→ Magnetic Helicity and Taylor Relaxation

- Physics of Magnetic Helicity
  - conservation / balance
  - physical meaning / picture
  - significance
- Taylor Hypothesis
  Arguments.
- Mean Field Theory.

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→ Magnetic Helicity → Relaxation Constraint

- another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x \; \underline{A} \cdot \underline{B}$$

{ V is taken to be the volume of a 'flux tube'.

- what, yet another invariant!?!

→ K is different ⇒ has $\boxed{\text{topological}}$ interpretation

$$K = \int_V d^3x \; \underline{A} \cdot \underline{\nabla} \times \underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a pseudo-scalar

∴ has orientation or "handedness"...

Proceed via:

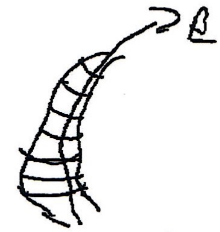
- show K conservation / balance
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x \, \underline{\nabla}\chi \cdot \underline{B}$$

$$= K + \int_V d^3x \, \underline{\nabla} \cdot (\underline{B} \chi)$$



$= 0$, to surface term. $\left\{ \underline{B} \cdot \hat{n} = 0 \right.$ on surface of tube

Now, consider a blob of MHD fluid in motion

can show $\frac{dK}{dt} =$

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{v} \times \underline{\nabla} \times \underline{A} - c\nabla\phi - c\eta \underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{v} \cdot \underline{\nabla} \underline{B} + \underline{B} \cdot \underline{\nabla} \underline{v} - \underline{B}\, \underline{\nabla} \cdot \underline{v} + \eta \nabla^2 \underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt} \int_V d^3x \, (\underline{A} \cdot \underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt} \cdot \underline{B} + \underline{A} \cdot \frac{d\underline{B}}{dt} \right) + \int \underline{A} \cdot \underline{B} \, \frac{d}{dt} d^3x$$

total derivatives!

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$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + (\underline{V} \cdot \nabla \underline{A}) \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial t} + \underline{A} \cdot (\underline{V} \cdot \nabla \underline{B}) \right) + \underline{A} \cdot \underline{B} \; \nabla \cdot \underline{V} \Big)$$

where $\frac{d}{dt} d^3x = \nabla \cdot \underline{V}$

i.e. $\frac{d}{dt} dV = \frac{d}{dt} d\underline{\sigma} \cdot d\underline{\ell} + d\sigma \frac{d}{dt} d\underline{\ell}$

$$= - d\underline{\ell} \cdot \nabla \underline{V} \cdot d\underline{\sigma} + (\nabla \cdot \underline{V})(d\underline{\sigma} \cdot d\underline{\ell}) + d\underline{\ell} \cdot \nabla \underline{V} \cdot d\underline{\sigma}$$

$$= \nabla \cdot \underline{V} \; d^3x$$

s.t. and $\underline{B} \cdot \hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \Big[\big(\underline{B} \cdot \cancel{\underline{V} \times \underline{B}} - c \cancel{\underline{B}} \cdot \nabla \phi - c\eta \, \underline{J} \cdot \underline{B} \big) + \underline{A} \cdot \big(\nabla \times (\underline{V} \times \underline{B}) \big) + \nabla \cdot \big((\underline{A} \cdot \underline{B}) \underline{V} \big) + \underline{A} \cdot \eta \nabla^2 \underline{B} \Big]$$

where: $\underline{A} \cdot (\underline{V} \cdot \nabla \underline{B}) + \underline{B} \cdot (\underline{V} \cdot \nabla \underline{A}) + \underline{A} \cdot \underline{B} \; \nabla \cdot \underline{V} = \nabla \cdot (\underline{V} \underline{A} \cdot \underline{B})$

$$\frac{dK}{dt} = \int d^3x \Big[\nabla \cdot \big((\underline{A} \cdot \underline{B}) \underline{V} \big) + \nabla \cdot \big((\underline{V} \times \underline{B}) \times \underline{A} \big) + (\underline{V} \times \cancel{\underline{B}}) \cdot (\nabla \times \underline{A}) - c\eta \, \underline{J} \cdot \underline{B} - \cancel{\eta} (\underline{A} \cdot \nabla \times \underline{J}) c \Big]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla} \cdot \left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta(\underline{A}\times\underline{J}) \right] - c\eta\,\underline{J}\cdot\underline{B} - c\eta\,\underline{J}\cdot\underline{B} \right]$$

$$= \int d\underline{S}\cdot\left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta\,\underline{A}\times\underline{J} \right]$$

