with a Doppler profile toward the low-frequency region. As before, part of the radiation occurs at frequency $\nu_0$; it is connected with the continuing operation of the maser. The intensity of this component falls since the Compton effect rapidly transfers the photons which are produced into the drifting line. When the line has shifted by an amount of order $\nu_{Di}$, the possibility arises of forming a new line close to the frequency $\nu_0$ and the process is repeated. It is assumed that the maser operates in a highly saturated regime so that the radiation at frequency $\nu_0$, even after being reduced, lies above the threshold: $n(\nu_0) \gg \bar{n}_H$ (the general case is treated in [98]). The number of photons in the moving line increases linearly with time until a shift by a distance of order $\nu_{Di}$ is achieved:

$$\left.\begin{aligned} n(\nu, t) &\approx \frac{B\bar{n}_H t}{\sqrt{2\pi}\Delta_M} \exp\left[-\frac{(\nu-\nu_*)^2}{2\Delta_M^2}\right], \\ \nu_* &= \nu_0 - \Delta_M^2 \nu_0^2 K'(0) B\bar{n}_H t^2/2. \end{aligned}\right\} \tag{3}$$

We now introduce special notation for the radiation flux density at frequency $\nu_0$ for which the time of stimulated scattering on plasma protons $t_c^{-1} = \nu_0^2\Delta_M^2 K'(0)n_*$ is comparable with the time for the exponential growth of the emission $t_M$ (obviously for an unsaturated maser)

$$I_* \equiv 8\pi h\nu_0^3 n_* \Delta_M \approx \frac{T_{0i}}{t_M \sigma_T} \cdot \frac{\nu_{Di}/\Delta_M}{N_{0i}\lambda^3}; \quad t_M = \sqrt{2\pi}\Delta_M/B; \quad \lambda = c/\nu. \tag{4}$$

Then during the time in which the line drifts,

$$t_R = \left[\frac{2\nu_{Di}}{\nu_0^2 \Delta_M^2 K'(0) B\bar{n}_H}\right]^{1/2} \approx \left(\frac{\nu_{Di}}{\Delta_M}\right)^{1/2} \left(\frac{n_*}{\bar{n}_H}\right)^{1/2} t_M, \tag{5}$$

the drifting line increases to the limiting value

$$n = n_H t_R/t_M \approx (n_H n_*)^{1/2} (\nu_{Di}/\Delta_M)^{1/2}; \quad \bar{n}_H < n_* \left(\frac{\Delta_M}{\nu_{Di}}\right)^{3/2} \tag{6}$$

The brightness temperature is related to the limiting value of the occupation number by the relation $T_b \sim h\nu n$.

▶ 2. Compute the pressure of the stimulated light in a plasma in the case of a wide radiation spectrum [105]; cf. also [98].

This force can be computed from the quasilinear equation (3.36) or from the loss of momentum of the electromagnetic waves.



In the case of a wide radiation spectrum, we multiply Eq. (3.43) by $\nu\mathbf{l}/c$ and integrate over phase space to obtain

$$F_j = \frac{3h^2 N_{0e}\sigma_T}{32\pi mc^3} \int d^2\mathbf{l}\, d^2\mathbf{l}' \int \nu^2 d\nu \Big\{\nu^2 n(\nu, \mathbf{l}, t)\, w_j(\nu, \nu; \Theta) \times$$
$$\times n(\nu, \mathbf{l}', t)\, 1 + n(\nu, \mathbf{l}, t)\Big\} \left[\frac{\partial}{\partial\nu'} \nu'^2 w_j(\nu, \nu', \Theta)\, n(\nu', \mathbf{l}', t)\right]_{\nu'=\nu} \nu(\mathbf{l}' - \mathbf{l}).$$

## Chapter 4

## ANOMALOUS RESISTIVITY IN A PLASMA

### § 4.1. Formulation of the Problem. Conservation Relations

In the first three parts of this monograph we have described nonlinear interactions between waves and waves and between waves and particles; all of these processes can be realized in a plasma which, as a consequence of an instability, goes from a laminar state to a turbulent state. The macroscopic consequences of this change in the state of the plasma are represented by a change in its transport properties (the transport coefficients) such as diffusion, thermal conductivity, electrical resistance, etc. Under these conditions one speaks of anomalous transport coefficients. The basic problem of the theory then is to relate the values of these anomalous transport coefficients to the underlying cause which produces the original instability (in other words, with the source of "free energy" which drives the instability).

The anomalous electrical resistivity is the most important example of a problem of this kind. In this chapter we shall show how the methods of the theory of plasma turbulence developed in Chapters 1-3 are applied to this problem.

The anomalous resistivity of a plasma usually arises when the magnitude of the electrical current that flows in the plasma exceeds some critical value. Sometimes this critical value, above which the plasma resistivity changes abruptly, is extremely small. The density of the flowing current is expressed in terms of the so-called drift velocity $\mathbf{V}_d$. If the electron distribution function is characterized by some velocity $\mathbf{V}_d$ with respect to the ion distribution function, and if this velocity exceeds the critical value, then an instability can arise. When this instability does arise, in addition to losing momentum by binary collisions, electrons also lose momentum because of the interaction with oscillations and waves of various kinds. It is convenient to start with a table of relevant instabilities which arise when a critical velocity is exceeded (cf. Table 1). In this table we have listed all of the basic instabilities which bear on the problem of anomalous resistivity in a plasma. The simplest instability is the so-called Buneman instability [106, 107]. In this case the original distribution functions for the electrons and ions are two $\delta$-functions which are shifted with respect to each other by the mean velocity $\mathbf{V}_d$. The instability is manifest in the excitation of longitudinal electrostatic plasma oscillations with a growth rate of the order of the ion-plasma frequency. The well-known dispersion equation for the Buneman instability is

$$1 - (\Omega_p^2/\omega^2) - \omega_p^2/(\omega - \mathbf{k}\mathbf{V}_d)^2 = 0. \tag{4.1}$$

The growth rate is

$$\operatorname{Im}\omega \approx \mathbf{k}\mathbf{V}_d\,(m/M)^{1/2} \lesssim \Omega_p \tag{4.2}$$

when $\mathbf{k}\mathbf{V}_d \ll \omega_p$ and reaches a maximum value

$$\operatorname{Im}\omega = \left(\frac{M}{4m}\right)^{1/6}\Omega_p \tag{4.3}$$

when $\mathbf{k}\mathbf{V}_d \approx \omega_p$.

Another example of an instability which arises in practice is the ion-acoustic instability. This instability appears when

| Instability | Threshold | Frequency | Growth rate |
|---|---|---|---|
| Buneman | $V_d \geqslant v_{Te}$ | $\sim \Omega_p$ | $\sim \Omega_p$ |
| Ion-acoustic | $V_d > c_s$ | $\lesssim \Omega_p$ | $\leqslant \Omega_p \dfrac{V_d}{v_{Te}}$ |
| Drummond–Rosenbluth | $V_d > c_s$ | $\sim \Omega_H$ | $\sim \Omega_H \dfrac{V_d}{v_{Te}}$ |
| Electric mode $k_\perp^2 \gg k_\parallel^2$ | Very low, sometimes $< v_{Ti}$ | $\ll \omega_H$ | $\sqrt{\omega_H \Omega_H}$ |
| Bernstein mode | | $l\omega_H$ | $\omega_H V_d / v_{Te}$ |



the electron drift velocity is smaller than the thermal velocity. The dispersion relation for the ion-acoustic instability is

$$0 = 1 - \frac{\Omega_p^2}{\omega^2} + \frac{\Omega_p^2}{k^2 c_s^2}\left\{1 + i\left(\frac{\pi m}{2M}\right)^{1/2}\left(\frac{\omega}{kc_s} - \frac{\mathbf{k}\cdot\mathbf{V}_d}{kc_s}\right)\right\}, \tag{4.4}$$

and the growth rate is

$$\gamma \approx \omega\sqrt{\frac{\pi m}{8M}}\left(\frac{\omega}{kc_s} - \frac{\mathbf{k}\mathbf{V}_d}{kc_s}\right)\Big/\left(1 + \frac{k^2 c_s^2}{\Omega_p^2}\right). \tag{4.5}$$

The growth rate for the ion-acoustic instability (the imaginary part of the frequency) is the ion-plasma frequency reduced by a factor equal to the ratio of the electron drift velocity to the electron thermal velocity. In the limiting case $V_d \to v_{Te}$, the ion-acoustic instability goes over into the Buneman instability.

In the presence of a magnetic field a number of new instabilities can appear. One of the instabilities which is also a consequence of the imaginary part of the electron term (the electron pole) in the ion-cyclotron mode is called the Drummond–Rosenbluth instability [108]. This instability arises when the current flows along the magnetic field, whereas the first two instabilities that have been considered are not affected by a magnetic field if the field is reasonably small ($\omega_H \ll \omega_p$). The Drummond–Rosenbluth instability is usually not discussed in connection with anomalous resistivity because it is characterized by a small growth rate and is evidently suppressed by simple quasilinear effects such as the formation of a plateau.

