Lecture 11 → Homogenization

→ Flux Expulsion and Homogenization – Non-Ideal, cont'd

– so far, have encountered:

→ S-P reconnection ⇒ weak dissipation ($R_m \gg 1$) has strong effect at singularity – BOUNDARY LAYER

→ Taylor Hypothesis ⇒ small flux tubes destroyed by stochasticity, leaving $\int d^3x\, \underline{A}\cdot\underline{B}$ as robust invariant

∴ diffusive dissipation most effective at breaking freezing-in on small scales

Another examples: { singular behavior in 2D closed-streamline flow

→ $\boxed{\text{Homogenization Theory}}$ → { Prandtl, Batchelor, Weiss, Rhines, Young

recall $\underline{\omega}$ evolution for $\underline{\nabla}\cdot\underline{v} = 0$

$$\frac{\partial}{\partial t}\underline{\omega} + \underline{v}\cdot\underline{\nabla}\,\underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v} + \nu\nabla^2\underline{\omega}$$

2D → $\underline{\omega}\cdot\nabla\underline{v} = 0$ $\qquad \omega = \omega\,\hat{z}$

$\qquad\qquad\qquad\qquad v = \nabla\phi\times\hat{z}$

c.f. P.A. Davidson 2004] – c.f.

A little viscosity ~~goes a long way~~ makes a global difference.

then $\partial_t \omega + \nabla\phi \times \hat{z} \cdot \nabla \omega = \nu \nabla^2 \omega$

more generally scalar q: $\left\{ \begin{array}{l} \text{active} \\ \text{or} \\ \text{passive} \end{array} \right.$

$$\partial_t q + \nabla\phi \times \hat{z} \cdot \nabla q = \nu \nabla^2 q$$

Now: $t \to \infty$, $\partial_t q \to 0$

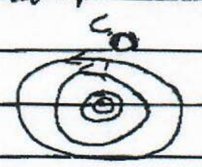
$$\nabla\phi \times \hat{z} \cdot \nabla q = \nu \nabla^2 q$$

$\nu \to 0$ $\qquad \nabla\phi \times \hat{z} \cdot \nabla q = 0$

i.e. $Pe \sim \frac{VL}{\nu} \to \infty$

$\sim Re$ $\qquad q = q(\phi)$

i.e. bounded domain, closed streamline solution

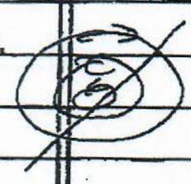


$\to q = q(\phi(r))$ is arbitrary solution

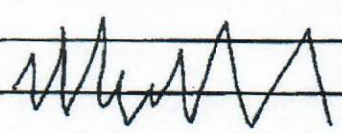
can develop arbitrarily fine scale $q(\phi)$ $\to$ closed streamlines $\Rightarrow$ perfect memory

$\Rightarrow$ $\therefore$ fine scale structure develops, no inter-streamline communication, & persists

i.e.



$\Rightarrow$



$\left\{ \begin{array}{l} \text{i.e. tag each} \\ \text{streamline} \\ \text{arbitrarily} \end{array} \right.$

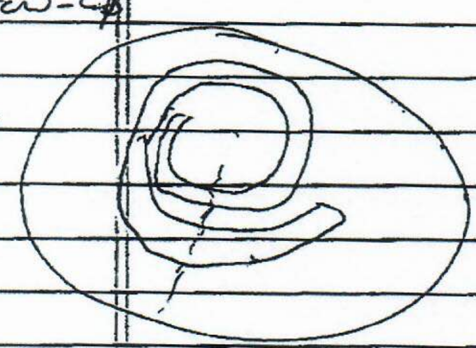
$\sim$ no smoothing of sharp gradients

"Not all solutions of the Navier-Stokes equations are realized in nature."

Landau & Lifshitz

→ i.e. of  → blob in concentric shear flow

blow-up



⇒ [non-diffusive stretching produces arbitrarily fine scale structure!

now, point is that for $\nu \neq 0$

$Re, Pe \gg 1$

instead of arbitrarily fine scale structure

must have: $\omega(\psi) \Rightarrow \text{const}$ as $t \to \infty$ { i.e. small ν ⇒ global behavior

⇒ i.e. finite ν at large Re ⇒ (P) vorticity homogenization! | $\omega \to$ const within C_0

⇒ highly singular behavior!