$$- 2\int d^3x \left[c\eta\,\underline{J}\cdot\underline{B} \right]$$

$$= \int d\underline{S}\cdot\left[\cancel{(\underline{A}\cdot\underline{B})\underline{v}} - \cancel{(\underline{A}\cdot\underline{B})\underline{v}} + \underbrace{(\underline{A}\cdot\underline{v})\underline{B}}_{0} \right] - c\eta\int d\underline{S}\cdot\underline{J}\times\underline{A}$$

$$-2c\eta\int d^3x(\underline{J}\cdot\underline{B}) \qquad \underline{B}\cdot\underline{\hat{n}} = 0, \text{ on } \underline{\text{tube}}$$

$$= -\int c\eta\, d\underline{S}\cdot\left[\underline{\nabla}\underline{B}\cdot\underline{A} - \cancel{\underline{A}\cdot\underline{\nabla}\underline{B}}^{\,0} \right] - 2c\eta\int d^3x\,\underline{J}\cdot\underline{B}$$

$$= -2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$\Rightarrow$ have shown: Helicity Balance

$$\boxed{\frac{dK}{dt} = -2c\eta\int d^3x\,(\underline{J}\cdot\underline{B})}$$

$\oint$ includes 'transport' $\to$ decay

$\int d^3x\,\underline{J}\cdot\underline{B} \equiv$ current helicity
also pseudoscalar.

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and clearly: $\frac{dK}{dt} \to 0$ as $\eta \to 0$

beware! for (non-singular $\underline{J}$) → current sheet

∴ Magnetic Helicity is conserved in ideal MHD.

→ $\boxed{\text{Magnetic Helicity conserved, but what does it mean?}}$

- helicity is non-trivial ⇒ more than just helical field lines.

safety factor → pitch of magnetic field line

interesting to note: $q(r) = \frac{r B_z}{R B_\theta(r)} = \frac{1}{R \mu(r)}$

$\mu(r) = \frac{B_\theta(r)}{r B_z}$ → field line pitch.

[(length scale over which winding varies)]

cylindrical plasma → $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r} \int_0^r r B_z \, dr$

$A_z = - \int_0^r B_\theta \, dr$

$\underline{B} = \nabla \times \underline{A}$

$B_r = 0$

$$\frac{dr}{B_r} = \frac{r d\theta}{B_\theta} = \frac{dz}{B_z}$$

so $\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z \, dr - B_z \int_0^r B_\theta \, dr$ $\qquad B_z$ const

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu} \, dr - B_z \int_0^r B_\theta \, dr$$

$$\underline{A}\cdot\underline{B} = B_z \left[\mu \int_0^r \frac{B_\theta}{\mu} \, dr - \int_0^r B_\theta \, dr \right]$$

$= 0$ for constant μ !

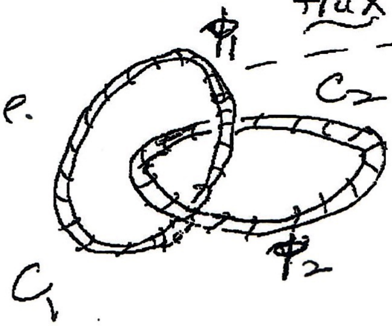
$\therefore$ $\boxed{\text{non-zero helicity requires } \mu = \mu(r)}$

i.e. – pitch varies with radius

$\Rightarrow$ $\boxed{\text{magnetic shear}}$ twist

– physically $\rightarrow$ helicity means self-linkage of 2 flux tubes

i.e.



tube 1: flux

$$\Phi = \int \underbrace{d\underline{A}}_{\text{x-section area}} \cdot \underline{B} = \underbrace{\phi_1}_{\text{const}}$$

tube 2: $\Phi = \phi_2$

field in loops, only

$$\Phi = \int da \cdot \underline{B}$$

2 linked tubes

Now, for volume V_1 of tube 1

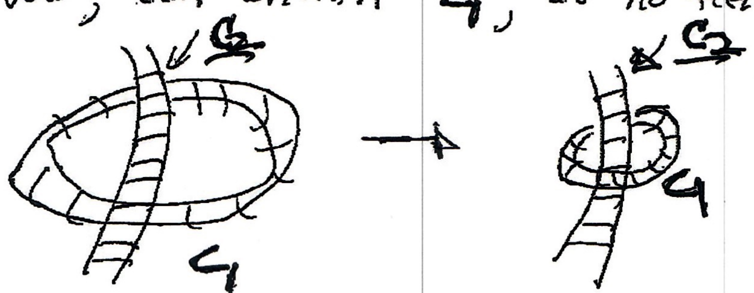
$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \underbrace{\oint_{C_1}}_{\text{along loop}} d\ell \underbrace{\int_{A_1}}_{\text{X-sect area}} \underline{dS}\ \underline{A}\cdot\underline{B}$$