A more important role is played by a class of instabilities associated with electrostatic perturbations for which the wave vector along the magnetic field is much smaller than the transverse component of the wave vector and for which the frequency is much smaller than the electron gyrofrequency, but larger than the ion gyrofrequency. This mode is reminiscent of the well-known Post–Rosenbluth mode which arises in the presence of a loss cone [109]:

$$1 + \frac{\omega_p^2}{\omega_H^2} + \frac{\Omega_p^2}{k^2}\int \frac{\mathbf{k}\cdot\frac{\partial f_i}{\partial \mathbf{v}}\,d^3\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v} + i0} + \frac{\omega_p^2}{k^2}\int \frac{k_\parallel \frac{\partial f}{\partial v_\parallel}\,\partial v_\parallel}{\omega - \mathbf{k}\cdot\mathbf{V}_d - k_\parallel v_\parallel + i0} = 0. \tag{4.6}$$

In the approximation in which $\omega \gg k v_{Ti}$ and $\mathbf{k}\mathbf{V}_d \gtrless k_\parallel v_{Te}$, Eq. (4.6) becomes the dispersion equation for the so-called modified

Buneman instability [110]

$$1+\frac{\omega_p^2}{\omega_H^2}-\frac{\Omega_p^2}{\omega^2}-\frac{\omega_p^2 k_\parallel^2/k^2}{(\omega-\mathbf{k}\cdot\mathbf{V}_d)^2-k_\parallel^2 v_{Te}^2}=0. \qquad (4.7)$$

The growth rate is

$$\gamma(\mathbf{k})=\sqrt{\omega_H\Omega_H}\ll\Omega_p;\quad \bar{k}r_H\approx 1;\quad \bar{k}_\parallel=\bar{k}\,(V_d/v_{Te}). \qquad (4.8)$$

The approximation in (4.7) is valid if the drift velocity is much greated than $v_{Ti}$. If this condition is not satisfied then we are dealing with the so-called electron-acoustic instability ($\omega_p \gg \omega_H$):

$$\mathrm{Re}\,(\omega-\mathbf{k}\cdot\mathbf{V}_d)\approx k_\parallel\left(\frac{T_i}{m}\right)^{1/2}\Big/\,1+k^2r_{H*}^2,\quad \mathrm{Im}\,\omega=\pi^{1/2}\frac{\omega}{2\,|\mathbf{k}|\,v_{Ti}}(\omega-\mathbf{k}\mathbf{V}_d). \qquad (4.9)$$

An instability of this kind arises when the current flows across the magnetic field. The instability in (4.7) has a very small growth rate and is important only at comparatively small currents, in which case the stronger instabilities such as the Buneman instability or the ion-acoustic instability are not excited.

Finally, recent interest has been given to an instability associated with Bernstein modes [111]. This instability is characterized by a relatively large growth rate and arises when the current flows across the magnetic field. The dispersion relation for this instability is rather complicated.

Up to the present time, primary attention has been given to two kinds of instabilities; the Buneman instability and the ion-acoustic instability. In his first paper Buneman proposed a heuristic expression for the nonlinear stage of the instability. He proposed that the effective collision frequency for the electrons should be of the order of the imaginary part of the frequency as determind from the linear theory, that is to say, the order of the ion-plasma frequency. This simple formulation, in which the ion-plasma frequency appears in place of $\nu_{eff}$ in Ohm's law, is called the Buneman conductivity. It will be clear that this formulation cannot give a very accurate description of experimental results; it can only give the appropriate order of magnitude.

A rigorous formulation of the problem of determining the conductivity $\sigma$ must be carried out taking account of the exchange



of momentum between the electrons and the waves. The well-known expression for the plasma conductivity

$$\sigma=Ne^2/m\nu \qquad (4.10)$$

contains $\nu$, the frequency of collisions of electrons with scattering centers (ions, neutrals) in terms of the loss of momentum. If the plasma electrons excite some kind of oscillation or wave as a consequence of the instability, there will be an anomalous loss of momentum (transfer by the oscillations, i.e., collective ion motion). In order to find $\nu_{eff}$ we can use the conservation of momentum for the system consisting of the electron and the wave. The mean momentum loss of the electrons per unit time is

$$\nu_{eff}mN\mathbf{V}_d\approx-\mathbf{F}. \qquad (4.11)$$

If this momentum is transferred to a wave with energy density W, then the change in the wave momentum is

$$\int\gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}, \qquad (4.12)$$

where $\gamma_{\mathbf{k}}^e$ is the electron contribution in the imaginary part of the frequency. Equating (4.11) and (4.12), we have

$$\nu_{eff}mN\mathbf{V}_d\approx\int\gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}, \qquad (4.13)$$

i.e.,

$$\nu_{eff}=\frac{1}{mNV_d}\int\gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}. \qquad (4.14)$$

Thus, the problem reduces to finding $W_k$; the quantity $\gamma_k^e$ is to be understood in the quasilinear sense.

The validity of Eq. (4.13) can be demonstrated through the use of the quasilinear diffusion equation for the electrons. For example, with ion-acoustic oscillations,

$$\frac{\partial f_e}{\partial t}=\frac{e^2}{m^2}\int\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\,|\Phi_{\mathbf{k}}|^2\,\pi\delta(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})\,\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\cdot\frac{\partial^3\mathbf{k}}{(2\pi)^3}. \qquad (4.15)$$

Multiplying this equation by mv and integrating over velocity we have

$$\nu_{\text{eff}}\, mN\mathbf{V}_d = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\gamma_{\mathbf{k}}^e\, \frac{\partial\varepsilon(\omega)}{\partial\omega_{\mathbf{k}}}\cdot\frac{k^2\,|\Phi_{\mathbf{k}}|^2}{8\pi}\,\mathbf{k} \tag{4.16}$$

since $W_{\mathbf{k}} = (\partial\omega\varepsilon/\partial\omega)\,(k^2|\Phi_{\mathbf{k}}|^2/8\pi)$, and the validity of Eq. (4.13) is demonstrated.

The existence of an anomalous resistivity leads to the anomalous generation of Joule heat in a plasma $j^2/\sigma_{an}$. This kind of plasma heating is frequently called *turbulent heating* since the mechanism responsible for the anomalous resistivity of the plasma is the turbulence associated with the instability. In the absence of binary collisions the turbulent heating is different for the electron and ion components of the plasma. Furthermore, it is not really valid to talk about increasing the "temperatures" of the electrons and ions if temperature is understood in the traditional sense (a Maxwellian particle distribution). Under the plasma conditions described here the term temperature is usually taken to mean the mean random energies of the components.

As a rule the electron temperature increases more rapidly in turbulent heating of a plasma. It is possible to establish a simple criterion which relates the rate of electron heating to the rate of ion heating. The derivation of this criterion is based on the conservation of momentum and energy in the interaction of electrons and ions with the waves. As we have seen, the electrons in the plasma experience a frictional force

$$\mathbf{F} = -\nu_{\text{eff}}\, Nm\mathbf{V}_d. \tag{4.17}$$

The work performed by this force goes into heating the plasma electrons:

$$\frac{d\mathscr{E}_e}{dt} \sim \nu_{\text{eff}}\, mNV_d^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\gamma_{\mathbf{k}}^e\, W_{\mathbf{k}}\, \frac{(\mathbf{k}\cdot\mathbf{V}_d)}{\omega_{\mathbf{k}}}. \tag{4.18}$$

In the stationary saturated state, which is reached when the growth of the instability is limited by nonlinear effects, the momentum of the waves (along with their energy) is transferred to the ions. Thus, in the saturated state the ions must absorb the oscillation energy at a rate of order $\int\gamma_{\mathbf{k}}^e W_{\mathbf{k}}\, d^3\mathbf{k}$. As a result, the ion heating proceeds at a rate



$$\frac{d\mathscr{E}_i}{dt} \sim \int \gamma_{\mathbf{k}}^e\, W_{\mathbf{k}}\, \frac{d^3\mathbf{k}}{(2\pi)^3}. \tag{4.19}$$

Now let us divide Eq. (4.18) by Eq. (4.19):

$$\frac{d\mathscr{E}_e}{d\mathscr{E}_i} \sim \int \gamma_{\mathbf{k}}^e\, W_{\mathbf{k}}\, \frac{(\mathbf{k}\cdot\mathbf{V}_d)}{\omega_{\mathbf{k}}}\, d^3\mathbf{k} \Big/ \int \gamma_{\mathbf{k}}^e\, W_{\mathbf{k}}\, d^3\mathbf{k}. \tag{4.20}$$

If we write

$$\int \gamma_{\mathbf{k}}^e\, W_{\mathbf{k}}\, \frac{(\mathbf{k}\cdot\mathbf{V}_d)}{\omega_{\mathbf{k}}}\, d^3\mathbf{k} \approx \frac{kV_d}{\omega_{\mathbf{k}}} \int \gamma_{\mathbf{k}}^e\, W_{\mathbf{k}}\, d^3\mathbf{k}$$

in Eq. (4.20), the ratio of the rate of electron heating to the rate of ion heating is easily obtained from the following estimate [112]:

$$d\mathscr{E}_e/d\mathscr{E}_i \sim V_d/\omega/k. \tag{4.21}$$

In the form in which it has been obtained here this relation is independent of the nature of the instability, and is thus a universal relation. For most instabilities this relation leads to a more rapid heating of the electrons. Thus, in the ion-acoustic and Buneman instabilities $V_d \gg \omega/k$ so that $d\mathscr{E}_e/d\mathscr{E}_i \gg 1$. The ratio $d\mathscr{E}_e/d\mathscr{E}_i$ is especially large [of order $(M/m)^{1/2}$] for the Buneman instability.

