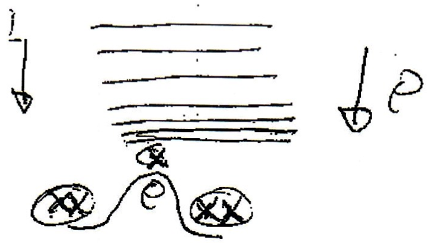
iii.) Interchange Instability

(basic confinement consideration)

→ consider plasma confined by magnetic pressure gradient



$$\underline{\nabla} p = \underline{J} \times \underline{B} + \rho \underline{g}$$

str. lines

$$\frac{dp}{dz} = -\nabla\left(\frac{B^2}{8\pi}\right) + \cancel{\frac{\underline{B}\cdot\underline{\nabla}\underline{B}}{4\pi}} + \rho \underline{g}$$

$\beta \ll 1$

$$\boxed{\underline{\nabla}\left(\frac{B^2}{8\pi}\right) = \rho \underline{g}}$$

$\underline{g} = -g\,\hat{z}$

$p \to 0$

equilibrium

$$\to \quad \delta W = \int d^3x \left[\frac{Q^2}{8\pi} + (\underline{\nabla}\cdot\underline{\xi})^2 \gamma p + \underline{j}_0 \cdot (\underline{\xi} \times \underline{Q}) + (\underline{\xi}\cdot\underline{\nabla} p_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)\,\underline{\nabla}\cdot(\rho_0 \underline{\xi}) \right]$$

$j_0 = 0$

$p_0 = 0$

$$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + (\underline{\xi}\cdot\underline{g})\left(\underline{\xi}\cdot\underline{\nabla}\rho_0 + \rho_0 \underline{\nabla}\cdot\underline{\xi}\right)\right]$$

Here, must address $\underline{Q}$,

$$\underline{Q} = \underline{B}_0 \cdot \underline{\nabla}\, \underline{\xi} - \underline{\xi} \cdot \underline{\nabla}\, \underline{B} - \underline{B}_0\, \underline{\nabla} \cdot \underline{\xi}$$

Now, can have $\underline{Q} = 0$ if:

→ $\underline{B}_0 \cdot \underline{\nabla}\, \underline{\xi} = 0$ i.e. $\underline{\xi}$ constant along $\underline{B}_0$

⇒ $k_{\parallel} = 0$

and

→ $\underline{\nabla} \cdot \underline{\xi} = - \dfrac{\underline{\xi} \cdot \underline{\nabla} B_0}{B_0}$

∴

$$\delta W = \int d^3x \left[(\underline{\xi} \cdot \underline{g})\, \rho_0 \left(\frac{\underline{\xi} \cdot \underline{\nabla}\, \rho_0}{\rho_0} - \frac{\underline{\xi} \cdot \underline{\nabla} B_0}{B_0} \right) \right.$$

$$= \int d^3x \left[(\underline{\xi} \cdot \underline{g} \rho_0)\; \underline{\xi} \cdot \underline{\nabla} \ln (\rho/B) \right]$$

$g < 0 \Rightarrow$ if $\nabla \ln (\rho/B) > 0$ anywhere

∴ instability there !

Now:

→ obvious parallel to Rayleigh-Taylor is

$\nabla \rho > 0 \quad \Leftrightarrow \quad \boxed{\nabla \ln(P/B) > 0}$

→ as $k_{\parallel} = 0$, field lines not bent

so can think of instability motion as interchange of flux tubes

Key question: Does interchange lower/raise potential energy?

interchange conserves magnetic flux

$$\Phi_2 = \int B_2 \, da = B_2 A_2$$

$$\Phi_1 = \int B_1 \, da = B_1 A_1$$

$$M_2 = \left(\frac{\rho}{B}\right)_2 \Phi_2$$

$M \Leftrightarrow$ m/length

$$M_1 = \left(\frac{\rho}{B}\right)_1 \Phi_1$$

but $\phi_1 = \phi_2$ $\Rightarrow$ flux frozen in! (not equal $\rightarrow$ violation)

$$M_{2,1} = (\rho/B)_{2,1}\,\Phi$$

so $\Delta M \gtrless 0 \Rightarrow \Delta(\rho/B) \gtrless 0$

$\Rightarrow$ if ρ/B increases interchange will liberate gravitational potential energy, i.e. instability, a la' R.-T.