$\nu = 0$ ⇒ Euler Eqn. (2D) → { $\omega = \omega(\psi)$ solves

$\nu \neq 0$ ⇒ [large Re 2D Navier-Stokes Eqn. ⇒ $\omega = \text{const}$ solves

note contrast!

Issues :

→ how long to homogenization ? (what means asymptotic)

→ where is $\nabla\omega \neq 0$ ⇒ boundary layer thickness ?

→ analogy in MHD ? — Flux Expulsion

$\underline{E} + \frac{\underline{V}\times\underline{B}}{c} = \eta \underline{J}$ $\underline{V} = \nabla\phi \times \hat{z}$, $\underline{B} = \nabla A \times \hat{z}$

$$-\frac{1}{c}\partial_t \underline{A} - \nabla\phi + \frac{(\nabla\phi\times\hat{z})}{c}\times(\nabla A\times\hat{z}) = \eta \underline{J}$$

$\hat{z}\cdot(\ \ )$

$$\Rightarrow -\frac{1}{c}\partial_t A_z - \cancel{\nabla\phi\cdot\hat{z}} + \hat{z}\cdot\left[\cancel{(\nabla\phi\times\hat{z})\cdot\hat{z}}\right]\nabla A$$

$$-\frac{(\nabla\phi\times\hat{z})\cdot\nabla A}{c} = \eta \underline{J}$$

$$\therefore \quad \partial_t \underline{A} + \underline{\nabla}\phi\times\hat{z}\cdot\underline{\nabla} A = \eta\nabla^2 A$$

⇒ 2 D convection $\begin{cases} \nabla\cdot V = 0 \\ \eta \neq 0 \end{cases}$

⇒ expect $\nabla A = 0$, except boundaries
$t\to\infty$

→ analogy of "flux expulsion"

E. Prandtl - Batchelor Theorem

* G. Batchelor, JFM 1, 177 (1956) (posted)
P.B. Rhines and W.R. Young, JFM 122, 347 '82 (posted)
JFM 133, 133 '83
J. Pedlosky, "Ocean Circulation Theory"
see also Springer 1996, sec. 3.8.

Prandtl - Batchelor Theorem

Thm: Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline C_0. Then, if diffusive dissipation,

i.e. $\partial_t \omega + \nabla\phi\times\hat{z}\cdot\nabla\omega = \nabla\cdot(\nu\nabla\omega)$

then, vorticity → uniform (homogenization), as $t\to\infty$, within C_0.

N.B.: finite ν ⇒ radically different final state

no comment on "how long"?

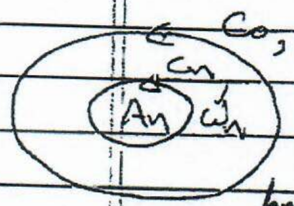
$f \to \infty$ before $\nu \to 0$

- $\nabla\phi \times \hat{z} \cdot \nabla\omega = \nabla \cdot \nu \nabla\omega$

for stationarity

[note $t \to \infty$ before $\nu \to 0$]

- choose arbitrary closed C_n within C_0.
  Here C_n a streamline



n.b. – assume simply connected region, i.e. no holes
– stationarity ⇒ ω constant along streamlines
– C_0 → specified on 1 bndry

∴ $\omega \to \omega_0$ on C_0 (ultimately C_0 sets b.c.)
$\omega \to \omega_n$ on C_n

if A_n is area enclosed by C_n,

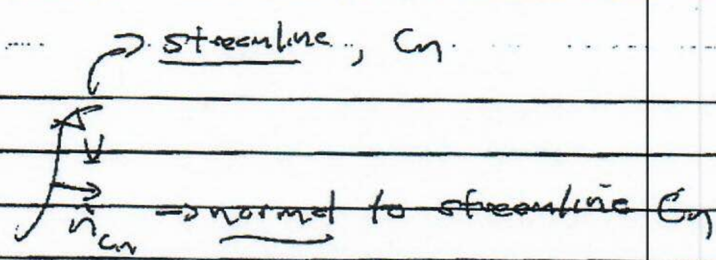
$$\int_{A_n} d^2x \; \underline{v}\cdot\nabla\omega = \int_{A_n} d^2x \; \nabla\cdot\left(\nu \nabla\omega\right)$$

but

$$\int_{A_n} d^2x \; \underline{v}\cdot\nabla\omega = \int d^2x \; \nabla\cdot[\underline{v}\omega]$$

$$= \int_{C_n} dl \; \underbrace{\hat{n}_{c_n}}_{\text{normal}} \cdot (\underline{v}\omega)$$