It is desirable to write Eq. (4.14) in a more useful form in dealing with the ion-acoustic and Buneman instabilities. For this purpose, in Eq. (4.14) we substitute the well-known value for the maximum growth rate of the ion-acoustic instability (4.5). The maximum growth rate obtains when $k \sim \lambda_D^{-1}$. Making the substitution $\gamma_{\mathbf{k}}^e \approx \omega V_d/v_{Te}$, we obtain the following relation:

$$\nu_{\text{eff}} = \omega_p W/N_0 T_e. \tag{4.22}$$

Thus, a knowledge of the energy density of the waves W in the saturated regime of the instability can be used to find $\nu_{\text{eff}}$. The quantity W in nonlinear plasma theory can be obtained by the usual methods of *weak plasma turbulence*. However, this method does not always apply. Even the simplest case of the Buneman instability must be treated from the viewpoint of strong turbulence. However, the available theories for strong turbulence can only hope to give estimates of orders of magnitude. In the case of the Buneman instability an estimate of this kind can be made, as an example, using the following approach. The ratio of the ion energy and electron energy is written in the form $\mathcal{E}_i/\mathcal{E}_e \sim \omega/kV_d$, where $\mathcal{E}_e \sim m\langle v^2\rangle/2$, while the energy density of the waves $W \lesssim \mathcal{E}_i$. Since $\omega/k \approx \sqrt{m/M}\langle v\rangle$, using Eq. (4.14) we find $\nu_{eff} \sim \Omega_p$.

## § 4.2. Anomalous Resistivity Due to the Ion-Acoustic Instability

The ion-acoustic instability furnishes a convenient example for the application of the methods of weak plasma turbulence. The imaginary part of the frequency is much smaller than the real part since the drift velocity can be much smaller than the mean thermal velocity of the electrons. The nonlinear theory of the ion-acoustic instability and the anomalous resistivity have been treated by many authors. We shall consider this question in some detail. The energy density of a mode $W_k$ characterized by wave vector **k** grows exponentially at small amplitudes. Then, at large amplitudes, effects associated with nonlinear saturation come into play and it is possible for a steady or quasisteady state situation to arise. We can then neglect the left side and find the spectrum $W_k$, equating the linear growth rate to one of the effects associated with the nonlinear saturation. The nonlinear effects can be written symbolically in the following form:

$$0=\left\{2\gamma_{\mathbf{k}}-A\omega_{\mathbf{k}}\left(\frac{W}{N_0T_e}\right)-B\omega_{\mathbf{k}}\left(\frac{W}{N_0T_e}\right)^2\right\}W_{\mathbf{k}}. \tag{4.23}$$

The quadratic effects (proportional to the square of the wave amplitude) represent wave–wave interactions. Resonant three-wave interactions are forbidden in the ion-acoustic instability so that the only effect which can give a term of order $W^2$ is nonlinear scattering of the wave on the ions [III]. This effect, associated with the presence of a denominator like $\omega - \omega' = (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}$, represents Landau resonances for the nonlinear beats associated with a given wave pair. These beat waves fall in resonance with the ions; one part of the energy is absorbed while another part goes to the wave with the lower frequency. The quadratic term actually represents a rather complicated integral expression (cf. § 3.2) in which the integral is taken over all wave vectors. The value of



this term can be estimated as follows. Since the effect is associated with the thermal motion of the ions, the operator **A** contains a small numerical factor $T_i/T_e$ (since we are discussing ion-acoustic waves, then by definition it is necessary that $T_i \ll T_e$).

The next effect, the cubic effect, relates to the four-wave interaction. The four-wave interaction has been solved and taking account of it leads to a rather complicated nonlinear operator which contains the energy density of the waves to the third power. In the theory of weak turbulence this effect is weaker than nonlinear ion scattering. Thus, the basic contribution, which is decisive, is the balance between the linear growth and the first (quadratic) nonlinear term. A problem of this kind has been solved approximately by Kadomtsev who has been able to convert a complicated integral operator to differential form by assuming that the nonlinear interaction only causes a small frequency change (cf. § 3.2). In solving the balance equation Kadomtsev has observed that in the region of wave numbers much smaller than the Debye wave number (wavelengths much larger than the Debye wavelength), there is a simple dependence: The energy density is proportional to $k^{-3}$ [III]. When $k\lambda_D \sim 1$ the integral operator, i.e., the collision term for the waves, does not reduce to a simple form. But it is possible to investigate the opposite limit of large wave vectors, that is to say, wavelengths shorter than the Debye radius (where the dispersion relation for ion-acoustic waves is simple: $\omega \approx \Omega_p$). It turns out that when $k\lambda_D > 1$ the spectrum falls off rapidly ($\sim k^{-13}$), [IA]. The Kadomtsev spectrum has a logarithmic divergence: The total wave energy diverges at small wave numbers. But this logarithmic divergence is not described because Eq. (4.14) for $\nu_{eff}$ does not contain the energy density but rather the momentum lost by the electrons; that is to say, in Eq. (4.14) we deal with another integral (k and $\omega$ are approximately proportional to each other for long wavelengths). There is an additional factor, the imaginary part of $\gamma_k$, a quantity which is proportional to the frequency; i.e., the quantity k appears again. Thus, now there is no divergence at small values of k. On the contrary, the contribution to the integral comes from the region of large $k\lambda_D \approx 1$. It is reasonable to make the assumption that a cut-off must be introduced at wave vectors of the order of the Debye radius (beyond this point the Kadomtsev spectrum is not valid and sharp damping occurs). The calculation of this integral leads to the following formula for the effective collision frequency [112]:

$$\nu_{\mathrm{eff}} = 10^{-2}\Omega_p (V_d/c_s)(T_e/T_i)\Theta^{-2} \tag{4.24}$$

The factor $10^{-2}$ arises in the calculations (cf. also [113]). Thus, if it is possible to propagate a current in the plasma (this current being considerably above the critical value) so that the electrons lose momentum because of coherent emission of phonons, i.e., ion-acoustic waves, ultimately a stationary spectrum will be established (more precisely, a quasistationary spectrum) and $\nu_{\mathrm{eff}}$ will be determined by Eq. (4.24). This relation has a deeper significance than Eq. (4.22) for the Buneman conductivity because of the fact that it reflects the specific nonlinear saturation of the instability. Nonetheless, it is only approximate in nature since the stationary Kadomtsev spectrum (3.25) is only an approximate description of the steady-state ion-acoustic waves. A solution of this kind could only be rigorous in the absence of an angular dependence in the expression for the growth rate of the ion-acoustic instability. This approximation is sometimes called the *isotropic growth rate approximation*. In the best case, the error incurred through the use of this approximation leads to an undetermined numerical factor of the order of unity. However, the danger exists that the nonlinear steady-state solution obtained by Kadomtsev may itself be unstable with respect to the formation of an elongated cone of unstable modes in **k**-space. This would lead to the reduction of the angle $\Theta$, whose square appears in the denominator of the expression for the effective collision frequency (4.24). At the present time this question has not yet been resolved, although it is shown in [95] that solutions exist in which the angle $\Theta_0$ varies in time about some mean value.

In conclusion we note another feature: If a current flows under conditions corresponding to the anomalous Buneman resistivity in a plasma in which the ions and electrons are initially isothermal (in which case ion-acoustic waves cannot be excited), sooner or later these conditions will become favorable for the ion-acoustic instability. This feature follows from the fact that the electrons are heated more rapidly than the ions by a factor of $kV_d/\omega$ in the Buneman instability; ultimately, therefore, the plasma must exhibit a difference between the electron and ion temperatures. In this sense, the ion-acoustic instability appears to be self-sustaining because when $V_d > c_s$ the electrons will always acquire more heat than the ions.



The ion-acoustic wave spectrum is a nondecay spectrum; however, as has been pointed out by Tsytovich [114], in the region of small wave numbers, where the dispersion relation is almost linear, there is a small imaginary frequency component which arises because of the nonlinear broadening of the line. It is then possible that the three-wave resonance condition can be satisfied. Tsytovich includes the three-wave resonance in the wave kinetic equation as a nonlinear term which leads to saturation, that is to say, as a decay process; the kinetic equation is then solved in this form. In this case one obtains a spectrum which is like the Kadomtsev spectrum, since the nonlinearity is quadratic; however, the small parameter $T_i/T_e$ which appears in nonlinear scattering on ions does not appear. It is obvious that a somewhat different value will be obtained for the quantity $W_k$. However, the Tsytovich spectrum can only obtain for sufficiently small wave numbers, in which case the small nonlinearity can cause overlapping of the resonances so that the three-wave interaction can be realized. However, the basic contribution in the momentum loss (in contrast with the wave energy) comes from the short wavelengths. At the short wavelengths (frequencies of the order of the ion-plasma frequency) the deviation from the linear dispersion relation becomes very large and a very strong nonlinearity would be needed in order to satisfy the three-wave resonance condition. Thus, the Tsytovich model has a very limited range of applicability, the more so since strong nonlinearities mean that it is necessary to treat the problem from the point of view of strong turbulence.

