Lecture 6 – Nonlinear Waves

Unit IV [Nonlinear Waves,] Shocks and Turbulence – An Introduction

Previously discussed:

→ basic waves in MHD, i.e. structure of MHD 'stiffness matrix'

→ δW and MHD instabilities (an introduction)

Now, are concerned with evolving waves and instabilities i.e. what happens? →

- nonlinear amplification of MHD waves, wavebreaking
- shocks and collisionless shocks in MHD,
- turbulence

Flow of development is:

waves (perturbation) → [instabilities; $\boxed{\text{nonlinear waves}}$ (amplification)] → [breaking, shocks, collisionless shocks, $\boxed{\text{turbulence}}$ (nonlinear evolution)]

→ [heating, mixing, transport, structure formation, disruption, ⋮] ↔ (practical consequences)

Need,

Wave Kinetics — [Quasi-particle Formulation

$\boxed{N(\underline{k}, \underline{x}, t) \rightarrow \text{effective distribution function of waves}}$

Why? →

"A wave is never found alone, but is mingled with as many other waves as there are uneven places in the object where said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions."

— Leonardo da Vinci

Codice Atlantico, c 1500.

⇒ Waves come in packets, spectra, etc.!

Read: { ① Kulsrud 5.5, 5.6
② Whitham, Chapt. 11
③ Landau Lifshitz Fluids Chapt. on Sound

→ <u>Nonlinear Waves</u>

→ have considered plane waves in uniform media

i.e. $\mathcal{E} \sim \mathcal{E}_{\underline{k}}\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

→ what if media <u>non-uniform</u>, but slowly varying?

i.e. $$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\tilde{\rho} = \nabla^2 \tilde{\rho}$$ (acoustics)

with $c^2 = c_0^2 / n^2(\underline{x})$

↦ index of refraction (can be time dependent)

then for $\left|\frac{\underline{\nabla} n}{n}\right| < |\underline{k}|$, can write

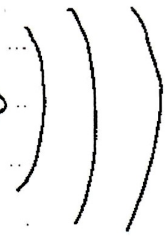
$$\tilde{\rho} = \rho_0\, e^{i\phi(\underline{x}, t)}$$

where $\phi \sim O(1/\epsilon)$
→ phase contains fastest variation

then have:

$$\boxed{\frac{n(\underline{x})^2}{c_0^2}\left(\frac{\partial \phi}{\partial t}\right)^2 = \left(\underline{\nabla}\phi\right)^2}$$

→ <u>eikonal equation</u> for phase front function ϕ

i.e. ⇒  iso-ϕ surfaces ⇒ $\underline{\nabla}\phi$ ⇒ direction of propagation clean analogy with plane waves $\Rightarrow$

$$\underline{\nabla}\phi \leftrightarrow \underline{k}$$

$$-\frac{\partial\phi}{\partial t} \leftrightarrow \omega$$

[if n time independent, $\omega =$ const, for linear wave]

so eikonal equation is :

$$\boxed{\frac{n(\underline{x})^2\omega^2}{c_0^2} = k^2}$$

→ local dispersion relation

so, have for medium with no explicit time dependence,

$$d\phi = \frac{\partial\phi}{\partial\underline{x}}\cdot d\underline{x} + \frac{\partial\phi}{\partial t}dt$$

$$= \underline{k}(\underline{x})\cdot d\underline{x} - \omega(\underline{k},\underline{x})\,dt$$

→ via Eikonal Equation

$$\therefore \frac{d\phi}{dt} = \underline{k}(\underline{x})\cdot\frac{d\underline{x}}{dt} - \omega(\underline{k},\underline{x})$$

$\Rightarrow$

$$\Phi = \int dt\left[\underline{k}(\underline{x})\cdot\dot{\underline{x}} - \omega\right]$$

but recall:

$$\underbrace{S}_{\text{action}} = \int dt \left(p\dot{q} - \underbrace{H}_{\text{Hamiltonian}} \right)$$

and $\delta S = 0 \Rightarrow$ equations of motion

Can immediately note analogy:

Hamiltonian Dynamics	Rays/Eikonal Theory
$\underline{p}$ → momentum ($= \partial L / \partial \dot{q}$)	$\underline{k}$ ($= \nabla \phi$)
$\underline{q}$ → gen. coord	$\underline{x}$ (phase front position)
H → Hamiltonian	ω (frequency)
ϕ → phase function	S → action

and recall Hamilton-Jacobi Equation:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0$$

(Eikonal Eqn. for S. E.)

⇒ phase evolution equation:

$$\frac{\partial \phi}{\partial t} + \omega(\underline{k}, \underline{x}) = 0$$

$$\underline{k} = \underline{\nabla}\phi$$

exact isomorphism

$\therefore$ just as advance Hamiltonian variables in time via Hamilton's Eqn. of Motion, i.e.

$$\frac{d\underline{p}}{dt} = -\frac{\partial H}{\partial \underline{q}}, \quad \frac{d\underline{q}}{dt} = \frac{\partial H}{\partial \underline{p}}$$

then can advance $\underline{k}$ and $\underline{x}$ analogously by:

Ray Eikonal Equations

$$\boxed{\frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}} \; ; \quad \frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}} = \underline{V}_{gr}}$$

Snell's Law — group velocity

$\underline{k} = \underline{\nabla}\phi \Rightarrow$ phase front orientation

$\underline{x} \rightarrow$ position of phase front.

check: If analogy is valid, should be able to derive eikonal equations from $\delta \underline{\Phi} = 0$

$$\underline{\Phi} = \int dt \left[\underline{k} \cdot \dot{\underline{x}} - \omega(\underline{k}, \underline{x})\right]$$

show!

$$\delta \underline{\Phi} = \int dt \left[\underline{k} \cdot \delta \dot{\underline{x}} + \delta \underline{k} \cdot \dot{\underline{x}} - \left(\frac{\partial \omega}{\partial \underline{x}} \cdot \delta \underline{x} + \frac{\partial \omega}{\partial \underline{k}} \cdot \delta \underline{k} \right) \right]$$

$\delta \underline{x} = \delta \underline{k} = 0$ at end-points.

$\Rightarrow$

$$\delta \underline{\Phi} = \int dt \left[\left(\underline{k} \cdot \frac{d}{dt} \delta \underline{x} - \frac{\partial \omega}{\partial \underline{x}} \cdot \delta \underline{x} \right) + \left(\dot{\underline{x}} - \frac{\partial \omega}{\partial \underline{k}} \right) \cdot \delta \underline{k} \right]$$

ibp

$$\delta \underline{\Phi} = \cancel{\underline{k} \cdot \delta \underline{x} \Big|_{t_1}^{t_2}} + \int dt \left[\left(\frac{d\underline{k}}{dt} + \frac{\partial \omega}{\partial \underline{x}} \right) \cdot \delta \underline{x} + \left(\dot{\underline{x}} - \frac{\partial \omega}{\partial \underline{k}} \right) \cdot \delta \underline{k} \right]$$

$\Rightarrow$

$\delta \underline{x}, \delta \underline{k} \neq 0, \Rightarrow$

$$\frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}, \qquad \frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}}$$

so → eikonal equations are Hamiltonian equations

→ eikonal equations extremize Φ.