Now



so $$\int_{C_n} dl \, (\hat{n}_{cn} \cdot \underline{V}) \, \omega = 0$$

as $\underline{V}$ is along streamline

∴

$$0 = \int_{A_n} d^3x \, \nabla \cdot (\nu \nabla \omega)$$

$$= \nu \int_{C_n} dl \, \hat{n}_{cn} \cdot \nabla \omega$$

now, in stationary state, must have [redacted]
$\omega \rightarrow$ const along streamline

∴ $\omega = \omega(\phi)$

so $\omega_{cn} = \omega(\phi_n)$

$$0 = \nu \int_{C_n} dl \, \hat{n}_{cn} \cdot \nabla \phi_n \frac{\delta \omega}{\delta \phi_n}$$

$$0 = \nu \frac{\delta \omega}{\delta \phi_n} \int_{C_n} dl \, \hat{n}_{cn} \cdot \nabla \phi_n$$

but $\Gamma = \int d\underline{l} \cdot \underline{v}$

$= \int d\underline{l} \cdot (\nabla\phi \times \vec{z})$

$= \int (\vec{z} \times \vec{n}) \cdot (\nabla\phi \times \vec{z})$

$= -\int dl \, (\nabla\phi \cdot \vec{n}) = -\int dl \, (\nabla\phi \cdot \vec{n})$

$\therefore$

$0 = \nu \frac{\delta \omega}{\delta \phi_n} \Gamma_n$

$\therefore$

$\delta\omega / \delta\phi_n = 0$

but ϕ_n arbitrary $\Rightarrow$ $\boxed{\delta\omega/\delta\phi = 0 \, , \, \text{all } \phi}$

arbitrary $\Rightarrow$ no variation from line to line

$\Rightarrow$ $\boxed{\boxed{\omega \text{ homogenized}}}$!

C fluid boundary stokes

so, expect $\nabla\omega$ larger at bounding contour C_0.

$\nabla\omega \to 0$, within $\Rightarrow$ $\nabla\omega$ held at boundary

Some Comments:

⇒ Homogenization theory looks 'magical' → caveat emptor!

i.e.

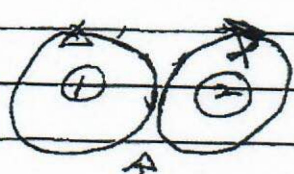
* 1.) note assumptions of:
  - $t \to \infty$ ⇒ time asymptopia
  - $q = q(\phi)$ ⇒ concentric streamlines

how long to achieve configuration?!

2.) simply connected domain ⇒ annulus? (2F!)

3.) single structure → expulsion from neighbors and possible interaction not addressed

i.e. what happens if? → 'interference' of boundary layers?



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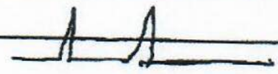


⇒ straining interaction

⇒ ② 'strips' ①, etc.

4.) Key Assumptions:

→ { closed, bounding streamline
  { viscous dissipation

i.e. can envision:

→ [exact streamline, molecular viscosity
or
→ [coarse-grained streamline, eddy viscosity

⇒ correspond to homogenization of
  → ~~total vorticity~~
  → mean/coarse-grained vorticity

⇒ time scales different

→ $\boxed{\dfrac{\tau_{circulation}}{\tau_{diffusion}} \ll 1 \Rightarrow R_e \gg 1}$ ← L →

then
- to establish concentric circulation lines
- diffuse across to homogenize → but slow!!

$$\frac{\tau_c}{\tau_d} = \frac{1}{(V/L)} \frac{D}{L^2} \ll 1 \Rightarrow \frac{D}{VL} \ll 1$$

i.e. $R_e \gg 1$

or equivalently $\frac{VL}{D} \gg 1$

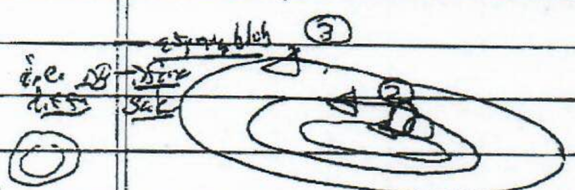
i.e. $(Re)_{eff\ cell} \gg 1$ $\quad Pe \gg 1$

related: – essential idea is that q constant along streamlines established on fast ($\sim \tau_c$) scale
few

– dissipation homogenizes on slower ($\sim \tau_D$) time scale
(but this is slow . . .)