Finally, we note that the formal application of perturbation theory, as in § 3.1, for computing the electron distribution function leads to a somewhat paradoxical result: The linear theory for the excitation of ion-acoustic waves must be treated for a level of turbulence which is much lower than that reached in the steady-state turbulence [115]. The point here is that the basic contribution to the nonlinear correction to the work in the field of the wave, as computed by means of the distribution function $f^{e(3)}_{\mathbf{k}^1,\mathbf{k},-\mathbf{k}'}(\mathbf{v})$ (cf. § 3.1), gives particles with velocities that are not higher than the wave phase velocity:

$$\mathbf{j}^{(3)}\mathbf{E} \approx \mathrm{Im}\int d^3\mathbf{v}\,(\mathbf{v}\cdot\mathbf{E})\left(\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}\,\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}}\right)^3 f_e^{(0)} \approx \mathbf{J}^{(1)}\cdot\mathbf{E}\,\frac{E^2}{4\pi N_0 m\,(\omega/k)^2}.$$

Thus, according to this estimate the nonlinear correction to the growth rate becomes larger than the linear growth rate when

$W > mN_0T_e/M$. The resolution of the paradox lies in the fact that in an earlier stage of the development of the instability it is necessary to take accout of the nonlinear broadening of the resonance velocities by an amount of order $\Delta v \approx (eE/mk)^{1/2}$. The Doppler broadening of the resonance due to this effect starts to play a role when $(eE/mk)^{1/2} > \omega/k$, that is to say, when $E^2/4\pi > mN_0T_e/M$. In view of these considerations it is not difficult to estimate the nonlinear correction to the growth rate [115]:

$$\delta\gamma/\gamma^L \approx E^2/4\pi m N_0 \Delta v^2 \approx (E^2/4\pi N_0 T_e)^{1/2} \ll 1.$$

Thus, the conclusion regarding the importance of taking account of the nonlinear electron contribution in the growth rate is not justified.

## § 4.3. Quasilinear Effects in Anomalous Resistivity Due to the Ion-Acoustic Instability

Although it is not yet completely reliable, indirect experimental evidence is available which shows that Eq. (4.24) has been verified in certain limiting cases. However, these experimental data must still be treated with caution. The point here is that no one of the four quantities which appear in this equation, $V_d$, $c_s$, $T_e$, i, has its usual physical significance because we are dealing with a plasma in which true binary collisions are replaced by scattering on fluctuations. Let us start with the electron temperature. If binary collisions do not occur it is difficult to believe that the distribution function will be a Maxwellian. Even if it is not required that the electron distribution function be a Maxwellian with some mean thermal spread, it is necessary that the function $f_e$ have a rather rapidly converging tail. Under these conditions one can discuss the idea of a single temperature for the electrons. The situation is more complicated for the ions because at the outset it is obvious that the ion distribution function will be rather unusual if the ions only interact with the waves (without binary collisions). Finally, there is the mean drift velocity. Usually the particle distribution in a plasma in which a current flows is like that shown in Fig. 30. Here, we have an ion distribution function and an electron distribution function which is displaced with re-



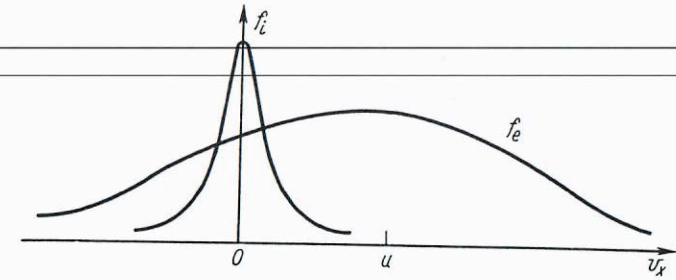



Fig. 30. Maxwellian particle distribution in a plasma with a current.

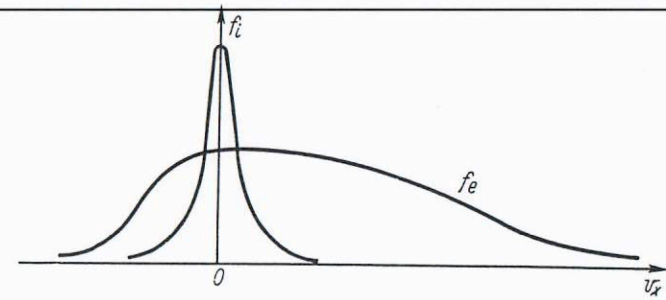



Fig. 31. Example of a stable electron distribution in a plasma with a current.

spect to the ion function under the assumption that the electron distribution is displaced as a whole. However, in principle, it is possible to have a situation (Fig. 31) in which the electron distribution has its maximum at the same point as the ion distribution, but in which the electron distribution is distorted so as to become highly asymmetric (cf. Fig. 31).

At this point it is useful to discuss the possible form of the distribution functions for the ions and electrons. It will be convenient to use the two-dimensional pattern shown in Fig. 32. Along the abcissa axis we have plotted the component of the particle velocity in the direction of current flow; the transverse component is plotted along the ordinate axis. Assume that initially we have the usual Maxwellian distribution of electrons and ions. With a Maxwellian distribution the curves corresponding to equal values of the distribution function in this plane will be circles. The interaction between waves and particles is especially strong when the Landau resonance is realized. A wave with phase velocity $\omega/k$

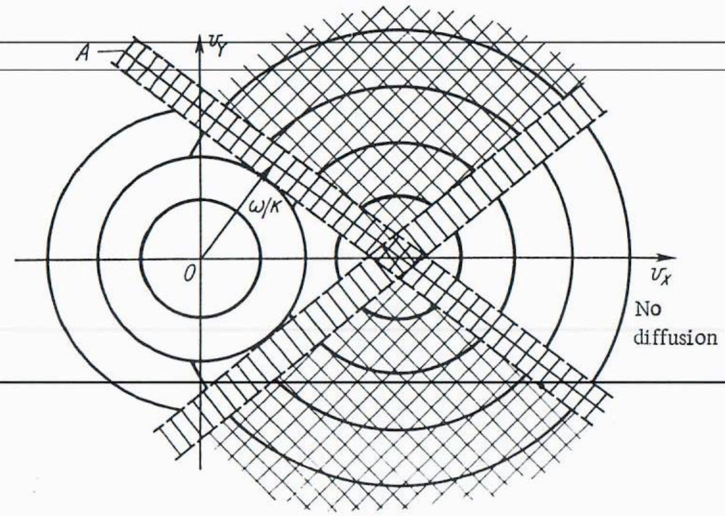



Fig. 32. Diffusion cone for ion-acoustic turbulence in a plasma with a current.

interacts with particles located close to the line A (cf. Fig. 32). These are the particles which take part in the resonance interaction. If we consider waves with all possible directions and phase velocities it is easy to show that all particles located in that part of the plane ($v_x$, $v_y$) in which it is possible to draw a line which corresponds to the Landau resonance condition will experience the effect of the random field of the wave. However, the ion-acoustic spectrum does not contain waves with velocities below some critical value ($\omega/k \lesssim v_{Ti}$). In the quasilinear approximation the particles noted above will not interact with waves. The interaction is found to be much weaker, being associated with nonlinear effects in the next approximation. The number of ions in the wave interaction region is rather small since only a small fraction of the ions experience a strong effect from the wave. In the zeroth approximation the distribution function in the primary region is then essentially undeformed. On the other hand, a strong deformation is produced in the resonance region. In the language of quasilinear theory, this deformation is nothing more than the diffusion of particles in velocity space. There is a large number of experiments which have been carried out in the intermediate regime in which this diffusion process is relatively slow and binary particle collisions are capable of producing something like a Maxwellian distribution. This is the case in which the plasma resistivity is somewhat higher than the classical value. We only discuss the most extreme case, in which binary collisions do not play a role. In this case the change in the ion distribution function is very important. It has a strong effect on the ion imaginary part (the ion pole, which is proportional to the number of resonance ions). As a rule, this is a very small number and is very sensitive to the features of the tail of the ion distribution. At the present time there is still not available a self-consistent theory which is capable of describing the change in the ion distribution function at long times (with the exception of the one-dimensional case, which is discussed below). It is possible, however, to make certain estimates which make use of the two-temperature approximation, that is to say, the division of the ions into two groups: cold ions, for which essentially nothing happens, and hot ions in the tail of the distribution function which are characterized by a high temperature.