→ ite eikonal equations satisfy Liouville's Theorem!

i.e. "flow" in phase space $\underline{k}, \underline{x}$ is incompressible

$$\frac{\partial}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} + \frac{\partial}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt} = -\frac{\partial^2 \omega}{\partial \underline{k} \partial \underline{x}} + \frac{\partial^2 \omega}{\partial \underline{x} \partial \underline{k}}$$

$$= 0$$

∴ so if define wave density $\rho(\underline{k}, \underline{x}, t)$

then $\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\underline{V}_T \rho) = 0$

but $\underline{\nabla} \cdot \underline{V}_T = 0$

$$\boxed{\underline{V}_T = \left[\frac{d\underline{x}}{dt}, \frac{d\underline{k}}{dt}\right]}$$

$\Rightarrow$ $\frac{\partial \rho}{\partial t} + \underline{V}_T \cdot \underline{\nabla} \rho = 0$

Phase Space Flow

$$\frac{\partial \rho}{\partial t} + \underline{V}_{gr} \cdot \frac{\partial}{\partial \underline{x}} \rho - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial \rho}{\partial \underline{k}} = 0$$

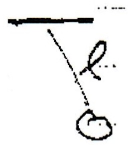
⇒ Vlasov-like equation for evolution of ρ!

but... what is ρ?

→ physical argument:

have $\frac{d\rho}{dt} = 0$ → conservation/invariance principle

Now, recall for oscillator with slowly varying parameters



$$\ell = \ell(t)$$

$$\frac{1}{\ell(t)} \frac{d\ell}{dt} < \omega = \sqrt{g/\ell}$$

then $\frac{d}{dt}\left(E/\omega\right) = 0$

$E/\omega \equiv$ Action (dims energy * time)

$$E = 2 \cdot \tfrac{1}{2} m \omega^2 \ell^2 \theta^2 = mg\ell\theta^2$$

so $$\frac{E}{\omega} = m\sqrt{g}\, \ell^{3/2}\, \theta^2$$

$$\Rightarrow d(E/\omega) = 0 \Rightarrow \frac{3}{2}\ell^{1/2}\frac{d\ell}{dt} + \ell^{3/2}\frac{d\theta^2}{dt}$$

$$d\theta^2/dt = -3/2\,\frac{1}{\ell}\frac{d\ell}{dt}$$

→ ℓ shortened ($\dot{\ell}<0$) amplitude increased

→ ℓ lengthened, amplitude decreased.

Now, for waves argue analogue of action is wave action density $\mathcal{E}/\omega = N$

$\mathcal{E}$ = energy density
N = action density

so wave kinetic equation is:

$$\boxed{\frac{\partial N}{\partial t} + \underline{v}_{gr}\cdot\frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}}\cdot\frac{\partial N}{\partial \underline{k}} = 0}$$

and analogy with Vlasov equation is evident, i.e.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0.$$

Wave Adiabatic Theory / Wave Kinetics

- frequently encounter problems with slowly varying parameters ⇒ adiabatic theory

⇒

- wave kinetic equation (consequence of Liouville Thm.)

$$\partial_t N + (\underline{V}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \partial_{\underline{x}} (\omega + \underline{k} \cdot \underline{V}) \cdot \partial_{\underline{k}} N$$

$$= C(N)$$; obvious analogy to Boltzmann Eqn.

$N \equiv \mathcal{E} / \omega_{\underline{k}}$ ≡ wave action density / wave quanta density

$\mathcal{E}$ ↓ wave energy density $\mathcal{E} = \left. \frac{\partial}{\partial \omega} (\omega \epsilon_k) \right|_{\omega_k} \frac{|E_k|^2}{8\pi}$, for e.s.

characteristics:

$$\frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}} \hat{k} + \underline{V} \quad , \quad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V})$$

refraction by shear (on $\underline{k}\cdot\underline{V}$)

refraction by parametric variation (on ω)

- need:

$$\omega < \frac{1}{\lambda} \frac{d\lambda}{dt} \qquad \lambda \equiv \text{parameter}$$

$\boxed{\text{space and time scale separation}}$

$$\frac{1}{N} (\underline{V}_{gr} \cdot \nabla N) \ll \omega \quad \Rightarrow \quad \underline{\kappa} \cdot \underline{V}_{gr} < \omega$$

CCN) → interactions with comparable scale.