→ What are the time scales? – how resolve slow time scale problem

– useful to consider differentially rotating sheared flow within closed pattern



$V_1 \neq V_2 \neq V_3$

need blob with finite l_y

what is the mixing time scale?

finer, finer scale . . .

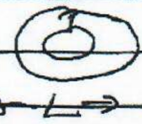
key: synergism between $\left\{ \begin{array}{l} \text{shear} \\ \text{diffusion} \end{array} \right.$!

$\boxed{\text{shear dispersion}}$

c.f. {H. Biglari, P.H. Diamond, P.W. Terry, Phys Fluids (B2), 1, 1990
(first noted by G.I. Taylor)

Mixing — Shear Dispersion

i.e. compare



$\leftarrow L \rightarrow$

time

(a) pure diffusion

$1/\tau \sim D/L^2$ $\quad \boxed{\langle \delta r^2 \rangle \sim D t}$ → usual diffusion

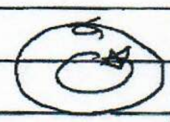
radial diffusion

$D_r \sim \tilde{V}_\theta^2 \tau_c$

any radial scattering process (unspecified)

(b) diffusion + shear hybrid

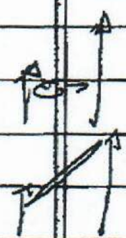
with



$V' \sim V/L$

shear

now $\frac{dr}{dt} = \tilde{V}_r$ → random walk



$r\theta = y$

$\frac{dy}{dt} = V_y(r)$ → streaming

shear

$\therefore \frac{d}{dt}\delta y = \left(\frac{\partial V_y}{\partial r}\right)\delta r$

$\delta y = \int \left(\frac{\partial V_y}{\partial r}\right)\delta r \, dt$

$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_y}{\partial r}\right)^2 \langle \delta r^2 \rangle t^2$

$\langle \delta r^2 \rangle \sim D_r t$

$\Rightarrow \boxed{\langle \delta y^2 \rangle \sim \left(\frac{\partial V_y}{\partial r}\right)^2 D_r t^3}$ →

shear dispersion

hybrid decorrelation

$\langle \delta y^2 \rangle \sim t^3$

scale of consideration

$$\langle \delta y^2 \rangle \sim L_y^2 \Rightarrow \underline{\text{arbitrary}}$$

$$1/\tilde{\tau}_{mix} = \left(\left(\frac{\partial V_y}{\partial r}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \left(\left(\frac{V_0}{L_0}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \frac{V_0}{L_y} \left(\frac{D}{V L_0}\right)^{1/3}$$

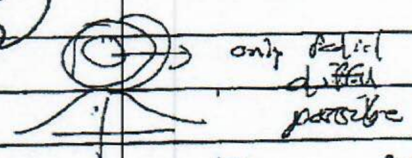
$$\therefore \quad \boxed{1/\tau_{mix} \sim \frac{1}{\tau_0} \underline{(Re)}^{-1/3}}$$

{ $Re \gg 1$ by construction → consistent ✓

so have:

→ mixing / homogenization on hybrid time scale → time to come to ⊙ symmetric state

$$1/\tau_{mix} = 1/\tau_c \left(\tau_c / \tau_D \right)^{1/3}$$



$$\Rightarrow \frac{1}{\tau_c} > \frac{1}{\tau_{mix}} > \frac{1}{\tau_D}$$ time to uniformize vs τ_D

⇒ $\boxed{\text{PV homogenization most relevant to closed eddys with sheared rotation}}$