The situation for the electrons is as follows: The region of forbidden velocities, within which there is no resonance between particles and waves, is relatively small because the acoustic velocity is $(M/m)^{1/2}$ times smaller than the mean thermal velocity of the electrons. In principle, it is possible to neglect effects which occur within this small group. However, the situation is more complicated. It is very difficult to think of a situation in which the current flowing in this direction can excite a wave which is almost perpendicular to the current. Actually, it is well known from the theory of the ion-acoustic instability that a wave with a large wave-vector component perpendicular to the direction of current flow has a small imaginary part; in practice, waves of this kind can be regarded as stable. Thus, we conclude that there is formed a small cone in velocity space in which there are no waves that can resonate with the electrons. These electrons are freely accelerated by the electric field which produces the current flow in the plasma. The contribution of these electrons can have a strong reduction effect on the plasma resistance. It is then of interest to ask what fraction of the electrons fall in this loss cone and are freely accelerated. The problem can be considered in two limiting cases. It is first useful to isolate the simpler case, in which there is a weak magnetic field $H_0$ in the plane perpendicular to $V_d$. This magnetic field causes a slow gyration of the electrons (slow as compared with the frequency of the plasma oscillations). However, it can be regarded as fast on the time scale in which electrons move into the "loss cone." Thus, on the average all of the electrons in one Larmor orbit will interact with the waves. No additional difficulty arises in this problem for the electrons. Although $f_e$ does not reduce to a Maxwellian distribution, it is still possible to speak of a mean electron temperature. Furthermore, under the rather general assumption that the phase velocity of the oscillations is much smaller than the mean thermal velocity of the electrons, without making any assumption as to the spectrum it is possible to obtain a simple formula for the electron distribution function. Assuming there is some time during which the electron energy becomes substantially greater than the initial thermal energy we find that there is a universal distribution given by $f_e \sim \exp(-\alpha v^5)$ (cf. § 2.3). In certain experiments distributions similar to this distribution have been observed. With a distribution of this kind one can talk of a mean temperature, and all of the calculations for the electrons go through in almost the same way as for a Maxwellian distribution when account is taken of small changes in the numerical coefficients. Thus, we shall assume that if there is a weak transverse magnetic field all of the results for the effective number of collisions can be carried over even for long times, in which case there might be a substantial distortion of the electron distribution which represents a deviation from a Maxwellian distribution. It is precisely this kind of situation which obtains in experiments on collisionless shock waves that propagate across a magnetic field. The current flows across the magnetic field in this kind of a shock wave.

However, there are still certain difficulties associated with the ion distribution. The point here is that ion mixing does not occur in the magnetic field, so that the ion distribution may acquire an unusual form which, ultimately, will be very different from a Maxwellian distribution. One expects that the bulk of the ions will be cold and that some fraction of the ions (starting from a velocity of the order of the acoustic velocity) will be heated. Without the use of numerical methods it will evidently not be possible in the near future to find the form of this complicated ion distribution. On the other hand, certain conclusions can be drawn from the following considerations. If there is no interaction with the walls during the process in which the current flows and if the electron and ion energies increase, that is to say, if there is no loss



of heat to an external sink, one expects some kind of self-similar forms for both the distribution functions and the spectra.

The situation should be much simpler in the case of a weak nonlinearity, in which it is possible to rely on the quasilinear approximation. In this approximation, saturation of the instability is reached as a consequence of the quasilinear deformation of the ion distribution function; as a result, even in a nonisothermal plasma there will be a group of ions with large velocities. These ions, which absorb the ion-acoustic waves in resonant fashion, then balance the excitation of the waves by electrons.

Let us consider the nonlinear saturation process in this case. Assume that the equation for the spectrum of unstable waves is given by the symbolic form [compare with Eq. (4.23)]

$$dW_k/dt = 2\gamma_k^e W_k - 2\gamma_k^i W_k - A\,(W/N_0T_e)W_k - B\,(W/N_0T_e)^2 W_k. \qquad (4.25)$$

The wave growth when $V_d > V_c$ leads to an increase in the resistance, that is to say, a friction force arises which acts on the electrons. If it is assumed that the electric field which drives the current is not too large, then as a consequence of the reaction on the electrons due to the increasing resistance, the velocity $V_d$ will continue to be reduced as long as the instability threshold is not reached. This means that the nonlinear terms in Eq. (4.25) will play a small role and that the saturation of the waves is determined by the condition $\gamma_k^e \approx \gamma_k^i$ for all originally unstable waves. In other words the following condition must be satisfied for all originally unstable waves:

$$\gamma_k = \gamma_k^e - \gamma_k^i \approx [df_e/dv + (m/M)(df_i/dv)] = 0. \qquad (4.26)$$

In this form, Eq. (4.25) does not contain the amplitudes of the steady-state waves $W_k$ so that it cannot be used to compute $\nu_{eff}$ directly. However, the effective collision frequency for this saturation regime of the instability (this is sometimes called the "threshold" or quasilinear regime) can be found from Ohm's law by substituting the expression found for $j = eNV_d \approx eNV_c$. Writing $j = eNV_c = \sigma E$, we have

$$\nu_{eff} = eE/mV_c. \qquad (4.27)$$

Now, using Eq. (4.14) which relates $\nu_{eff}$ to the wave energy, we can find W:

$$W/N_0T_e \approx (eE\lambda_{De}/N_0T_e)(v_{Te}/V_c). \tag{4.28}$$

As E increases the quantity W also increases, and at sufficiently large fields E, it is no longer possible to neglect the nonlinear terms in Eq. (4.25). At this point the apparently simple formula for the threshold (quasilinear) regime becomes complicated. At first sight it might appear that in finding $V_d \approx V_c$ it is sufficient to use the linear expression for the imaginary part of the frequency (4.26). However, $\gamma_k^i$ is very sensitive to the form of the ion distribution for large ion velocities, that is to say, it is sensitive to the tail of the distribution function. These ions, which damp the waves, increase their energy and rapidly change (in quasilinear fashion) the form of the distribution function. As a result, $\gamma_k^i$ changes rapidly, as does $V_c$.

If the thermal energy absorbed by the plasma particles is not transferred to an external sink, one expects that ultimately a universal self-similar ion distribution function will be established. The assumption that such a self-similar solution (corresponding to the quasilinear regime) actually exists is verified in [116]. The self-similar variables in which the equations of weak turbulence assume a simple form are found in this work; however, these equations still cannot be used to solve the general case. Nonetheless we can proceed as follows. When the current flows across the magnetic field and the wave spectrum is three-dimensional, the ion distribution can be divided into cold ions and hot ions. In this two-group approximation we can find the quantities which characterize the current flow. The group of ions which are in resonance with the ion-acoustic waves and are then accelerated is relatively small. We denote the fractional concentration of these hot ions by X. The effective temperature of these resonance ions will be called $T_{hi}$. Then from Eq. (4.21) we have

$$T_e/T_{hi} \approx XV_d/c_s. \tag{4.29}$$

Estimating the ion damping coefficient as $\gamma_i \approx (\omega^3/k^3)\,(Xc_s/(T_{hi}/M^{3/2})$ and comparing it with the electron coefficient (4.5), we have

$$X = (m/M)^{1/4}\,(T_{hi}/T_e)^{1/4}, \tag{4.30}$$

$$V_d \approx c_s\,(M/m)^{1/4}\,(T_e/T_{hi})^{5/4}. \tag{4.31}$$



We now consider the momentum of the resonance ions. The momentum lost by the electrons in scattering is transferred to the ions, $\dot{\mathbf{P}}_i = Nm\mathbf{V}_d\nu_{\text{eff}}$. Since $T_{hi} \gtrless T_e$, the ion distribution function can be written in the form $f_i(\mathbf{v}, \vartheta) = f_{0i}(\mathbf{v}) + f_{1i}(\mathbf{v}, \vartheta)$ (cf. § 2.3), where the anisotropic part $f_{1i} \sim [c_s/(T_{hi}/M^{1/2}]\, f_{0i} \lesssim f_{0i}$. Thus, $|\mathbf{P}_i| \approx \approx |\int M\mathbf{v}_s f_{1i} d^3\mathbf{v}| \approx N_0XMc_s$. Finally we have

$$V_d \approx c_s\,(M/m)^{1/4}, \quad T_{hi} \approx T_e, \quad X \approx (m/M)^{1/4}. \tag{4.32}$$

Thus, the mean random energy of these hot ions is approximately the same as the random energy of the bulk of the electrons. Obviously, Eq. (4.32) contains factors of the order of unity; without knowing the exact solution, these factors can only be determined by comparison with experimental data. The only exception is the idealized case of the one-dimensional spectrum, which allows an exact analytic solution. This solution is of great interest from a procedural point of view, and we shall consider it in some detail.