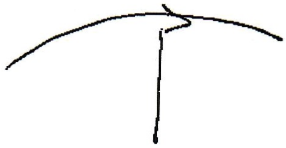
$\Rightarrow$ why care?:

- interchange instability severely degrades plasma confinement

- curing interchange stability is key element in device design $\rightarrow$ "minimum-B"

iv.) Interchange without Gravity — Expansion Free Energy

- in the context of magnetic confinement, "g" is a crutch, to represent curved field lines

— i.e. 

$$\underline{g} = \frac{V^2}{R_c} \rightarrow \underline{g}_{eff}$$

[centrifugal acceleration on particle]

— $\boxed{\text{natural to investigate interchanges without "}\underline{g}\text{" } \Rightarrow \text{ pressure gradient drive (expansion free energy)}}$

- now

$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + \gamma p (\underline{\nabla}\cdot\underline{\xi})^2 + \underline{\xi}\cdot\underline{\nabla} p\, (\underline{\nabla}\cdot\underline{\xi}) + \underline{j}\cdot\underline{\xi}\times\underline{Q} \right]$$

[will pressure gradient relax?]

Now, $\underline{Q} = 0 \rightarrow$ avoid bending, etc.

$$\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0) = 0$$

$$\Rightarrow \quad \underline{\xi}\times\underline{B}_0 = \nabla\phi$$

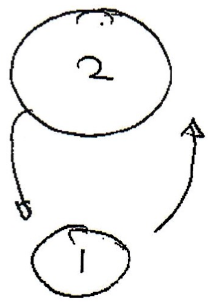
↳ some scalar potential

and $\underline{B}_0\cdot\underline{\nabla}\phi = 0 \Rightarrow \phi$ constant along lines of force ---

and can formulate dW in terms ϕ, or....

$\Rightarrow$ consider interchange, with flux conservation

$$\Phi_1 = \Phi_2$$



$\boxed{\text{Does interchange raise or lower energy?}}$

ΔE = [final energy of ①] − [initial energy of ①]
+ [final energy of ②] − [initial energy of ②]

where interchange

a) "puts" ① into ② slot
"puts" ② into ① slot

b) keeps $P\rho^{-\gamma} = PV^{\gamma} = \text{const}$

$V \equiv$ volume of flux tube

$\Rightarrow$ final energy of ① $\rightarrow$ (new P_1) $V_2/(\gamma-1)$

final energy of ② $\rightarrow$ (new P_2) $V_1/(\gamma-1)$

so ...

$$\Delta E = \Delta W = \frac{1}{(\gamma-1)}\left[(p_1' V_2 - p_1 V_1) + (p_2' V_1 - p_2 V_2)\right]$$

and

$$\left.\begin{aligned} p_1' V_2^{\gamma} &= p_1 V_1^{\gamma} \\ p_2' V_1^{\gamma} &= p_2 V_2^{\gamma} \end{aligned}\right\}$$

from eqn. state
$p' \equiv$ pressures of displaced flux tubes
(argument akin to Schwarzschild)

$\Rightarrow$

$$(\gamma-1)\Delta W = \left\{ p_1\left[\left(\frac{V_1}{V_2}\right)^{\gamma} V_2 - V_1\right] + p_2\left[\left(\frac{V_2}{V_1}\right)^{\gamma} V_1 - V_2\right]\right\}$$

$$V_2 = V_1 + \delta V$$
$$p_2 = p_1 + \delta p$$

$$(\Delta W)(\gamma-1) = \left\{ p_1\left[\left(\frac{V_1}{V_1+\delta V}\right)^{\gamma}(V_1+\delta V) - V_1\right] + (p_1+\delta p)\left[\left(\frac{V_1+\delta V}{V_1}\right)^{\gamma} V_1 - (V_1+\delta V)\right]\right\}$$

$$(\gamma-1)\,\Delta W = \Big\{ P_1 V_1 \Big[\Big(1+\frac{\delta V}{V}\Big)^{-(\gamma-1)} - 1 \Big]$$

$$+ P_1 V_1 \Big(1+\frac{\delta P}{P}\Big) \Big[\Big(\cancel{1}+\frac{\delta V}{V}\Big)^{\gamma} - \Big(\cancel{1}+\frac{\delta V}{V}\Big) \Big] \Big\}$$

$$= P_1 V_1 \Big\{ \Big[\cancel{1} - (\gamma-1)\frac{\delta V}{V} + \frac{(\gamma-1)(\gamma)}{2} \Big(\frac{\delta V}{V}\Big)^2 - \cancel{1} \Big]$$