Some Points

i.) Time scales

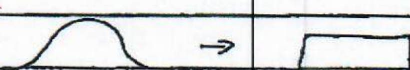
have Re, Pe $\gg 1$ $\Rightarrow$ $\frac{\tau_D}{\tau_c} \gg 1$

but

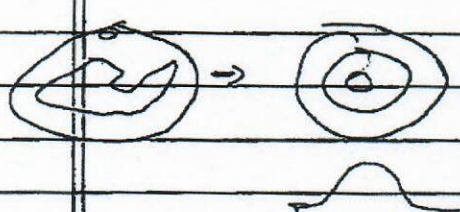
$\tau_{mix} < \tau_D$ $\qquad \tau_{mix} \sim Re^{1/3}\tau_c \sim \frac{\tau_D}{Re^{2/3}}$

$\downarrow$ $\rightarrow$ time to homogenize

time to establish ~ azimuthally symmetric state



i.e



but radially profiled

Compare with NS

ii.) $\boxed{\text{Point of theorem is global impact of small dissipation.}}$

iii.) Interesting to note that P-B theorem applies to both active, passive scalar.

ii) Observe: all that is really required for applicability of theory is:

- → incompressible advection – 2D: $\nabla\phi\times\hat{z}\cdot\nabla$
- → closed streamline → $\left\{\begin{array}{l}\text{fine}\\ \text{coarse}\end{array}\right.$
- → diffusive dissipation → $\left\{\begin{array}{l}\text{molecular}\\ \text{eddy}\end{array}\right.$

i.e. can apply to magnetic potential, as noted previously, i.e.

2D M.H.D. $\underline{U} = \nabla\phi\times\hat{z}$, $\underline{B} = \nabla A\times\hat{z}$

$$\frac{\partial A}{\partial t} + \underline{V}\cdot\nabla\underline{A} = \eta\nabla^2 A$$

$E + \underline{v}\times B = \eta J$

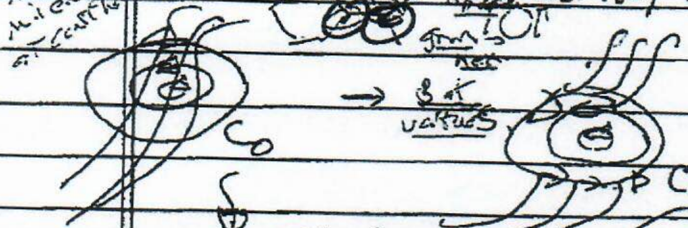
$-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi + (\nabla\phi\times\hat{z})\times(\nabla A\times\hat{z}) = \eta J$

$\hat{z}\left\{\left(\frac{-1}{c}\frac{\partial A}{\partial t}\right) + (\nabla\phi\times\hat{z})\cdot\hat{z} - (\nabla\phi\times\hat{z}\cdot\nabla A)\hat{z}\right\} = \eta J$

Why? → ~~sun spots~~

→ Famous problem of Flux Expulsion (i.e. N. Weiss Proc. Roy Soc., Series A 293 310 1966)

Act Mil eq... fix



→ $\nabla A \to 0$ within

→ B expelled to boundary (i.e. $B = \nabla A\times\hat{z}$)

→ $B \sim 0$ in cell.

specified convective cell (hat magnetoconvection as aim)

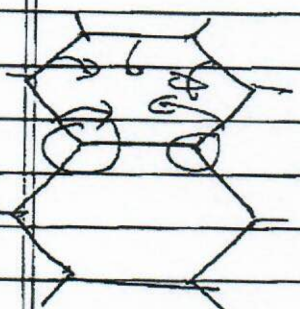
see 15a

~~Outcome~~ of Magneto Convection T_B

→ showing that above argument can be recycled, so

$\delta A_n/\delta\phi_n = 0$, for all C_n in C_0, treat kinematically → why?

i.e. top view → solar granulation

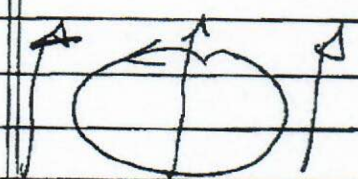


~ hexagonal pattern

~ field strongest at $\boxed{\text{vertices}}$

result of (~ expulsion

suggests → side view → toy problem of flux expulsion



→ i.e.