Let us assume that there are waves which only propagate in the direction of the current ($\parallel X$). The ion distribution function for the ions which interact with these waves is also one-dimensional. However, because of the magnetic field the electron distribution will be axially symmetric in the plane $v_x$, $v_y$ around the point $V_d$, 0. The interaction of the electrons with the waves in this problem corresponds to the case that has been treated in § 2.3. The electron distribution function is of the form $f_e \sim \exp(-\alpha v_\perp^5)$ with the origin taken at the point $V_d$, 0. However, not all of the electrons interact with the waves. This point can be made clear from consideration of Fig. 33. For example, if only the usual ion-acoustic

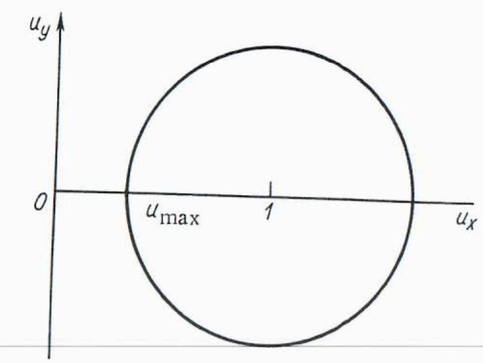



Fig. 33. Interaction of one-dimensional waves with electrons in a magnetic field.

waves are excited in the plasma the phase velocity cannot be larger than $\omega/k_{max}$. Since the spectrum is one-dimensional, for drift velocities $V_d > \omega/k_{max}$ some of the electrons within the circle (cf. Fig. 32) will not interact with the waves. Hence, the electrons in this region remain cold and the corresponding value of the distribution function will become larger and larger as compared with the value of $f_e$ in the resonance region. This process will continue as long as the dispersion relation for the ion-acoustic waves does not change so much that waves with phase velocities in the range from the initial value $\omega/k_{max}$ to $V_d$ do not appear. In this case the circle (cf. Fig. 33) is elongated and the electrons in the circle can be described by a distribution in the form of a $\delta$-function $f_e \sim X_e \delta(v_x - V_d)\delta(v_y)$. The relative fraction of these nonresonant electrons $X_e$ will be determined below.

In terms of the dimensionless variables $u_x = v_x/V_d$, the quasilinear kinetic equation for the ions can be used to write the distribution function $f_i(v_x, t) = (N/V_d)g_i(u_x)$ in the form

$$-(d/du_x)u_x g_i = (m/M)^2 (d/du_x)D(u_x)(dg_i/du_x), \tag{4.33}$$

where

$$D(v_x) = \frac{8\pi^2 e^2}{m^2}\int W_k \delta(\omega_k - kv_x)dk/(2\pi).$$

As in the preceding problem, it is expected that the bulk of the ions will not interact with the waves. The relative fraction of resonant ions described by Eq. (4.33) is $(1 - X_i)$. The condition $\gamma_i + \gamma_e = 0$ in the range of phase velocities between 0 and $V_d$ $(0 < u_x < 1)$ now assumes the form

$$dg_i/du_x = (M/m)(dh_e/du_x), \tag{4.34}$$

where $h_e(u_x)$ is the electron distribution (of the form $\exp[-\alpha V_\perp^5]$) in the self-similar variables, integrated over $v_y$. The system of equations in (4.33) and (4.34) can be solved (cf. Problem 1 at the end of this section).

It is interesting to note that the ratio of the directed electron velocity $V_d$ to the electron thermal velocity, and the number of resonant ions $v_{Te}$, can be estimated from simple considerations based on the conservation relations. Scattering of electrons on waves is associated with an electron-ion frictional force $F_{fr}$ which transfers momentum from the electrons to the ions. The momentum of the ions is denoted by $P_i$ and we have $dP_i/dt = F_{fr}$. This relation is equivalent to the result obtained by computing the



first moment of the ion kinetic equation. And since $P_i \sim (1 - X_i)MV_d$, then $F_{fr} = (1 - X_i)M dV_d/dt$. The work of the frictional force goes into electron heating: $dT_e/dt \approx V_d F_{fr}$. It then follows that $v_{Te}^2 \sim (1 - X_i)MV_d^2/m$. Comparing the electron growth rate with the ion damping rate we find the relation $(m/M)(1 - X_i)/V_d^2 \sim (V_d/v_{Te}^3)$, whence follows immediately that $1 - X_i \approx (m/M)^{1/5}$, $V_d/v_{Te} \sim (m/M)^{2/5}$. As might be expected, this result is verified by the exact solution (cf. Problem 1 at the end of this section).

In this analysis we have not taken account of oblique waves, which propagate at an angle with respect to the direction of current flow. It is not difficult to show that the existence of a sharp maximum in the electron distribution function at the point $v_x = V_d$ leads to an ion-acoustic instability with wave vector **k** directed almost perpendicularly to the current. The kernel of the electron distribution function becomes smeared out in this case, and the situation is very similar to that described in the preceding paragraphs [cf. Eqs. (4.29) and (4.32)].

As we have already noted, the quasilinear approximation can be applied for weak nonlinearities. This approximation is valid if the electric field is small so that the frictional force due to coherent emission of ion-acoustic waves retards the electrons and prevents them from acquiring a mean velocity greater than the critical velocity needed for the instability. In other words, at all times the plasma is essentially at the threshold of the instability. In this case the nonlinear regime described in § 4.2 must correspond to the case of large electric fields. The quantity $E_c$, that is to say, the limiting value which separates the two regimes being considered, can be found as follows. Let $V_c$ be the electron drift velocity which corresponds to the instability threshold $(\gamma_i + \gamma_e = 0)$. The case of large electric fields (4.24) is realized if $V_d$, as computed from the relation $V_d = eE/m\nu_{eff}$, exceeds $V_c$ $(V_d \gg V_c)$. Using Eq. (4.24) for $\nu_{eff}$, we have .

$$E \gg 10^{-2}(mM^3)^{1/4}\Omega_p c_s/e. \tag{4.35}$$

The functional dependence $j = j(E)$ can be represented qualitatively by a curve like that shown in Fig. 34. Here, there is a classical region which obtains at low electric fields, in which the plasma is far from the unstable regime; at moderate electric fields $j = eNV_c$ (quasilinear regime); $j \sim E^{1/3}$ in the nonlinear regime, in which case $\nu_{eff}$ is determined from Eq. (4.24).

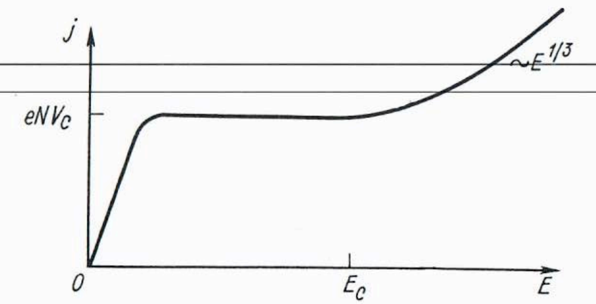



Fig. 34. Ohm's law with ion-acoustic turbulence.

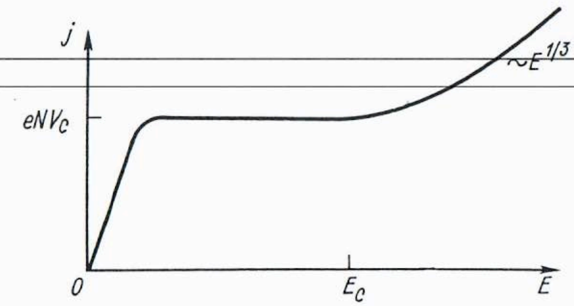
All of the discussion in this section refers to the case in which the current flow is perpendicular, or almost perpendicular, to the magnetic field. Now let us consider the situation in which the magnetic field is directed in the direction of current flow or in which there is no magnetic field. In this case the mechanism which mixes the electrons no longer operates and the problem of determining the electron distribution function becomes more complicated. The question arises as how to proceed in this case. We shall assume here that after some time a self-similar distribution is established [117]. The electron distribution function assumes some universal form: Further heating of the electrons occurs and increases their mean velocity, but the form of the function remains the same. In practice, however, it appears that these self-similar variables cannot be used: It is evidently not possible to introduce a two-temperature electron distribution because there are no two clearly defined groups of electrons, as in the ion case. On the contrary, there is a smooth transition from the slow electrons to the fast electrons; in the course of time, as is shown by a qualitative analysis of the equations in the self-similar variables, a significant fraction of the electrons fall into the velocity region in which there are essentially no waves. This effect is reminiscent of electron runaway in a gas with Lorenz collisions, in which the collision frequency falls off with velocity as $v^{-3}$. The interaction of electrons with ion-acoustic waves exhibits precisely the same properties [63].

The question of the ultimate form that is assumed by Ohm's law in such a plasma remains open at the present time. One of the points of view that is presently held is the following: A significant fraction of the electrons fall into the runaway regime; the



electron distribution function as projected along the parallel velocity is then found to be highly elongated in the direction of current flow, and the ratio of mean electron drift velocity to mean electron thermal velocity can be equal to unity [117]. It is difficult to predict the exact numerical value. So far data obtained from laboratory experiments have not been very useful. The point is that most experiments in which anomalous resistivity is investigated are carried out in discharges with so-called *open ends*, that is to say, discharges in which the electrons in the plasma can move freely and be lost along the lines of force of the magnetic field. Under these conditions it is impossible to obtain an extended self-similar regime since heat is lost continuously. Furthermore, since the particle mean free path for high-velocity particles is large, the fast particles are lost rapidly from the system. For this reason a continuous truncation of the tail of the electron distribution function occurs. The problem is extremely complicated and is very sensitive to the boundary conditions, and for this reason loses much of its general application. It can be shown that the ratio $V_d/v_{Te}$ will remain much smaller than unity in this case. In principle it is not possible to exclude a number of other mechanisms which terminate the runaway of electrons and thus lead to a ratio $V_d/v_{Te} \ll 1$. The combined effect of ion-acoustic and cyclotron instabilities on the electrons is discussed in [118]. Along the same lines, in [119] the authors have introduced the notion of the extended existence of so-called *macroparticles* [120]. However, these ideas are essentially of phenomenological nature.