$$+ \Big(1+\frac{\delta P}{P}\Big) \Big[\cancel{1} + \gamma\frac{\delta V}{V} + \frac{\gamma(\gamma-1)}{2}\Big(\frac{\delta V}{V}\Big)^2 - \cancel{1} - \frac{\delta V}{V} \Big] \Big\}$$

$$= P_1 V_1 \Big\{ -\cancel{(\gamma-1)\frac{\delta V}{V}}^{\,1} + \frac{(\gamma-1)(\gamma)}{2}\Big(\frac{\delta V}{V}\Big)^2$$

$$+ \cancel{\gamma\frac{\delta V}{V}}^{\,1} - \cancel{\frac{\delta V}{V}}^{\,1} + \frac{\delta P}{P}(\gamma-1)\frac{\delta V}{V} + \gamma\frac{(\gamma-1)}{2}\Big(\frac{\delta V}{V}\Big)^2 \Big\}$$

<0

$$\boxed{\frac{\Delta W}{P_1 V_1} = \gamma\Big(\frac{\delta V}{V}\Big)^2 + \frac{\delta P}{P}\frac{\delta V}{V}}$$

→ generic expression for interchange δW

>0   >0 or <0

$$\gamma\frac{\delta V}{V}\Big[\gamma\frac{\delta V}{V} + \frac{\delta P}{P}\Big]$$

$+ \quad + \quad -$

$\frac{\delta P}{P}$ beat $\gamma\frac{\delta V}{V}$!

$\delta W =$

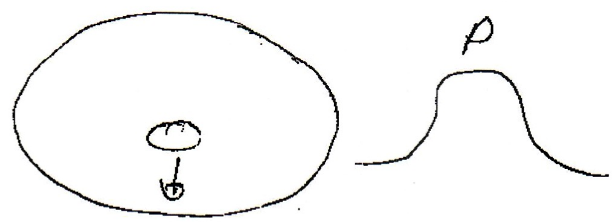
Clearly,

$\frac{\delta V}{V} \sim (\nabla \cdot \xi)$, $\frac{\delta p}{p} \sim$ $\xi \cdot \nabla p$

and

→ expansion free energy relaxation ⇒

$\delta P < 0$ → i.e.



pressure higher in center so occurs.

$\delta P < 0$ ⇒ relaxation

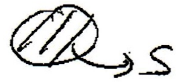
∴ Key is sign $\frac{\delta V}{V}$ → > 0 → instability (inflate)

< 0 → stability (compress)

→ Now, for flute perturbation ($k_\parallel = 0$)

Volume ↔ B

$V = \int S \, dl$ $S \equiv$ cross-sectional area of tube.



but $\Phi = B(l) \, S(l)$ = const

⇒

$$\bar{V} = \Phi \int \frac{d\ell}{B} \quad \Rightarrow \quad \frac{\delta V}{V} < 0$$

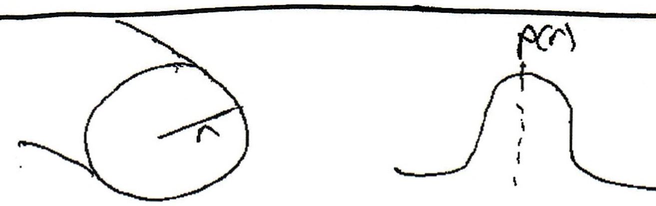
⇒ $$\boxed{\delta \int \frac{d\ell}{B} < 0}$$ → condition for interchange stability ($\delta p < 0$ disp pt.)

$\delta p \; \delta V > 0$

$< 0 \;\; < 0$

→ content of criterion is that configuration should have a minimum in B in the core, to confine pressure

i.e.



then stable if:

$$\boxed{\delta \int \frac{d\ell}{B} < 0}.$$

⇒ "minimum B" criterion for stability.

min B → max volume

→ if define ψ → label of surface enclosing const flux Φ

$\therefore$ $V(\psi) \equiv$ volume enclosed by flux surface

$p(\psi) \equiv$ pressure enclosed

$dp/d\psi < 0$ ⇒ need $\dfrac{d^2V}{d\psi^2} > 0$ expand

$\nabla p < 0$ ⟺ minimum-B ⟺ minimum

exp free energy!