Examples :

- linear theory of Langmuir turbulence
  i.e. when will phonon grow ?
- QL theory of Langmuir turbulence
  i.e. determine evolution of plasma
  energy → net impact ?!
- drift waves and sheared flow.

⋮

Fundamentals of Wave Kinetics

→ where does conservation of action
  emerge from ?

→ answer: phase symmetry (of wave train) underlies
  wave kinetics

→ approach via variational principle.

c.f. Whitham: "Linear and Nonlinear Waves"
Chapt. 14.

→ Derivation

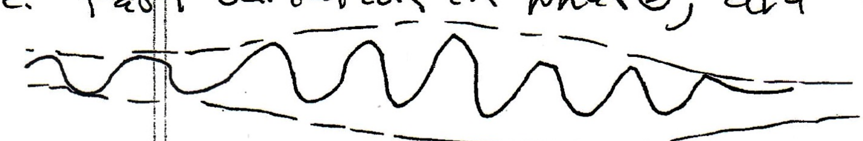
Consider a system, like ideal MHD, which can be described in terms of displacement $\underline{\xi}$:

i.e. $\underline{\xi} = \mathrm{re}\left\{ A\, e^{i\phi} + A^{*} e^{-i\phi} \right\}$

then wave equation arises from:

$$\delta S = \delta \int dt \int d\underline{x}\; \mathcal{L}(\xi)$$

- Envision a wave train, with slowly varying amplitude, so eikonal approach optimal i.e. fast variation in phase, ala WKB:



$$S = \int dt \int d\underline{x}\; \mathcal{L}(\omega, \underline{k}, \underset{\text{amplitude}}{a}) \qquad \underline{k} = \nabla\phi, \quad \omega = -\partial_t \phi$$

$$= \int dt \int d\underline{x}\; \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

→ neglect all corrections to eikonal theory.

⇒ - here L corresponds to period-averaged Lagrangian
- ϕ undetermined to const → phase symmetry!

∴ to vary:

$$\delta S/\delta a = 0$$

$$\delta S/\delta \phi = 0$$

Now, in linear theory:

$$\left[G(k,\omega) \equiv \frac{\partial G}{\partial \omega} \right]$$

$$\mathcal{L} = G(\omega, k)\, a^2$$

i.e. for MHD, as in wave section:

$$\mathcal{L} = \frac{1}{2}\rho \dot{\underline{\xi}}^2 - \frac{1}{2}\rho \left[\underline{\underline{D}}(k, x, t)\right]^2 \underline{\xi}^2$$

concrete form of Lagrangian

↳ eikonal form of stiffness matrix (→ potential energy)

$$\Rightarrow \underline{\xi} \cdot \underline{\underline{M}} \cdot \underline{\xi}$$

$M(k, \omega, x)$, as for linear waves

If: $\underline{\xi} = \underline{A} e^{i\phi} + \underline{A}^* e^{-i\phi}$

so $\quad G(\omega, \underline{k}) = \frac{1}{2}\rho\left[\left(\frac{\partial \phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2\right]$

Now, 1) $\delta S/\delta a = 0$

$\Rightarrow \quad G(\omega, k) = 0$

but
$$G(\omega, k) = \rho\left(\frac{\partial \phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2$$
$$= \rho\omega^2 - D^2$$

$\Rightarrow$ dispn. relation

2) $\delta S/\delta\phi = 0$

$$\delta S = \int dt \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(-\phi_t)} \delta(-\phi_t) + \frac{\partial \mathcal{L}}{\partial(\phi_{\underline{x}})} \delta(\phi_{\underline{x}}) \right\}$$

end pts. fixed, ibp:

$$= \int dx \int d^3x \left\{ \partial_t\left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\partial \mathcal{L}}{\partial(\underline{\nabla}\phi)}\right)\right\} \delta\phi$$

$\delta S = 0 \Rightarrow$

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)}\right) - \underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