There do exist, however, certain idealized limiting cases in which the weak turbulence equations can be solved almost exactly in treating anomalous resistivity. These are one-dimensional cases. Just as there is a class of one-dimensional models amenable to solution in statistical thermodynamics, in the theory of weak turbulence the one-dimensional models are found to be very much simpler. In certain cases the one-dimensional models can have actual physical meaning. For example, if the magnetic field is so large that the electron gyrofrequency is much greater than the plasma frequency, electron motion across the lines of force of the magnetic field is essentially forbidden and one deals with what is essentially a one-dimensional motion. In these cases the one-dimensional theory can give an adequate description of the actual situation. Self-similar equations in the one-dimensional formulation for the quasilinear regime have been solved exactly [117].

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Let us consider a constant uniform electric field parallel to the magnetic field in a uniform plasma. After a certain time interval the mean-square velocity of the plasma can become so large that the plasma "forgets" its original state; in this case the further evolution of the system is of a universal nature, being independent of the initial conditions. Formally, this regime corresponds to the transition to the case in which quasilinear equations with self-similar variables can be used. It follows from simple dimensional arguments that the particle velocities must be measured in units of $eEt/m$ and the wave vectors must be measured in units of $m\omega_p/eEt$.

The electron and ion distribution functions $f_{e,i}$ and the spectral density of the electrostatic wave energy W are then of the form

$$\left.\begin{aligned} f_e = mNg_e(u)/eEt;\ \ f_i = mNg_i(u)/eEt;\ \ W(k,t) = m\omega_p^4 t^2 U(q);\\ u = mv/eEt;\ \ q = keEt/m\omega_p. \end{aligned}\right\} \quad (4.36)$$

Substituting these functions in the quasilinear equations written in the reference system that moves with the freely accelerating ions, we have

$$(-d/du)(u - 1 - \mu)g_e = (d/du)D(u)(dg_e/du), \quad (4.37)$$

$$-(d/du)ug_i = \mu^2 (d/du)D(u)(dg_i/du), \quad (4.38)$$

where D(u) is the quasilinear diffusion coefficient and $\mu = m/M$. Taken together with the marginal stability criterion

$$(d/du)(g_e + \mu g_i) = 0, \quad (4.39)$$

Eqs. (4.37) and (4.38) form a closed system with the following solution:

$$\left.\begin{aligned} g_e = Cu/(u+\mu^2),\ g_i = C\mu(1-u)/(u+\mu^2),\ D = u^2(1-u)/\mu^2\\ \text{for } 0 < u < 1;\\ g_e = g_i = D = 0 \quad \text{for } u < 0,\ u > 1, \end{aligned}\right\} \quad (4.40)$$

where C is an arbitrary positive constant. The functions $g_e(u)$ and $g_i(u)$ must be supplemented by a certain number of freely accelerating electrons and ions which, in the self-similar solution, correspond to δ-functions at the point u = 1 for the electrons and



u = 0 for the ions. Denoting the fraction of freely accelerating particles by $X_e$ and $X_i$, from the normalization condition we find that $X_e + C = 1$, $X_i + 2C\mu \ln \mu^{-1} = 1$. Knowing the functions $g_e$ and $g_i$ we can then easily write the dispersion relation

$$\varepsilon(q,\omega) = 1 - (1 - C)/(\omega - q)^2 - \mu/\omega^2 + C/\omega q - C/(\omega - q)q.$$

The function $\varepsilon(q, \omega)$ must satisfy the following requirements: All waves must be stable; all waves must have phase velocities in the range (0, 1). From these conditions it is possible to determine the constant C uniquely (which is found to be equal to $2\mu^{1/2}$) and, thus, the distribution function $g_{e,i}$.

Thus, ultimately one finds that a universal self-similar electron distribution is formed. In this case there is a plateau region from V = 0 extending up to the velocity of free acceleration characteristic of the bulk of the electrons.

Numerical experiments in the one-dimensional case give distribution functions of approximately the same form [121]. However, there is some difference between the distribution functions in the numerical experiments and the self-similar experiments. In particular, in the self-similar analysis the number of electrons in the plateau region is proportional to $N_0(m/M)^{1/2}$. On the other hand, in the numerical experiments a significant fraction of electrons is found in the plateau region. This difference stems from the fact that the quasilinear theory ignores nonlinear effects.

## PROBLEM

▶ 1. Find the exact solution for the quasilinear equations (4.33) and (4.34) for the problem of an anomalous resistance with respect to a current which flows across H in the one-dimensional problem [117].

The electron distribution function $f_e(v_\perp, t) = (N/v)g_e(u_\perp)$ can be expressed in terms of the quasilinear diffusion coefficient $g_e = C_1 \exp(-u_\perp^5/5\bar{D})$, where $\bar{D} = \pi^{-1}\int_0^1 D(u_x)(u_x - 1)^2\,du_x$, while the constant $C_1$ is found from the normalization condition $2\pi\int_0^\infty u_\perp g_e du_\perp = 1$

$$C_1 = 1/\pi\Gamma(7/5)[5\bar{D}]^{2/5}$$

From the condition $\gamma = 0$ in the phase velocity range (0, 1) we have

$$\frac{dg_i}{du_x} = -\frac{M}{m}\cdot\frac{dg_e}{du_x} \approx \frac{2M}{m}(1-u_x)\int_0^\infty \frac{du_\perp}{u_\perp}\cdot\frac{dg_e}{du_\perp} =$$

$$= -\frac{M}{m}(1-u_x)(5\overline{D})^{-3/5}\frac{5\Gamma(9/5)}{2\pi\Gamma(7/5)}$$

(in computing $dg_e/du_x$ we have taken account of the fact that the electron thermal velocity is much greater than the electron drift velocity which, in the variables used here, is equal to unity). We then find the ion distribution function

$$g_i = (M/m)(1-u_x)^2(5\overline{D})^{-3/5}[5\Gamma(9/5)]/[4\pi\Gamma(7/5)]$$

and, from Eq. (4.33), the diffusion coefficient $\overline{D} = 1/40\pi\,(m/M)^2$. Now, using the relation

$$\overline{u_\perp^2} = \int_0^\infty u_\perp^3 g_e\,du_\perp \bigg/ \int_0^\infty u_\perp g_e\,du_\perp = (5\overline{D})^{2/5}\,\Gamma(9/5)/2\Gamma(7/5)$$

we can find the mean-square electron velocity $(\overline{u_\perp^2})^{1/2}$, which is found to be $0.38(M/m)^{2/5}$. Thus, the ratio of the directed electron velocity to the thermal velocity is $2.65(m/M)^{2/5}$ in this model.

As before, the number of ions that interact with the waves is small,

$$1 - X_i = \int_0^1 g_i(u_x)\,du_x = (m/M)^{1/5}\frac{10\Gamma(9.5)}{3(8\pi)^{2/5}\Gamma(7/5)} \approx 0.95\,(m/M)^{1/5},$$

and in determining $X_e$ we make use of the dispersion relation

$$\varepsilon(\omega, q) = 1 - \frac{1}{q^2}\int_{-\infty}^{+\infty}\frac{du_x}{u_x}\cdot\frac{dg_e}{du_x} - \frac{X_e}{(\omega-q)^2} - \frac{mg_i(0)}{M\omega q} - \frac{mX_i}{M\omega^2} = 0$$

or

$$1 + \frac{8.2}{q^2}\left(\frac{m}{M}\right)^{4/5} = \frac{X_e}{(\omega-q)^2} + \frac{m}{M\omega^2} + \frac{2.86}{\omega q}\left(\frac{m}{M}\right)^{6/5}$$

From the requirements on stability and the existence of waves with all phase velocities in the range (0, 1), which is completely



analogous to the case of propagation of the current along H [cf. Eqs. (4.36-4.40)], we find the number of particles in the electron root $X_e \approx 8.2(m/M)^{4/5}$.

## § 4.4. Anomalous Resistivity Caused by Other Instabilities

Let us now return to the Table 1 in §4.1 and consider other instabilities. The Drummond–Rosenbluth instability leads to a relatively small imaginary frequency component; furthermore, from the point of view of the quasilinear approximation, this is a one-dimensional instability. Hence, an electron plateau arises which is similar to the plateau that arises in the one-dimensional ion-acoustic model (4.3); this plateau causes rapid saturation of the instability. The current can increase further, but in a small region of velocity space, the electron distribution will have a plateau and the plasma will not be unstable.