→ Can re-write instability criterion:

$$\delta W = p_1 \, \delta V \left(\gamma \frac{\delta V}{V_1} + \frac{\delta p}{p_1} \right)$$

$$= p_1 \, \delta V \left[\delta \ln (p V^\gamma) \right]$$

so $\boxed{\delta (p V^\gamma) < 0}$ → inst. (akin Schwarzschild)

Also, if tube ① has flux ψ, then

$$v = u\psi$$

$\Rightarrow$

$$\frac{\delta W}{\psi} = \rho\,\delta u\,\frac{\delta(\rho u^{\gamma})}{\rho u^{\gamma}} < 0$$

$$\begin{aligned}\delta W &= \rho\,\delta v\left[\delta\left(\ln(\rho v^{\gamma})\right)\right]\\ &= \rho\psi\,\delta u\left[\delta\left(\ln(\rho u^{\gamma})\right)\right]\end{aligned}$$

✓

→ What does it Mean?

$$\underset{\text{volume}}{V} = \int dl\, A = \Phi \int \frac{dl}{B}$$

now $\underline{\nabla} p$ → "expansion free energy"

$\delta V > 0 \Rightarrow \delta \int \frac{dl}{B} > 0$ → fluid element expands
⇒ tends reduce W_p

$\delta V < 0 \Rightarrow \delta \int \frac{dl}{B} < 0$ → fluid element compresses
⇒ tends increase W_p

$\delta V > 0$ → "maximum B"
$\delta V < 0$ → "minimum B".

Can then define:

$$\underset{\substack{\text{potential} \\ \text{energy of tube}}}{E_p} = -p\, U \quad , \quad U \equiv -\int \frac{dl}{B}$$

(i.e. for sign convention)

∴ → can argue tube tends to move in direction of lower U.
→ equilibrium for $p = p(U)$

then, not surprisingly, can develop parallel between convection and interchange

i.e.

Convection	Interchange
gravitational potential energy	$E_p \rightarrow$ expansion energy
blob	flux tube.
displace blob	displace tube
$\rho' < \rho_{ambient}$ $\Rightarrow$ buoyant rise	$\frac{\delta V}{V} > 0$ $\Rightarrow$ expansion continues (with $dP/dU < 0$) (squeezed out)
adiabatic profile $\frac{\nabla P}{P} = \gamma \frac{\nabla \rho}{\rho}$	adiabatic displacement $-\gamma P \frac{\delta U}{U} = \frac{dP}{dU} \delta U$
Schwarzschild Criterion $\frac{\nabla P}{P} < \gamma \frac{\nabla \rho}{\rho}$ for instability	Interchange Criterion $\frac{dP}{dU} \delta U < -\gamma P \frac{\delta U}{U}$ $\Rightarrow \left\lvert \frac{dP}{dU} \right\rvert > -\gamma \frac{P}{U}$ ∇P sufficiently steep $\rightarrow$ overcome compr.

$\therefore$ for instability: $\left|\frac{dp}{du}\right| > \frac{-\gamma p}{u}$

$\left|\frac{dp}{du}\right|$: change from relaxation

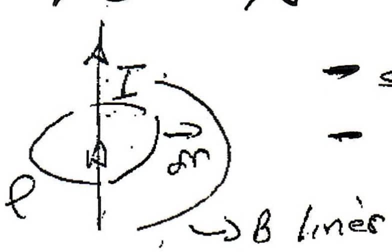
$\frac{-\gamma p}{u}$ → adiabatic pressure change

for stability, need:

$$\boxed{\left|\frac{dp}{du}\right| < \frac{\gamma p}{|u|}}$$

→ Consider some configurations (magnetic)

a.) single wire



- stability to displacement δr
- $\nabla p/p$ limit ?

now $\delta \int \frac{d\ell}{B}$

$d\ell = 2\pi r$

$B = 2I/r$

$d\ell/B \sim \frac{\pi}{I} r^2$ [→ "wire is not "minimum -B"" ! i.e. actually maximal → will have a ∇p crit.

for ∇p limit: $$\frac{dp}{dU} < \frac{\gamma p}{|U|}$$

$$U \equiv -\int \frac{d\ell}{B} \sim -r^2$$

$$\frac{dp}{dU} = \frac{dp}{dr}\frac{dr}{dU}$$

$\boxed{U \text{ scalar} \Rightarrow \text{I cancels}}$

$$= \left|\frac{dp}{dr}\right|\left(\frac{1}{2r}\right) \qquad \Rightarrow \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{\gamma(2r)}{r^2}$$

$$\therefore \quad \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{2\gamma}{r} \quad \Rightarrow \quad \boxed{\left|\frac{d\ln p}{d\ln r}\right| < 2\gamma}$$