Now, have:
$$\begin{cases} G(\underline{k}, \omega) = 0 & \text{(dispn. reln.)} \\ \partial_t\left(\dfrac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla\cdot\left(\dfrac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0 \end{cases}$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}} \cdot d\underline{k} = 0$$

$$\therefore \quad V_{gr} = \frac{d\omega}{d\underline{k}} = \frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}$$

(akin ω)

$\therefore$

$$\partial_t\left((\partial G/\partial \omega) a^2\right) + \nabla\cdot\left[\frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}\frac{\partial G}{\partial \omega} a^2\right] = 0$$

and so:
$$\boxed{N \equiv \frac{\partial G}{\partial \omega} a^2}$$

$$\frac{\partial N}{\partial t} + \nabla\cdot\left(\underline{v}_{gr} N\right) = 0$$

(N not yet action)

→ Also note energy is conserved ⟺ G invariant to time translations.

so, Noethers thm ⇒ there exists an [redacted] energy conservation equation.

have $\mathcal{L} = G(k,\omega)\, a^2$

$$\partial \mathcal{L}/\partial a = 0 \Rightarrow G(\omega, k) = 0$$

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

and of course:

$$\underline{\nabla} \times \underline{k} = 0 , \quad \text{as } \underline{k} = \underline{\nabla} \phi$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} , \quad \text{as } \partial_t \underline{\nabla} \phi = -\underline{\nabla}\left(-\frac{\partial}{\partial t}\phi\right) \checkmark$$

Now, $\mathcal{L} = 0$, as $G(k,\omega) = 0$

as expect $\frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow N$, $\omega \frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow \mathcal{E}$

⇒ 0, [redacted] creatively

⇒ $\frac{\partial}{\partial t}\Big(\omega \frac{\partial \mathcal{L}}{\partial \omega} -$ [redacted] $\mathcal{L}\Big) + \underline{\nabla} \cdot \Big[-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \Big] = 0$ [energy cons.

$$-\frac{\partial G/\partial k}{\partial G/\partial \omega} \frac{\partial G}{\partial \omega} a^2$$

[redacted] $\underline{v_{gr} \mathcal{E}}$

$$\partial_t \left(\omega \mathcal{L}_\omega - \mathcal{L} \right) + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

check:

$$(\partial_t \omega)\mathcal{L}_\omega + \omega\, \partial_t(\mathcal{L}_\omega) - \partial \mathcal{L}/\partial t$$

$$+ \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

⇒ but $\partial_t \mathcal{L}_\omega = \nabla \cdot (\mathcal{L}_{\underline{k}})$

∴ $(\mathcal{L}_\omega)(\partial_t \omega) + \cancel{\omega\, \nabla \cdot (\mathcal{L}_k)} - \cancel{\omega\, (\underline{\nabla} \cdot \mathcal{L}_{\underline{k}})}$

$$-\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) \cdot \underline{\nabla} \omega - \frac{\partial \mathcal{L}}{\partial t}$$

but $\partial_t \underline{k} = -\underline{\nabla} \omega$

$$(\partial_t \omega)(\mathcal{L}_\omega) + (\partial_t \underline{k}) \cdot \frac{\partial \mathcal{L}}{\partial \underline{k}} - \frac{\partial \mathcal{L}}{\partial t} = 0 \quad \checkmark$$

(identity)

$$\Rightarrow \boxed{\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0}$$

But $G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

$\therefore$

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

so

$$\mathcal{E} \equiv \omega \frac{\partial \mathcal{L}}{\omega} \rightarrow \text{energy density}$$

so

$$\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega \Rightarrow \text{action density } \mathcal{J}$$

$$= N$$

so have:

$$\boxed{\partial_t (N) + \nabla \cdot (\underline{v}_{gr} N) = 0}$$

wave - kinetics

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \underline{\nabla} N - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla \cdot (\underline{v}_{gr} N) + \nabla_{\underline{k}} \cdot \left(-\partial \omega / \partial \underline{x} \, N \right) = 0$$