B along tube

$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\, dA$$

$$= \Phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops.



re-oriented

→ in X section:

but $\displaystyle\int_{C_1} \underline{A}\cdot d\underline{\ell} = \int_{A\ \text{enclosed}} \underline{B}\cdot d\underline{S} = \Phi_2$

see Moffatt paper for
Fluid helicity $\int d^3x\, \underline{V} \cdot \underline{\omega}$
and knot connection

so . . . $K_1 = \phi_1 \phi_2$ → product of fluxes

similarly
$K_2 = \phi_2 \phi_1$

∴ $\boxed{K = 2\phi_1 \phi_2}$

if n windings $K = K_1 + K_2 = \pm 2n\phi_1\phi_2$.

⇒ $\boxed{\text{helicity is measure of self-linkage of magnetic configuration.}}$ ('complexity').

Can ~~only~~ change ~~Helicity~~ only by reconnection.

Why care → Taylor Conjecture (1974) (J. B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

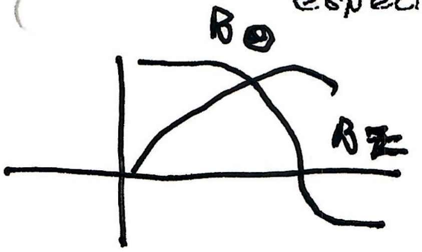
Reversed Field Pinch
- RFP

~ toroid
~ toroidal current

well fit by $B_z = B_0 J_0(\alpha r)$
$B_\theta = B_0 J_1(\alpha r)$

$\underline{J} \times \underline{B} = 0$
⇕
force free

⇒ why so robust?,
especially since RFP so turbulent



contrast $B_z \sim$ const. in cylindrical model tokamak $B_\theta < B_z$.

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point - helicity conserved in flux tubes, to η
- toroidal plasma → many small tubes

 etc.

$$V/L \simeq (V_A M)^{1/2}/L^{3/2}$$

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.
- smallscale linkage most fragile!

⇒ Taylor Conjecture: At finite η, helicity of small tubes dissipated but global helicity ~ conserved. (most rugged)

i.e.

$$\int_{\text{plasma volume}} \underline{A}\cdot\underline{B}\, d^3x = K_0 \rightarrow \text{~ conserved.}$$

∴ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\boxed{\delta\left[\int_V d^3x\,\frac{B^2}{8\pi} + \lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right] = 0}$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity.

Comments:

→ it works! – indeed amazingly well – for RFPs, spheromaks, etc. Departures only recently being discovered.

→ inspired idea of helicity injection as way to maintain configurations ⟷

→ it is a conjecture → no proof.

Hypothesis: Selective Decay

- energy cascades → small scale
- helicity cascades → large scale (less dissipation)

– Relevance to driven system?
i.e. in real RFP, transformer on.

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing, Mean Field electrodynamics

→ Taylor Conjecture – Why Global Helicity?

① → in ideal plasma, $\int_V d^3x\, \underline{A}\cdot\underline{B}$ conserved for all V (modulo BC's).

i.e.

any tube around a line (does line exist, how many?)

$\exists\ \alpha, \beta$ s/t $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$

$\int_{tube} d^3x\, \underline{A}\cdot\underline{B} = \text{const}$

→ microtube

if $\delta \int_{tube} d^3x \left[\frac{\underline{B}^2}{8\pi} + \lambda \underline{A}\cdot\underline{B} \right] = 0$

$\underline{\nabla}\times\underline{B} = \lambda(\alpha,\beta)\,\underline{B}$, $\underline{B}\cdot\underline{\nabla}\lambda = 0$

force free in micro-tube along line.

but $\lambda(\alpha,\beta) \neq \lambda(\alpha',\beta')$

- each tube/line defines conserved helicity.

- ∞ of invariants, due freezing in.

c.f. { E.N. Parker on Tubes/Current Sheets.
Moffatt on singularity formation

② → Relaxation occurs in a resistive, turbulent plasma.

n.b. "turbulent" = broadband excitation energy transfer between scales."

- small tubes destroyed by reconnection, faster

i.e $\tau_R \sim L/V_{in} \sim L^{3/2}$

(conventional trend) $(V_{in} \to V_A)$

- $t \to \infty$, only largest tube survives

$\Rightarrow$ global helicity — time asymptotic survivor.