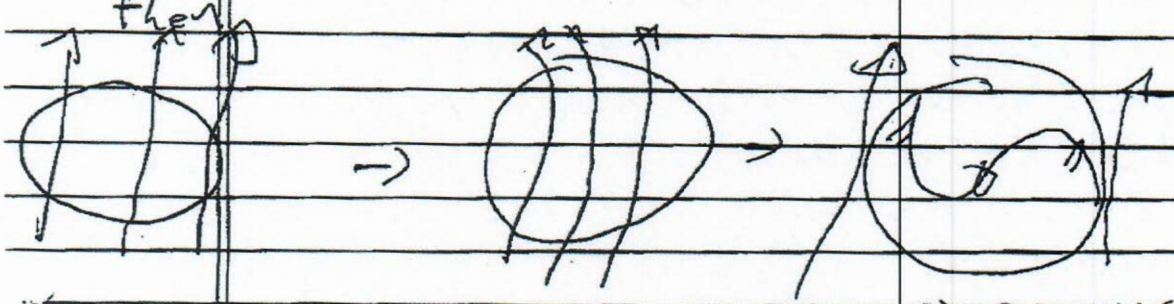
what happens?

→ cell linked by field

cell in uniform field

what happens

then



see Weiss, Proc. Roy Soc A 293, 310. 1966

also Moffatt, 3.7 – 3.10

→ here, requirement is $R_m = \frac{LV}{\eta} \gg 1$

and

$$1/\tau_{homog} \sim 1/\tau_{mix} \sim \frac{V_0}{L_0}(R_m)^{-1/3}$$

↓

time scale for flux expulsion

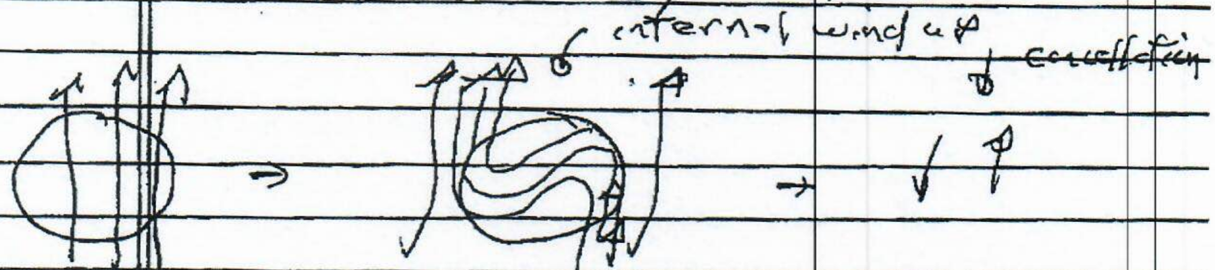
→ physically, can see relevant time scale by noting:

→ [redacted] wind up must conserve Flux volume/mass

→ $\boxed{\text{wind up must conserve flux}}$

→ $\boxed{\text{irreversibility sets in when } R_m|_{local} \sim 1}$

⇒ dissipation of field matches drive by wind-up



internal cancellation → { field expulsion, flux homogenization } so $B_0 L \sim \ell B$ (B ↑ upon flux conservation/winding)

$\Rightarrow$ $B \sim n B_0$

wind-up = dissip'n
in out

$\Rightarrow$ now, expect freezing-in lost when compressed field / amplified

$(R_m)_{eff} \sim 1 \Rightarrow \frac{V B_0}{L_0} \sim \eta \frac{B}{\ell^2} \sim \eta \frac{n B_0}{L_0^2/n^2}$

$\Rightarrow$

$n^3 \sim \left(\frac{V_0 L_0}{\eta}\right) \Rightarrow \boxed{n \sim R_m^{1/3}}$

\# of turns to render boundary diffusive

not $R_m^{1/2}$
[CHKD not realize field amplified]

{ ~~thickness of layer~~
$\frac{V}{L} B \sim \frac{\eta B}{\delta^2}$ $\frac{\delta}{L} \sim 1/\sqrt{R_m}$

but: – diffusion in boundary layer ⇔ homogenization within

so

– $n \sim R_m^{1/3}$ $\Rightarrow$ # turns for homogenization

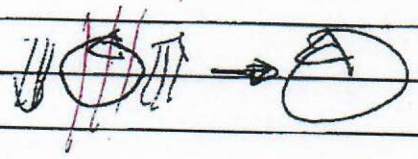
– $\boxed{\tilde{\tau}_{hom} \sim \tilde{\tau}_c R_m^{1/3}}$ → time ✓

agree above

⇒ expect flux expelled from closed cell
→ field stagnant at boundary → spots

→ possibly explain why stagnant cells in 2D

magneto convection often
reach a state independent of B
while neighbors quenched.