$\int d\underline{k}$, and assume narrow spread in $\underline{k}$ (i.e. wave packet) $\Rightarrow$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left[\underline{V}_{gr} N \right] = 0$$

Observe:

→ Vlasov-like equation in eikonal phase space $(\underline{x}, \underline{k})$:

$$\frac{\partial N}{\partial t} + \underline{V}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

→ continuity-type equation in $\underline{x}$-space, for packet.

$$\frac{\partial N}{\partial t} + \nabla \cdot (\underline{V}_{gr} N) = 0$$

Also observe:

→ seeming issue re:

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

Now $\partial_t \underline{k} = -\partial\omega/\partial\underline{x}$ is (Eulerian)
(partial) relation in $\underline{x}, t$

$d\underline{k}/dt = -\partial\omega/\partial\underline{x}$ is (Lagrangian)
(total) relation, following packet)

(here $\omega = D(\underline{k}, \underline{x}, t)$, as $\sigma = 0$)

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr}\cdot\nabla \underline{k}$$

$$= -\frac{\partial\omega}{\partial\underline{x}} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\underline{k}}{\partial\underline{x}}$$

$\therefore$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial D}{\partial \underline{x}}$$ agree!

→ Now, can convert from N to E

i.e. $N = E/\omega$

$$\left.\frac{dN}{dt}\right|_{rays} = \frac{d}{dt}\left(E/\omega\right) = 0$$

$$\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\bigg|_{rays} - \frac{1}{\omega^2}\,\mathcal{E}\,\frac{d\omega}{dt}\bigg|_{rays} = 0$$

Now $$\frac{d\omega}{dt} = \partial_t\omega + \frac{\partial\omega}{\partial \underline{x}}\cdot\frac{d\underline{x}}{dt} + \frac{\partial\omega}{\partial \underline{k}}\cdot\frac{d\underline{k}}{dt}$$

From eikonal eqns:

$$= \partial_t\omega + \cancel{\frac{\partial\omega}{\partial \underline{x}}\cdot\frac{\partial\omega}{\partial \underline{k}}} - \cancel{\frac{\partial\omega}{\partial \underline{k}}\cdot\frac{\partial\omega}{\partial \underline{x}}}$$

so if $\partial_t\omega = 0$

$\therefore$ $$\frac{dN}{dt} = 0 \Rightarrow \frac{d\mathcal{E}}{dt} = 0.$$

so $$\partial_t\mathcal{E} + \underline{v}_{gr}\cdot\underline{\nabla}\mathcal{E} - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_k\mathcal{E} = 0$$

and exploiting Liouville Thm, etc $\Rightarrow$

$$\boxed{\frac{d\mathcal{E}}{dt} = \partial_t\mathcal{E} + \nabla\cdot\left[\underline{v}_{gr}\,\mathcal{E}\right] = 0}$$

so, for conservative case i.e. $\partial_t \omega = 0$

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot \left[\underline{V}_{gr} \mathcal{E} \right] = 0}$$

If stationary, $\partial_t \mathcal{E} = 0$

$\Rightarrow$

$$\boxed{\underline{\nabla} \cdot \left[\underline{V}_{gr} \mathcal{E} \right] = 0}$$

incompressible wave energy flux!

$\Rightarrow$ V_{gr} drops $\Rightarrow$ $\mathcal{E} \uparrow$ $\Rightarrow$ blocking, breaking

Summary

Recall:

→ Hamiltonian structure of eikonal theory, etc. ⇒

$$\frac{\partial}{\partial t}\rho(\underline{k},\underline{x},t) + \underline{v}_{gr}\cdot\underline{\nabla}\rho(\underline{k},\underline{x},t) - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_k\,\rho(\underline{k},\underline{x},t) = 0$$