- equivalent: in turbulent state, expect stochastic lines

then 1 field line filling volume

$\Rightarrow$ 1 tube filling volume

$\Rightarrow$ global helicity only invariant

- another equivalent $\rightarrow$ in 3D MHD
  - magnetic energy and fluid energy forward [redacted] cascades $\Rightarrow$ small scale
  - magnetic helicity inverse cascades $\Rightarrow$ large scale.

3 arguments.

Heuristic:

$W \to$ magnetic energy $\int d^3x \, \frac{B^2}{8\pi}$

$$\dot{W} \sim -2\eta \langle (\nabla B)^2 \rangle$$

$$\sim -2 \frac{\eta}{L_{eff}^2} \langle B^2 \rangle$$

$$\dot{H} \sim -2c\eta \langle \underline{J} \cdot \underline{B} \rangle$$

$$\sim -2\eta \frac{\langle B^2 \rangle}{L_{eff}}$$

if $L_{eff} \sim \Delta \sim L/\sqrt{R_m} \sim \eta^{1/2}$

$\Rightarrow$ $\dot{W} \sim \eta^{(0)} \to$ finite, indep dissipn, as in turbulence.

$$\dot{H} \sim \eta^{1/2} \to 0, \text{ as } \eta \to 0.$$

W dissipated, $H \sim$ conserved,

$\rightsquigarrow$ selective decay.

→ Relaxed State and How Gets There

$$\delta \int d^3x \left[\frac{\underline{B}^2}{8\pi} + \lambda\, \underline{A}\cdot\underline{B} \right] = 0$$

$\Rightarrow$

$$\frac{\underline{B}\cdot\delta\underline{B}}{4\pi} + \lambda\, A\cdot\delta B$$

$$B_z = B_0 J_0(\mu r)$$

$$B_\theta = B_0 J_1(\mu r)$$

$$\frac{\underline{\nabla}\times\underline{A}}{4\pi} + \lambda\, \underline{A} = 0$$

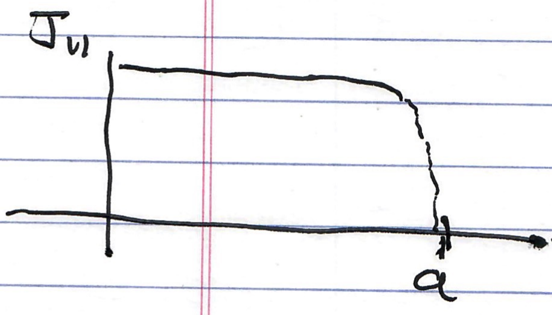
$$\Rightarrow \quad \underline{J} = \mu\, \underline{B}$$

$$\underline{\nabla}\times\underline{B} = \mu\, \underline{B}$$

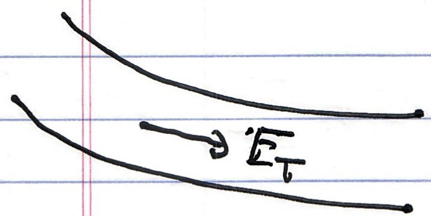
force free

$$J_\parallel = \frac{\underline{J}\cdot\underline{B}}{B^2} = \underbrace{\mu}_{\text{const}}$$

$$\Rightarrow \nabla_r J_\parallel \to 0 \quad \to \quad \left[\text{parallel current homogenized} \right]$$



→ How realize this?



- how drive poloidal currents to maintain reversal?
- why flat?

- What dynamics underpins Relaxation?

⇒ Welcome to Mean Field Electrodynamics!

c.f. Moffatt '78 ←

Moffatt and Dormy, recent

- idea is that mean field produced by fluctuations is what drives relaxation.

- akin quasilinear theory

- no comments on how fluctuations produced → instability(B).

Recall: $\underline{E} + \dfrac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$

$$\langle \underline{E} \rangle + \frac{\langle \underline{v} \rangle \times \langle \underline{B} \rangle}{c} = - \underbrace{\frac{\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle}{c}}_{\text{Mean EMF, driven by fluctuations}} + \eta \langle \underline{J} \rangle$$

$$- \frac{\langle \underline{v} \times \underline{B} \rangle}{c} = \underbrace{\langle \underline{S} \rangle}_{\text{un-resolved EMF "something".}}$$

What is S? → Form??

Point:

(i) $\langle \underline{S} \rangle$ must conserve H_M.

(ii) $\langle \underline{S} \rangle$ must dissipate E_M.