Magnetoconvection = convection in B field

Now: ⇒ why care about this?
Why homogenization important
→ similarity phenomenon

① identifies a trend, i.e. in spirit of
Taylor Theory (E_{mag} minimized s/t
$\int \underline{A} \cdot \underline{B}\, d^3x$ conserved), homogenization
theory identifies a trend, i.e.

Relax. Principle

if F [
- conserved locally by 2D flow, $\nabla \cdot v = 0$
- diffused
- enclosed

⇒ F homogenized

strong B_0
→ ~~[illegible]~~
Alfven

② trend applies to
- ~~non-passive~~ → vorticity
- passive → $A_{\parallel}$ ...

In particular,

③ trend severely constrains form of **vorticity flux**, flow evolution

i.e. zonal flows → 2D closed streamline flows ....

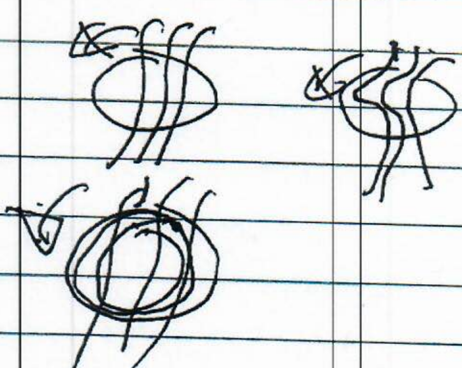
→ Do zonal flows tend homogenize PV ?

→ if noise (i.e. emission), what scale selected ?

→ When does kinematic expulsion end?

(MGH)

~ qualitatively, when magnetic tension grows to point of competition with vortex evolution.



Now, $b\ell = B_0 L_0$

$$\eta \frac{b}{\ell^2} = \frac{B_0 U_0}{L_0}$$

so $$\eta \frac{b}{\left(\frac{B_0 L_0}{b}\right)^2} = \frac{B_0 U_0}{L_0}$$

$$\Rightarrow \boxed{b \sim B_0 \left(\frac{U_0 L_0}{\eta}\right)^{1/3}} \quad \rightarrow \frac{b}{B_0} \sim \frac{L_0}{\ell} \sim R_m^{1/3}$$

For tension:

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = -\frac{|B|^2}{r_c}\hat{n} + \frac{d}{ds}\overbrace{\left(\frac{|B|^2}{2}\right)}^{\text{stretched field}}\hat{t}$$

$$|\underline{B}\cdot\underline{\nabla}\,\underline{B}| \approx \frac{b^2}{L_0} \qquad \begin{cases} r_c \sim L_0 \\ \frac{d}{ds} \sim L_0^{-1} \end{cases}$$

then:

$$\rho \frac{d\omega}{dt} = \left[\nabla \times (\underline{B}\cdot\nabla\underline{B})\right]\cdot\hat{z}$$

picks up **short** lengthscale

$$\rho\, u\cdot\nabla\omega \sim \frac{b^2}{\ell L_0}$$

$$\frac{u}{L_0}\omega \sim \frac{b^2}{\rho_0 L_0 \ell}$$

but

$$\frac{u}{L_0}\omega \sim \frac{u^2}{L_0^2} \sim \frac{b^2}{\rho L_0 \ell}$$

$$\sim \frac{B_0^2 L_0^2}{\ell^3 L_0 \rho_0}$$

$\therefore$ for given L_0, critical B_0 to prevent expulsion:

$$\frac{u^2}{L_0^2} \sim \frac{B_0^2 L_0}{\ell^3 \rho_0} \Rightarrow \left(\frac{V_{A0}}{u}\right)^2 \sim \left(\frac{L_0^3}{\ell^3}\right)^{-1}$$

$$\frac{b}{B_0} \sim \left(\frac{u_0 L_0}{\eta}\right)^{1/3} \sim R_m^{1/3} \qquad \sim R_m^{-1}$$

so $$\boxed{R_m\left(\frac{V_{A0}}{u}\right)^2 \sim 1}$$

is feedback criterion!

→ akin feedback criterion for magnetic flux transport in 2D

→ Can easily have $V_{A0} \ll u$ with $R_m \gg 1$, and still encounter feedback.

⟺ weak field is sufficient.