→ Physical arguments suggest $\rho = \dfrac{\varepsilon}{\omega} = N$

wave action density

→ Variational Approach

$$S = \int dt \int d^3x\, \mathcal{L}\,, \qquad \mathcal{L} = G(\omega,\underline{k})\,a^2$$

$$\omega = -\partial\phi/\partial t = -\phi_t$$

$$\underline{k} = \underline{\nabla}\phi = \phi_{\underline{x}}$$

$$\delta S = 0$$

but two parameters varied $\left\langle \begin{array}{l} a \\ \phi \end{array}\right.$

$\delta S/\delta a = 0 \Rightarrow G(\omega,\underline{k}) = 0$ → dispersion relation

$$\delta S/\delta\phi = 0 \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\omega}\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial\omega}a^2\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial G}{\partial\underline{k}}a^2\right) = 0$$

and time translation symmetry and $G = 0$ ⇒

$$\mathcal{E} = \omega \frac{\partial G}{\partial \omega} a^2 \quad \Rightarrow \quad N = \frac{\mathcal{E}}{\omega} = \frac{\partial G}{\partial \omega} a^2$$

and $\qquad \dfrac{\partial G}{\partial \underline{k}} a^2 = \underline{V}_{g_0} \cdot N$

→ Helpful Reminder:

recall, for electrostatic plasma waves

if $\quad \epsilon(\omega, \underline{k}) = 0 \quad \Rightarrow$ dispersion relation

then
$$\mathcal{E}_k = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$= \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi} \quad \rightarrow \text{wave energy density}$$

$$\therefore \quad N_k = \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

and
$$\underline{\Gamma}_k = -\frac{\partial \epsilon}{\partial \underline{k}}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi} \quad \rightarrow \text{wave energy density flux}$$

$$= \underline{V}_{gr} N_k$$

since $G(\underline{k}, \omega) = 0$, so along rays

$$dG = d\omega \frac{\partial G}{\partial \omega} + d\underline{k} \cdot \frac{\partial G}{\partial \underline{k}} = 0$$

$$d\omega / d\underline{k} = -(\partial G/\partial \underline{k}) / (\partial G/\partial \omega)$$

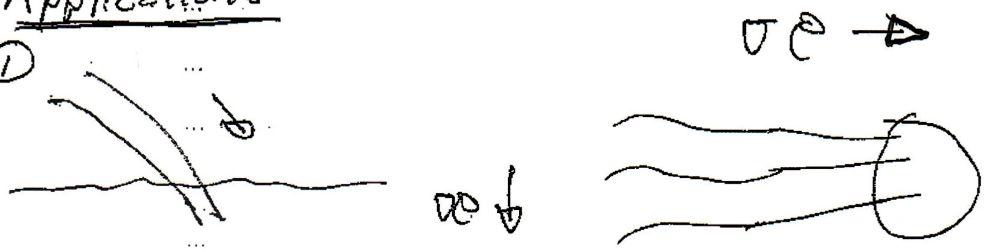
etc.

$$\frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla}\cdot\left[\underline{V}_{g} \, \mathcal{E}\right] = 0$$

applies to conservative case.

Applications

(1)



Alfven wave packet incident on region with density increasing, field fixed.

i.e.

$$\underline{\nabla}\cdot\left(\underline{V}_{gr}\,\mathcal{E}\right) = 0 \qquad \underline{B} = B\hat{z}$$

$$\frac{\partial}{\partial z}\left(V_A\,\mathcal{E}\right) = 0 \qquad V_A = B\big/\sqrt{4\pi\rho(z)}$$

$$V_{A\infty}\,\mathcal{E}_{\infty} = V_A(z)\,\mathcal{E}(z)$$

$\S$

Inflow I

$$I = V_A(z)\,\mathcal{E}(z)$$

$$\cong V_{A\infty}\sqrt{\frac{\rho_\infty}{\rho(z)}}\;\mathcal{E}(z)$$

$$\Rightarrow\; \mathcal{E}(z) = \left(\rho(z)/\rho_\infty\right)^{1/2}\mathcal{E}_\infty$$

→ wave energy density increases in high density region

→ point is $V_{gr}\,\mathcal{E}$ = const

$V_{gr} = V_A \downarrow$ while $\rho \uparrow$, so $\mathcal{E}$ does increase

How about displacement?

