrophysical fluid flows are invariably characterized by extremely high values of Rm. Perhaps the most important consequence of this is that weak large-scale fields can be stretched by the flow to generate strong small-scale fields, with the amplification being some positive power of Rm [17]. Once the small-scale fields are dynamically significant, the resulting evolution is essentially magnetohydrodynamic—rather than hydrodynamic—leading to dramatically different characteristics, despite the large-scale magnetic field being very weak. Such behavior has been identified in the suppression of turbulent transport [18–24], in the suppression of jets in β-plane turbulence [25], and in the inhibition of large-scale vortex formation in rapidly rotating convection [26].

The vortex disruption investigated here, which builds on Ref. [27], and can be contrasted with Ref. [28], depends on just such high Rm dynamics. Given that many astrophysical flows are rotating and stratified, such that the vortices are essentially two-dimensional, it is natural to investigate vortex disruption in the context of two-dimensional MHD. To quantify when a weak large-scale field can become dynamically significant, we first construct a scaling argument for a quite general setting with a single vortex. We first estimate the amplification of the large-scale field due to stretching

*Current address: Department of Physics, University of Oxford, Clarendon Laboratory, Parks Road Oxford, OX1 3PU; julian.c.l.mak@googlemail.com



ω

$M = \frac{V_{A0}}{U}$

B

$t \rightarrow$

FIG. 1. Snapshots of (a) vorticity and (b) magnetic field lines for the shear layer at different field strengths (for Rm = Re = 500), shown for the central half of the channel $(-L_y/2 \leqslant y \leqslant L_y/2)$.

observe the formation of regions of positive vorticity. The magnetic field is no longer confined to kinematic boundary layers, and the resulting stresses are strong enough to modify the resulting evolution to a certain extent. That said, the vortex is only mildly disrupted and maintains its integrity; there is only a slight decrease of vortex size by the end of the simulation at $t = 150$. For $M = 0.05$, the evolution is radically different to the other two cases, with a significant disruption of the vortex and an unconfined magnetic field. By the end of the simulation, only small remnants of the parent vortex persist; vorticity filaments and a complex magnetic field are now the dominant features in the domain.

Vortex disruption also has a signature in the time evolution of the kinetic and magnetic energies. Shown in Fig. 2 are time series for the three control runs of the mean kinetic energy $\overline{E}_k$ and mean magnetic energy $\overline{E}_m$ (defined as the energy content in the $k_x = 0$ Fourier mode), along with the perturbation energies E_k' and E_m' (defined as the energy content in the remaining Fourier modes). The evolution is similar up to $t \approx 60$ (cf. Fig. 1), at which time the field amplification is close to being arrested by diffusion; the scalings (2) and (3) then apply for the small-scale field, implying $E_m' \sim b^2 l L_v$ and $\overline{E}_m \sim B_0^2 L_0^2$, so that $E_m'/\overline{E}_m \sim \mathrm{Rm}^{1/3} \approx 8$ here, consistent with Fig. 2. However, for $t \gtrsim 80$, vortex disruption (if it occurs) changes the evolution of the energy. For the undisrupted case ($M = 0.01$), the evolution becomes one of complete flux expulsion (see Fig. 1), with E_m' decreasing to less than $\overline{E}_m$. (This is different to the well-known theory of Ref. [29], in which $E_m' \sim \mathrm{Rm}^{1/2} \overline{E}_m$ in the flux-expelled state, but that kinematic single-vortex theory may not apply to this dynamic regime with a periodic array of vortices and remote boundaries.) For the strongly disrupted case ($M = 0.05$), we enter a different regime, with E_m' staying close to $\overline{E}_m$ throughout the evolution. This regime with persistent small spatial scales results in stronger dissipation: whereas the total dissipation is small and comparable with that of the hydrodynamic case for $M = 0.01$ and 0.03, it is about three times higher when $M = 0.05$. Further, even though $\overline{E}_m \ll \overline{E}_k$ throughout the i.e.

then: $$\frac{d}{dt}\nabla_\perp^2\phi = B\cdot\nabla J$$

$$\rho\frac{d\omega}{dt} = \left[\nabla\times(B\cdot\nabla B)\right]\cdot\hat{z}$$

picks up short lengthscale

$$\rho\, u\cdot\nabla\omega \sim \frac{b^2}{\ell L_0}$$

tension held in edge band

$$\frac{u}{L_0}\omega \sim \frac{b^2}{\rho_0 L_0 \ell} \quad \checkmark$$

but

$$\frac{u}{L_0}\omega \sim \frac{u^2}{L_0^2} \sim \frac{b^2}{\rho L_0 \ell}$$

$$\sim \frac{B_0^2 L_0^2}{\ell^3 L_0 \rho_0}$$

∴ For given L_0, critical B_0 to prevent expulsion:

$$\frac{u^2}{L_0^2} \sim \frac{B_0^2 L_0}{\ell^3 \rho_0} \Rightarrow \left(\frac{V_{A0}}{u}\right)^2 \sim \left(\frac{L_0^3}{\ell^3}\right)^{-1}$$

$$\frac{b}{B} \sim \left(\frac{u_0 L_0}{\eta}\right)^{1/3} \sim R_m^{1/3} \qquad \sim R_m^{-1}$$

$\Rightarrow$ $\boxed{R_m \left(\frac{V_{A0}}{u}\right)^2 \sim 1}$

is feedback criterion!

$\boxed{\text{Not } \frac{V_{A0}}{u} \sim 1}$

→ akin feedback criterion for magnetic flux transport in 2D

→ Can easily have $V_{A0} \ll u$ with $R_m \gg 1$, and still encounter feedback.

⟺ weak field is sufficient.

Key contrast:

$$(V_{A0}/u)^2 \sim 1 \quad \underline{\text{vs}} \quad R_m \left(\frac{V_{A0}}{u}\right)^2 \sim 1.$$

Why: Stretching:

$\tilde{B}^3 \sim B_0^3 R_m$

$R_m \gg 1 \rightarrow$ weak critical field can fall outside kinematics