[Unanswered: How <u>obtain</u> $\langle S \rangle$ from primitive equations by some <u>systematic</u> procedure? — c.f. Moffatt] mean field helicity

Now,

$$\partial_t \int d^3x \, \langle \underline{A} \rangle \cdot \langle \underline{B} \rangle = \partial_t \int d^3x \left[\langle \underline{A} \rangle \cdot \langle \underline{\nabla} \times \underline{A} \rangle \right]$$

$$= -2c \int d^3x \left[\left(\langle \underline{E} \rangle + \langle \underline{\nabla} \phi \rangle \right) \cdot \langle \underline{B} \rangle \right]$$

$\underline{B} \cdot \underline{\nabla} \langle \phi \rangle = 0$

$$= -2c \int d^3x \left[\langle \underline{E} \rangle \cdot \langle \underline{B} \rangle \right]$$

$$= -2c \int d^3x \left[\underbrace{\langle \underline{S} \rangle \cdot \langle \underline{B} \rangle}_{1} + \eta \underbrace{\langle \underline{J} \rangle \cdot \langle \underline{B} \rangle}_{2} \right]$$

Now, to conserve helicity, 1 must integrate to surface term, so:

$$\underbrace{\langle \underline{S} \rangle}_{1} = \frac{\langle \underline{B} \rangle}{\langle B \rangle^2} \, \underline{\nabla} \cdot \underline{\Gamma}_H \rightarrow \frac{\underline{B}}{B^2} \, \underline{\nabla} \cdot \underbrace{\underline{\Gamma}_H}_{\text{flux of helicity}}$$

$$\langle \underline{S} \rangle \cdot \langle \underline{B} \rangle = \frac{\cancel{\langle B \rangle^2}}{\cancel{\langle B \rangle^2}} \, \underline{\nabla} \cdot \underline{\Gamma}_H \quad \checkmark$$

[redacted]

so

$$\langle \underline{E} \rangle = \frac{\langle \underline{B} \rangle}{B^2} \underline{\nabla} \cdot \underline{\Gamma}_H + \eta \langle \underline{J} \rangle .$$

$\langle \underline{B} \rangle \rightarrow \underline{B}$

For the form of Γ_H, consider energy.

$$\partial_t \int d^3x \frac{B^2}{8\pi} = \int d^3x \frac{\underline{B}}{4\pi} \cdot \partial_t \underline{B}$$

$$= - \int d^3x \frac{\underline{B}}{4\pi} \cdot c \, \underline{\nabla} \times \underline{E}$$

$$= - \int d^3x \, \underline{E} \cdot \underline{J}$$

$$= - \int d^3x \left[\eta J^2 + \frac{\underline{J} \cdot \underline{B}}{B^2} \underline{\nabla} \cdot \underline{\Gamma}_H \right]$$

$$= - \int d^3x \left[\eta J^2 - \underbrace{\underline{\Gamma}_H}_{\text{flux}} \cdot \underbrace{\underline{\nabla} \left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right)}_{\text{Force}} \right]$$

akin $\quad \dfrac{dS}{dt} = \alpha \left(-\underline{\nabla} T \cdot \underline{\Gamma}_T \right) = \alpha D_T (\nabla T)^2$

apart η

$$\partial_t E_M = +\int d^3x \, \underline{\Gamma}_H \cdot \underline{\nabla}(J_\parallel / B)$$

so

$$\Gamma_H = -\underset{\text{param}}{\lambda} \nabla(J_\parallel / B) \quad \text{assures}$$

$$\partial_t E_M = -\int d^3x \; \underset{\text{param}}{\lambda} \left[\nabla(J_\parallel / B)\right]^2$$

energy dissipated

and

$$\boxed{\langle \underline{E} \rangle = \eta \langle \underline{J} \rangle - \frac{\underline{B}}{B^2} \underline{\nabla} \cdot \left[\lambda \, \underline{\nabla}\left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right) \right]}$$

and in simplified form:

$$\langle E_\parallel \rangle = \eta \langle J_\parallel \rangle - \underline{\nabla}_\perp \cdot \lambda \, \underline{\nabla}_\perp J_\parallel$$

$$\lambda \equiv \begin{cases} \text{hyper-resistivity} \\ \text{electron viscosity} \end{cases}$$

structurally;

$$\lambda = \frac{c^2}{\omega_{pe}^2} D_J \qquad , \qquad \eta = \frac{c^2}{\omega_{pe}^2} \nu_{ei}$$

$\lambda \equiv \mu$, previous

Where from D_J?

→ MHD
→ multi-field
→ stochastic field

Recall:

- S-P reconnection, with $E_\parallel = -\mu \nabla_\perp^2 J_\parallel$

$$V_R / V_A \sim 1/(S_\mu)^{1/4} \qquad S_\mu = \frac{V_A L^3}{\mu}.$$

- Derive D_J from ensemble stoch. field
(shifted Maxwellian → $J_\parallel(x)$).

⋮