- very roughly speaking:

as wave is linearization, and assumes/predicts certain phase relation;

→ linear wave theory valid for ?

$$|k\tilde{\xi}| < 1$$

↳ wave slope

If $k\tilde{\xi} \sim 1 \Rightarrow$ expect strongly nonlinear behavior, breaking, mixing etc.

n.b. though for Alfven waves, need add parallel compressibility....

Now $\mathcal{E}(z) = 2\frac{\rho}{2}\overline{\tilde{\dot{\xi}}^2}$

$= \rho(z)\,\omega^2\,\tilde{\xi}^2$

Now – ω = const.

so – $\tilde{\xi}^2 = \frac{1}{\rho(z)\omega^2}\left(\frac{\rho(z)}{\rho_\infty}\right)^{1/2}\mathcal{E}_\infty$

$= \frac{1}{\sqrt{\rho(z)\rho_\infty}}\,\frac{\mathcal{E}_\infty}{\omega^2}$

$\therefore$ displacement drops $\sim \rho(z)^{-1/4}$

as wave propagates into high density region.

but – slope $s \sim |k\tilde{\xi}|$

$k = \frac{\omega}{V_A} = \frac{\omega}{V_{A\infty}}\sqrt{\frac{\rho(z)}{\rho_\infty}}$

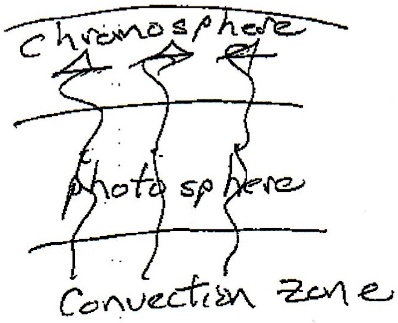
so $$|k\tilde{\mathcal{E}}| \sim \frac{\omega}{V_{A0}} \sqrt{\frac{\rho(z)}{\rho_{00}}} \, \frac{(\mathcal{E}_{00})^{1/2}}{\omega} \, \frac{1}{(\rho(z)\rho_{00})^{1/4}}$$

$$\boxed{\sim \rho(z)^{1/4}}$$

⇒ wave slope increases in high density region, as V_A changes

⇒ Nonlinearity increases

② Sound propagating in chromosphere



$\rho \sim e^{-z/H}$ → density decreases with height

Sound waves emitted from convection zone (compressible convection) → propagate into chromosphere

Take T = const ⇒ $\boxed{c_s = \text{const.}}$

Then $c_s \mathcal{E} = \text{const.}$

$\mathcal{E}(z) = \text{const.}$

and $k = \omega / c_s = \text{const.}$

so $\quad \cancel{k}\,\dfrac{\rho}{\cancel{k}}\,\tilde{\dot{\xi}}^2 = \text{const}$

$$\rho(z)\,\omega^2\,\tilde{\xi}^2 = \text{const}$$

$$\Rightarrow \quad \tilde{\xi} = \left(\mathcal{E}_{00}/\rho(z)\,\omega^2\right)^{1/2} \sim 1/(\rho(z))^{1/2}$$

as $\quad k = \text{const}, \qquad k\tilde{\xi} \sim 1/(\rho(z))^{1/2}$

$\Rightarrow$ – wave displacement <u>increases</u> in chromosphere

– sound wave simple $\Rightarrow$ wave steepens and can shock

– physical picture is that of a whip $\Rightarrow$ inertia at tip low, due tapering.

– constitutes simple argument for chromospheric and possibly coronal heating by sound waves propagating from convection zone into upper layers.