The instabilities with the lowest excitations threshold are the electrostatic instabilities with $k_\parallel \ll k_\perp$ in a plasma in which the current flows across the magnetic field. When $\omega \gtrsim kv_{Ti}$, $V_d > k_\parallel v_{Te}/k$ these oscillations are described by (4.6). The nonlinear saturation of this kind of instability is not amenable to analysis within the theory of weak turbulence. For example, we may consider the case of the modified Buneman instability [cf. Eq. (4.7)]. The dispersion relation in (4.7) differs from the usual Buneman equation only in that $\Omega_p$ is replaced by $\Omega_p/(1+\omega_p^2/\omega_H^2)^{1/2}$ and $\omega_p$ by $\omega_p k_\parallel/k(1+\omega_p^2/\omega_H^2)^{1/2}$. It is possible to estimate the wave amplitude in the saturation regime, as is frequently done in the theory of strong turbulence. We compare the linear term $\partial\mathbf{v}/\partial t$ and the nonlinear term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the electron equation:

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v} = e\{\mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{H}]\}.$$

In the nonlinear stage these terms compete, and this competition leads to a quasistationary regime in which the instability is saturated. Equating these terms (by order of magnitude) we find $kV_d \sim (kc/H_0)\sum_q q\Phi_q$. It is then possible to obtain the following estimate for the energy density of the waves:

$$\sum_k N_0 e^2 |\Phi_k|^2/2T_e \approx mNV_d^2, \quad kr_{He} \approx 1. \tag{4.41}$$

Now, using Eq. (4.14) we have [I]

$$\nu_{\text{eff}} = \omega_H V_d / v_{Te}. \tag{4.42}$$

It is useful to estimate the ratio of electron heating to ion heating in this instability making use of Eq. (4.21):

$$\dot{T}_e / \dot{T}_i \approx V_d / v_{Ti}. \tag{4.43}$$

This instability, as a rule, is a slower instability than the ion-acoustic instability (for $T_e \gg T_i$); however, it can occur in a plasma with a high ion temperature ($T_i \sim T_e$) in which the ion-acoustic instability cannot be excited. When the ratio $T_e/T_i$ is reduced below some critical value we can no longer neglect the ion thermal motion in the dispersion relation (4.6). In this limiting case one is dealing with a so-called electron acoustic mode [cf. Eqs. (4.6) and (4.9)]. It is possible to estimate $\nu_{\text{eff}}$ by again carrying out a procedure similar to that used in strong turbulence.

When $\omega_p \gg \omega_H$, the Bernstein modes can be unstable in a plasma that supports a current. This kind of instability can be described as follows. The Bernstein modes are waves for which the wave vector is perpendicular, or almost perpendicular, to the magnetic field and in which the frequencies are grouped close to the harmonics $l\omega_H$. These are pure electron oscillations at rather high frequencies, and no interaction with the ions occurs. Let us assume that a current flows through a plasma and that there are no waves in a reference system that moves with the electrons. Because of the Doppler effect, in the laboratory reference system in which the ions are at rest the oscillation frequency is shifted $l\omega_H - \mathbf{k} \cdot \mathbf{V}_d$. If the drift velocity is large enough, then with sufficiently large k (and k can be chosen as large as the Debye wave vector) the frequency can be reduced significantly in the ion reference system, so that these oscillations can interact with the ions $(l\omega_H - \mathbf{k} \cdot \mathbf{V}_d) \sim kv_{Ti}$. In this case an instability characterized by a negative energy will be excited. The usual Maxwellian ion distribution can then be unstable if one takes account of the imaginary part of the ion interaction with the Bernstein modes due to the Landau resonance. It turns out that this instability has a rather large imaginary part, this value being of the order of the electron Larmor frequency reduced by a factor equal to the ratio of the electron drift velocity to the electron thermal velocity. This instability is not very sensitive to the temperature ratio. In contrast with the ion-acoustic instability, this instability does not require that the electron temperature be much higher than the ion temperature. One would expect a high anomalous resistivity in this case. It turns out, however, that a very small nonlinearity (small effective collision frequency) which arises in the development of this instability is sufficient to completely suppress it. The electron inertia is important in these waves. This means that electron collisions can make a large contribution to the imaginary part. It will be evident that the quantity $\nu_{\text{eff}}$ is subtracted from the growth rate. It would appear at first sight that one could find $\nu_{\text{eff}}$ by equating these two quantities. However, collisions give a still larger contribution. To understand this it should be recalled that the oscillating part of the distribution function contains a factor $\exp(i[\mathbf{k} \times \mathbf{v}]/\omega_H)$ so that $\nu_{\text{eff}}$ will appear with the Pitaevskii factor $k^2 r_{He}^2$ [122]. Since we are discussing very large values of $kr_{He} \approx v_{Te}/V_d$ (short wavelengths) this factor plays an important role. As a result even a small nonlinearity is sufficient to suppress the Bernstein instability.



The effective collision frequency $\nu_{\text{eff}}$ for unstable Bernstein modes can be found approximately by the following method. The well-known dispersion equation from the linear theory of the instability is modified in order to include collisions with the frequency being determined, $\nu_{\text{eff}}$. This can be done by adding a Fokker–Planck collision term $\partial^2 f / \partial v_\perp^2$ in the linearized kinetic equation for the correction to the electron distribution function. In the dispersion equation which results we assume that nonlinear effects (which take account of $\nu_{\text{eff}}$) lead to saturation of the instability:

$$\gamma_k - \nu_{\text{eff}}\, k^2 r_{He}^2 = 0. \tag{4.44}$$

Then, substituting the growth rate $\gamma = \omega_H V_d / v_{Te}$ and the wave number $k \approx \omega_H / V_d$ for the growing waves, we have [123]

$$\nu_{\text{eff}} = 10^{-1} \omega_H (V_d / v_{Te})^3. \tag{4.45}$$

The approach described here, in which nonlinear effects are taken into account by introducing in the linear analysis turbulent transport coefficients with values such that the system returns to the stability threshold, has been used frequently in problems on anomalous diffusion and thermal conductivity in inhomogeneous plasmas [X]. Equation (4.39) can also be obtained from the nonlinear theory of the instability by replacing the unperturbed particle trajectories with trajectories that correspond to the appropriate approximations in the turbulent fields [124]. Formally, this procedure corresponds to taking account of damping due to turbulent diffusion of the particles. As a result, we find that Eq. (4.44) is replaced by

$$\gamma_k - k^2 D = 0, \tag{4.46}$$

where

$$D \approx \frac{c^2}{H_0^2} \sum_{\mathbf{k}} \gamma_k^{-1} |\mathbf{E}_k|^2 \frac{1}{\sqrt{\pi}\, k r_{He}}.$$

It is then an easy matter to estimate the level of the turbulence:

$$W/N_0 T_e \approx (\omega_H^2/\omega_p^2)(V_d/v_{Te})^3. \tag{4.47}$$

The estimate for the effective collision frequency then coincides with that in Eq. (4.45), since $D_\perp \approx \nu_{eff} r_{He}^2$.

## CONCLUSION

The relation between the various elements of the theory of weak plasma turbulence can be given in the form of a chart derived from the equations that are used.

### Theory of Weak Plasma Turbulence

Vlasov equation (for each particle species) + Maxwell's equations

Use of statistical approach

Particle kinetic equation | Kinetic equation for each plasma wave

The general symbolic forms for these equations are

$$\boxed{\frac{df(\mathbf{v})}{dt} = \mathrm{St}\,[f(\mathbf{v})]} \qquad \boxed{\frac{dn(\mathbf{k})}{dt} = \mathrm{St}\,[n(\mathbf{k})]}$$



The collision term can be written as follows:

In the first approximation

$$\boxed{\mathrm{St} = \mathrm{St}_{QL}[\bar{f}(\mathbf{v})]} \qquad \boxed{\mathrm{St}\,[n(\mathbf{k})] = 2\,\mathrm{Im}\,\omega_{\mathbf{k}}\,[f(\mathbf{v})]\cdot n_{\mathbf{k}}}$$

The quantity $\mathrm{Im}\,\omega_k\,[f(\mathbf{v})]$ symbolizes the dependence of the growth rate on the distribution function $f(\mathbf{v})$. This approximation corresponds to the quasilinear approximation of Chapter 2 and only takes account of the linear interaction between the waves and resonant particles, in accordance with the resonance relation $\omega - \mathbf{k}\cdot\mathbf{v} = 0$.

In the second approximation

(a) Wave–wave interaction (Chapter 1)

$$\omega_1 + \omega_2 = \omega_3,$$
$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3.$$

The collision term in the particle kinetic equation describes the adiabatic interaction of waves with particles which participate in the wave motion

St (n) = n · n is a symbolic notation which reflects the quadratic nature of the three-wave interaction

(b) Nonlinear wave–particle interaction (Chapter 2)

$$\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{v}.$$

The collision term describes the resonance interaction between particles and "beat" waves

St (n) – n · n · $f$ is a symbolic notation which reflects the fact that the particles also participate in the interaction

In the third approximation

Again, only the adiabatic interaction between particles and waves is included

St (n) = n · n · n. These processes are important for the nondecay spectrum (cf. § 1.3)