→ imposes limit on pressure gradient for interchange stability. ⇒ $\boxed{\text{"}\beta \text{ limits"}}$

b)
can approach point dipole similarly → i.e. earth

i.e. $B \sim 1/r^3$

$d\ell \sim r$ $\Rightarrow$ $U \sim -r^4$

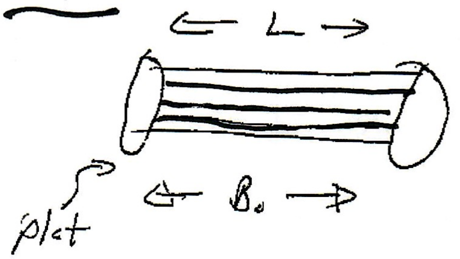
similar reasoning $\Rightarrow$ $-\frac{d\ln p}{d\ln r} < 4\gamma$

→ Line Tying and Conducting End Plates

→ Till now, have ignored boundary

⇒ consider plasma between two conducting end plates

i.e.



$E_t = 0$ on plate

$$E + \frac{v \times B}{c} = 0$$

$$\Rightarrow \xi\Big|_{\text{plate}} = 0$$

$$V_T = c\frac{E_T \times B_0}{B_0^2}$$

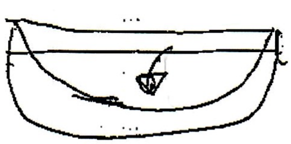
lines are "tied"!

$E_T = 0$, $V_T = 0$

$\partial_t \xi = 0$

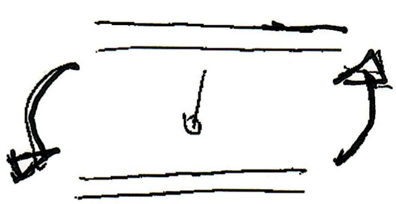
$\xi\big|_{pl} = 0$

i.e. displacement has form:



→ end points fixed.

not



ideal interchange

⇒ field lines bent!

now, $J_0 = 0$ ⇒

$$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + \gamma p (\nabla \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \nabla p_0)(\nabla \cdot \underline{\xi}) \right]$$

$$\underline{Q} = \nabla \times \underline{\xi} \times \underline{B}_0$$

$$= \underline{B}_0 \cdot \nabla \underline{\xi} - \underline{\xi}_0 \cdot \nabla \underline{B} - \underline{B}_0 \, \nabla \cdot \underline{\xi}$$

$\nabla \cdot \underline{\xi} \neq 0$

new stabilizing effect

$$\delta W = \int d^3x \left[\frac{(\underline{B}_0 \cdot \nabla \underline{\xi} - \underline{B}_0 \, \nabla \cdot \underline{\xi})^2}{8\pi} + \gamma p (\nabla \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \nabla p_0) \, \nabla \cdot \underline{\xi} \right]$$

i.e. can't take $\underline{B}_0 \cdot \nabla \underline{\xi} = 0$ any more!

so $\quad Q \sim B_0 \dfrac{\partial \xi_r}{\partial z}$ $\quad$ i.e. can make $(\nabla \cdot \underline{\xi}) \underline{B}_0$ smaller .. i.e. $\nabla \cdot \underline{V} \cong 0$

new

$$\delta W \sim V \left[\frac{B_0^2}{8\pi} \left(\frac{\partial \xi_r}{\partial z} \right)^2 + \underbrace{\gamma p \left(\frac{\delta u}{u} \right)^2 + \delta p \frac{\delta u}{u}}_{\text{old}} \right]$$

i.e. schematic ....

$$\frac{\partial \xi_r}{\partial z} \sim \frac{\xi_r}{L} \qquad \frac{\delta u}{u} = \frac{\nabla u}{u} \xi_r$$

$$\delta p = \nabla p \, \xi_r$$

⇒ heuristic:

$$\delta W \sim V\left\{\left(\frac{B_0^2}{8\pi L^2} + \gamma p\left(\frac{\nabla\psi}{\psi}\right)^2 + \nabla p\frac{\nabla\psi}{\psi}\right)\varepsilon^2\right\}$$

$\therefore$ $\delta W < 0 \to$ instability ⇒

instability if $\quad -\frac{\nabla p\,\nabla\psi}{\psi} < \gamma p\left(\frac{\nabla\psi}{\psi}\right)^2 + \underbrace{\frac{B^2}{8\pi L^2}}_{\text{additional stabilizing effect}}$

⇒ line tying raises critical pressure gradient

⇒ clearly stabilizing ⇒ β limit!

Physics → fixing end points forces bending of field lines

→ lose interchange structure

→ energy expended coupling to plucking magnetic field lines.