Problem Set III: due TBA

1) Kulsrud 5.4

2) Kulsrud 7.2

3) Consider a fluid in hydrostatic equilibrium with a vertical entropy gradient $\partial S/\partial z < 0$. Take $\underline{g} = -g\hat{z}$.

   a) Starting from the basic equations, derive the growth rate of ideal Rayleigh-Bernard instability. You will find it helpful to relate the density perturbation $\tilde{\rho}/\rho_0$ to the temperature perturbation $\tilde{T}/T_o$ by exploiting the fact that the instability develops slowly in comparison to the sound transit time across a cell. Relate your result to the Schwarzschild criterion discussed in class.

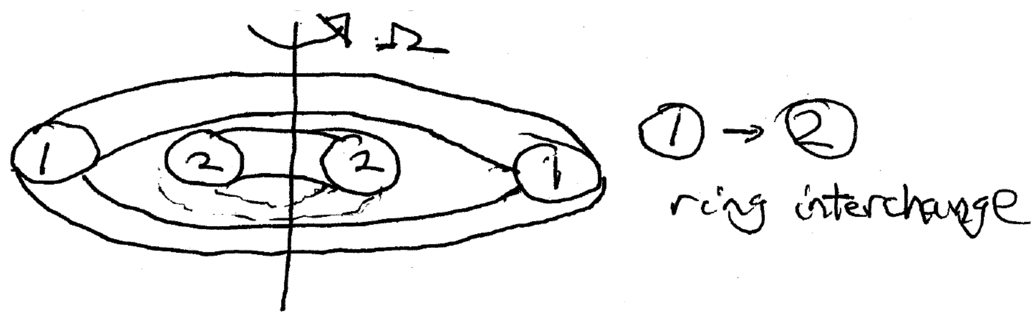
   b) Now, include thermal diffusivity (χ) and viscosity (ν) in your analysis. Calculate the critical temperature gradient for instability, assuming $\chi \sim \nu$. Discuss how this compares to the ideal limit. What happens if $\nu > \chi$?

4) Now again, consider the system of Problem 2, now immersed in a uniform magnetic field $\underline{B} = B_0\hat{z}$.

   a) Assuming ideal dynamics, use the Energy Principle to analyze the stability of a convection cell of vertical wavelength k_z. Of course, $k_z L_p \gg 1$, where L_p is a mean pressure scale length. What is the effect of the magnetic field? Can you estimate how the growth rate changes?

   b) Now, calculate the growth rate using the full MHD equations. You may assume $\underline{\nabla} \cdot \underline{V} = 0$. What structure convection cell is optimal for vertical transport of heat when B_0 is strong? Explain why. What happens when $B_0 \to \infty$? Congratulations - you have just derived a variant of the Taylor-Proudman theorem!

5) Consider a rotating fluid with mean $\underline{V} = r\Omega(\mathrm{r})\hat{\theta}$. Your task here is to analyze the stability of this system to interchanges of 'rings', i.e.



In all cases, assume $\underline{\nabla} \cdot \underline{V} = 0$ and $k_\theta = 0$, so the interchange motions carry no angular momentum themselves and the cells sit in the *r*-*z* plane.

Optional

6) Consider magnetic buoyancy interchange instabilities as discussed in class. Assume entropy stratification is neutral, so $dS_0/dz = 0$. Take η small, but non-zero.

a) Use quasilinear theory to calculate the vertical flux of magnetic intensity. Since, $\Gamma \sim -\partial_z \ell n\left(\langle B\rangle/\rho\right)$, show that Γ may be written as

$$\Gamma = -D\frac{\partial\langle B\rangle}{\partial z} + V\langle B\rangle.$$

Calculate *D*, *V*. Interpret your result. For $\rho = \rho_0(z)$. What profile corresponds to the zero flux state?

b) What is the origin of the pinch velocity *V*? Explain its significance.

c) As a related example, consider evolution of the particle density according to

$$\partial n/\partial t + \nabla \cdot \left(n\underline{\mathrm{V}}\right) = 0.$$

Take $n_0 = n_0(x)$, $\underline{B} = B_0(x)\hat{z}$ and $\underline{V} = -\nabla\phi x\hat{z}/B_0(x)$.

Show that density evolution can be related to the incompressible advection of the field n/B:

$$\partial n/\partial t + \underline{\mathbf{V}}_{eff} \cdot \underline{\nabla}(n/B) = 0$$

where $\underline{\nabla} \cdot \underline{\mathbf{V}}_{eff} = 0$.

Show that the mean field equation for $\langle n \rangle$ obeys:

$$\frac{\partial}{\partial t}\langle n \rangle = \frac{\partial}{\partial x} D \frac{\partial}{\partial x}\left(\frac{\langle n \rangle}{\langle B \rangle}\right)$$

where we took $\langle n/B \rangle \cong \langle n \rangle / \langle B \rangle$. Discuss the zero flux state here. What are its implications for the density profile?

Re-write the mean field equation as

$$\frac{\partial \langle n \rangle}{\partial t} = -\frac{\partial}{\partial x}\left[-D\frac{\partial \langle n \rangle}{\partial x} + \mathrm{V}\langle n \rangle\right].$$

Relate D and V, here. Under what circumstances will V be inward, i.e. *up* the density gradient?

d) Relate the results of parts b.), c.) here. What is the lesson?

Congratulations! You have just developed the basics of TEP pinch theory!